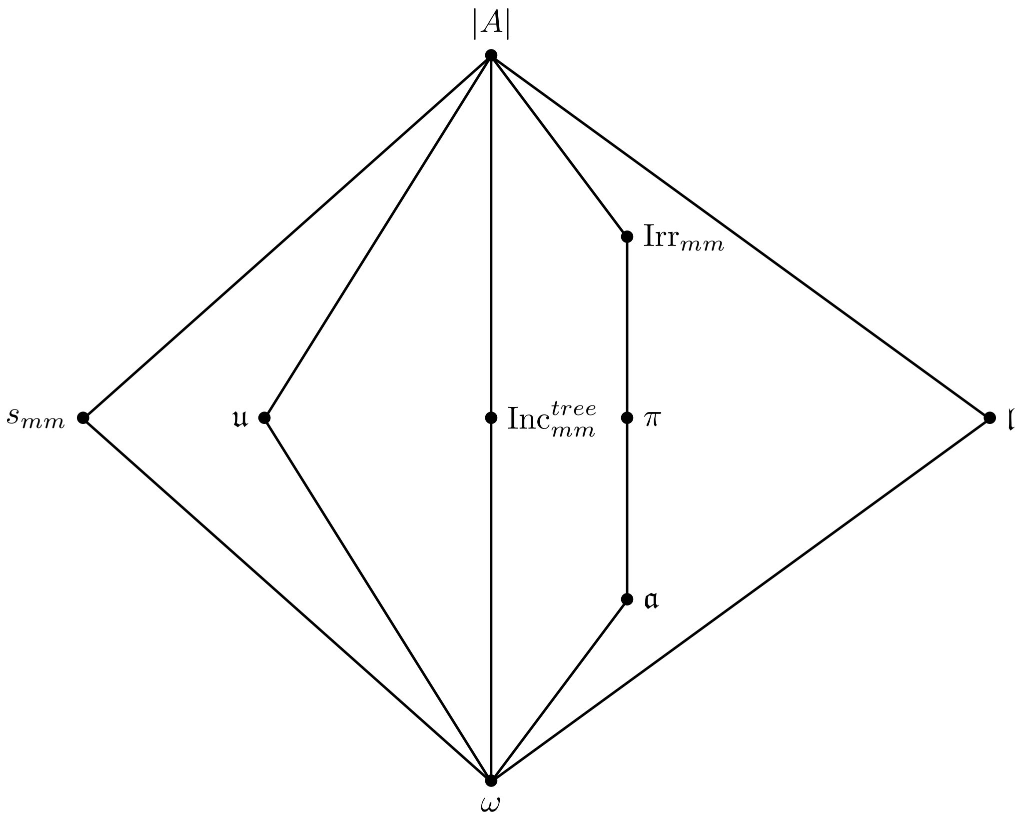
We consider small cardinal functions on the following classes of BAs; these were not considered in Monk [14] or Monk [18]: interval algebras, complete BAs, tree algebras, atomless tree algebras, pseudo-tree algebras, atomless pseudo-tree algebras.

1. Interval algebras



Interval algebras

Several of the invariants considered in Monk [18] do not exist in some interval algebras. These are omitted in the above diagram. Thus $\mathfrak{h}$ does not exist in an atomic interval algebra, since such algebras are completely distributive. $\mathrm{tow}(A)$ does not always exist; see Monk [09]. $\mathfrak{s}(A)$ does not exist if A has an atom. Some of the invariants always exist, but are trivial (equal to 1) in some interval algebras. By Proposition 2 of Monk [18], the existence of atoms implies that $\mathfrak{r}(A) = \pi_{\min}(A) = \mathfrak{i}(A) = \mathfrak{f}(A) = 1$ for interval algebras which have an atom. Moreover, $\mathrm{Inc}_{\mathrm{mm}}(A) = \mathfrak{i}_n(A) = \mathfrak{q}(A) = 2$ if A has an atom. $\mathrm{Card}_{H-}(A) = \omega$ for any interval algebra, by Koppelberg, Monk [92]. Now $\mathfrak{r}(A) \leq \mathfrak{i}(A)$ in general, and $\mathfrak{i}(A)$ is countable for any interval algebra A by Corollary 15.15 of Koppelberg [89]; so $\mathfrak{i}(A) = \omega = \mathfrak{r}$ for any interval algebra. $\mathfrak{u}(A) = \mathfrak{p}(A)$ for any interval algebra A by Theorem 14.18 of Monk [14], and $\mathfrak{f}(A) = \mathfrak{u}(A)$ for any atomless interval algebra A by Proposition 4.6 of Monk [12]. Hence $\mathfrak{f}(A) \leq \mathfrak{u}(A)$ for any interval algebra A. Clearly $\mathrm{Card}_{\mathrm{H}-}(A) = \omega$ for any infinite interval algebra; see Lemma 9.10 of Monk [14].

2. Complete BAs

First we check that the implications in the diagram on the next page hold. $\mathrm{cf}(A) = \mathrm{alt} = \mathrm{p\text{-}alt}(A) = \omega_1$ and $\mathrm{Card}_{\mathrm{H}-}(A) = 2^\omega$ by Monk [14], Chapter 9 and Koppelberg [77].

Clearly $\mathfrak{p} = \mathrm{tow} = \mathfrak{a} = \omega$. We have $\pi\chi_{\mathrm{inf}}(A) = \mathfrak{r}(A)$ by Balcar, Simon [91]. $\mathfrak{h}(A) \le \mathfrak{r}(A)$ by Proposition 25 in Monk [01]. We have $\omega_1 \le \mathfrak{i}_n(A)$ by Bruns [13], 6.6. The other implications hold for BAs in general.

We now turn to other possible implications, and we first indicate a construction of a complete atomless BA A such that $\mathfrak{l}(A) > 2^\omega$.

Lemma 2.1. *Let A be a complete atomless BA, X a maximal chain in A of size 2^ω, and $x \in X$. Then there exists a BA B such that*

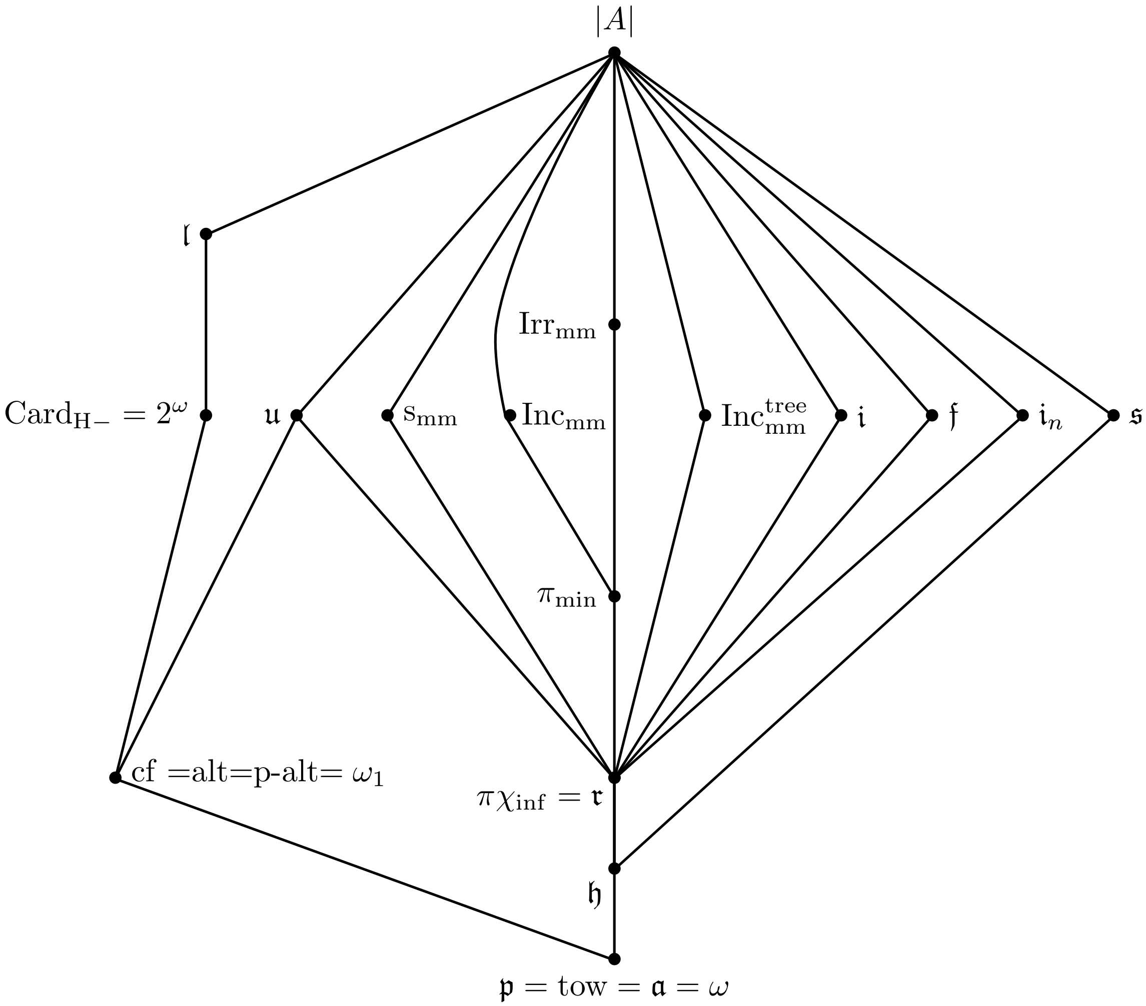
(i) A is a subalgebra of B;

(ii) B is complete and atomless;

(iii) X is not a maximal chain in B.

Proof. Let $A(y)$ be a free extension of A by an element y. Let I be the ideal in $A(y)$ generated by $\{x \cdot -y\} \cup \{y \cdot -z : z \in X, x < z\}$.

(1) $A \cap I = \{0\}$.



Complete BAs

In fact, suppose that $a \in A \cap I$. Then we can write

(2) $$a \le x \cdot -y + y \cdot -z_0 + \cdots + y \cdot -z_{m-1}$$

with $z_0, \ldots, z_{m-1} \in X$ and $x < z_0 < \ldots < z_{m-1}$. Since X is dense, there is an element $u \in X$ such that $x < u < z_0$. Fixing A pointwise and mapping y to u, (2) yields $a = 0$.

(3) $y \cdot -x \notin I$.

In fact, otherwise we get $y \cdot -x \leq$ the right side of (2) for some $z_0, \ldots, z_{m-1}$ as in (2). With u as above, fixing A pointwise and mapping y to u, we get $u \cdot -x = 0$, contradiction.

Now take $C = A(y)/I$, let D be an atomless BA such that $C \leq D$, and let B be the completion of D. $\square$

Lemma 2.2. *If A is a complete atomless BA, then there is a complete atomless BA B such that $A \leq B$, and no maximal chain in A of size 2^ω is maximal in B.*

Proof. Let $\kappa = |A|^{2^\omega}$, and let $\langle X_\alpha : \alpha < \kappa \rangle$ enumerate all of the maximal chains of A of size 2^ω. By an obvious transfinite construction using Lemma 2.1, and at limit steps taking the completion of th union of previous algebras we obtain the desired algebra B. $\square$

Proposition 2.3. *There is an atomless complete BA A such that $\mathfrak{l}(A) > 2^\omega$.*

Proof. Iterate Lemma 2.2 $(2^\omega)^+$ times, taking the completion of the union at limit stages. $\square$

For κ uncountable and regular, let $A = \overline{\mathrm{Fr}(\kappa)}$. We claim that $\mathfrak{l}(A) \leq 2^\omega$ and $\mathfrak{i}_n(A) \geq \kappa$. In fact, we claim that every chain in A has size at most 2^ω. Suppose that X is a chain in A with $|X| > 2^\omega$. Let $\prec$ be a well-order of A, and define $f : [X]^2 \to 2$ by setting, for any distinct $x, y \in X$, say with $x \prec y$,

$$f(\{x, y\}) = \begin{cases} 1 & \text{if } x < y, \\ 0 & \text{if } y < x. \end{cases}$$

By the Erdös, Rado theorem $(2^\omega)^+ \to (\omega_1)^2_\omega$ there exist $Y \in [X]^{\omega_1}$ and $\varepsilon \in 2$ such that $f[[Y]^2] \subseteq \{\varepsilon\}$. Say $\varepsilon = 1$. Then $x \prec y$ with $x, y \in Y$ implies that $x < y$. This contradicts ccc.

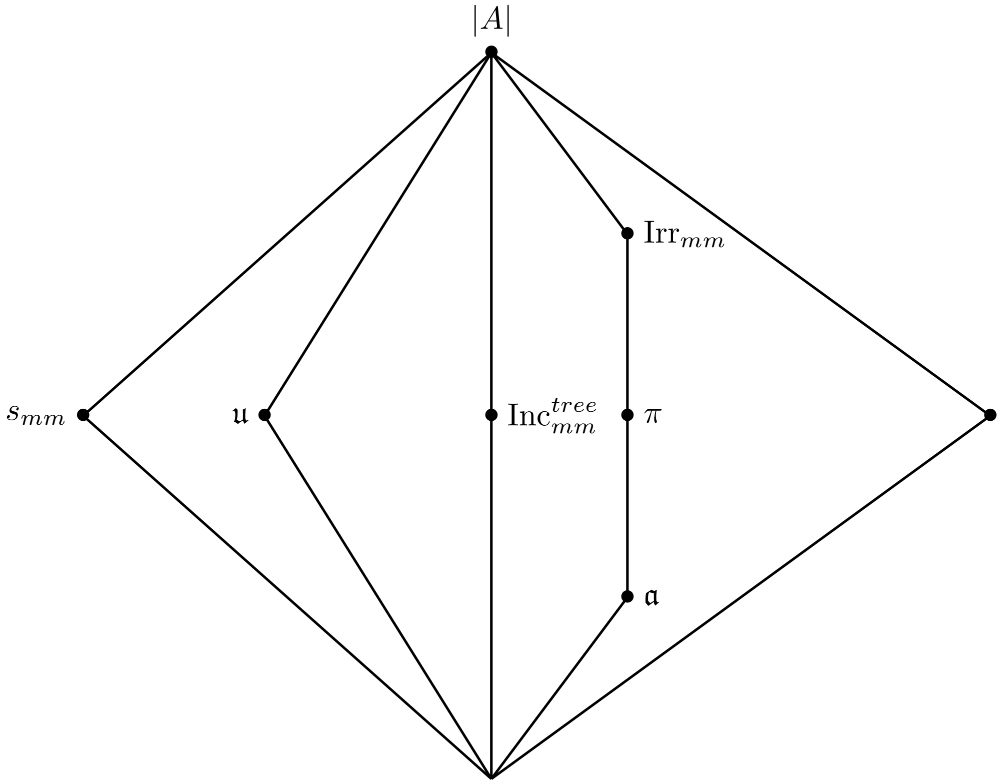
Now we also claim that $\mathfrak{i}_n(A) \geq \kappa$. For, suppose that Z is n-independent and $|Z| < \kappa$. For each $z \in Z$ let M_z be a countable subset of $\mathrm{Fr}(\kappa)$ such that $z = \sum M_z$. Let x be a free generator of $\mathrm{Fr}(\kappa)$ not in the support of any element of $\bigcup_{z \in Z} M_z$. Then $z \cdot x \neq 0$ for all $z \in Z$, and it follows that $Z \cup \{x\}$ is n-independent. Thus Z is not maximal, and hence $\mathfrak{i}_n(A) \geq \kappa$.

We have $\mathfrak{i}_n(\overline{\mathrm{Fr}(\omega)}) = \omega$. In fact, let X be a maximal n-independent subset of $\mathrm{Fr}(\omega)$ containing all the free generators of $\mathrm{Fr}(\omega)$. If $y \in \overline{\mathrm{Fr}(\omega)} \backslash \mathrm{Fr}(\omega)$, then there is a monomial z such that $z \leq y$. Then $z \cdot -y$ shows that $X \cup \{y\}$ is not n-independent.

This example gives a BA A such that $\mathfrak{i}_n(A) < \mathrm{cf}(A)$.

3. Tree algebras

$\mathfrak{i}(A)$, $\mathfrak{f}(A)$, $\mathrm{Inc}_{mm}(A)$, $\mathfrak{i}_n(A)$ can be finite; see Monk [18]. $\mathfrak{s}(A)$ is undefined if A has an atom. $\mathfrak{p}(A)$ is undefined. For $\mathrm{Card}_{H-}(A) = \omega$, see Koppelberg, Monk [92].



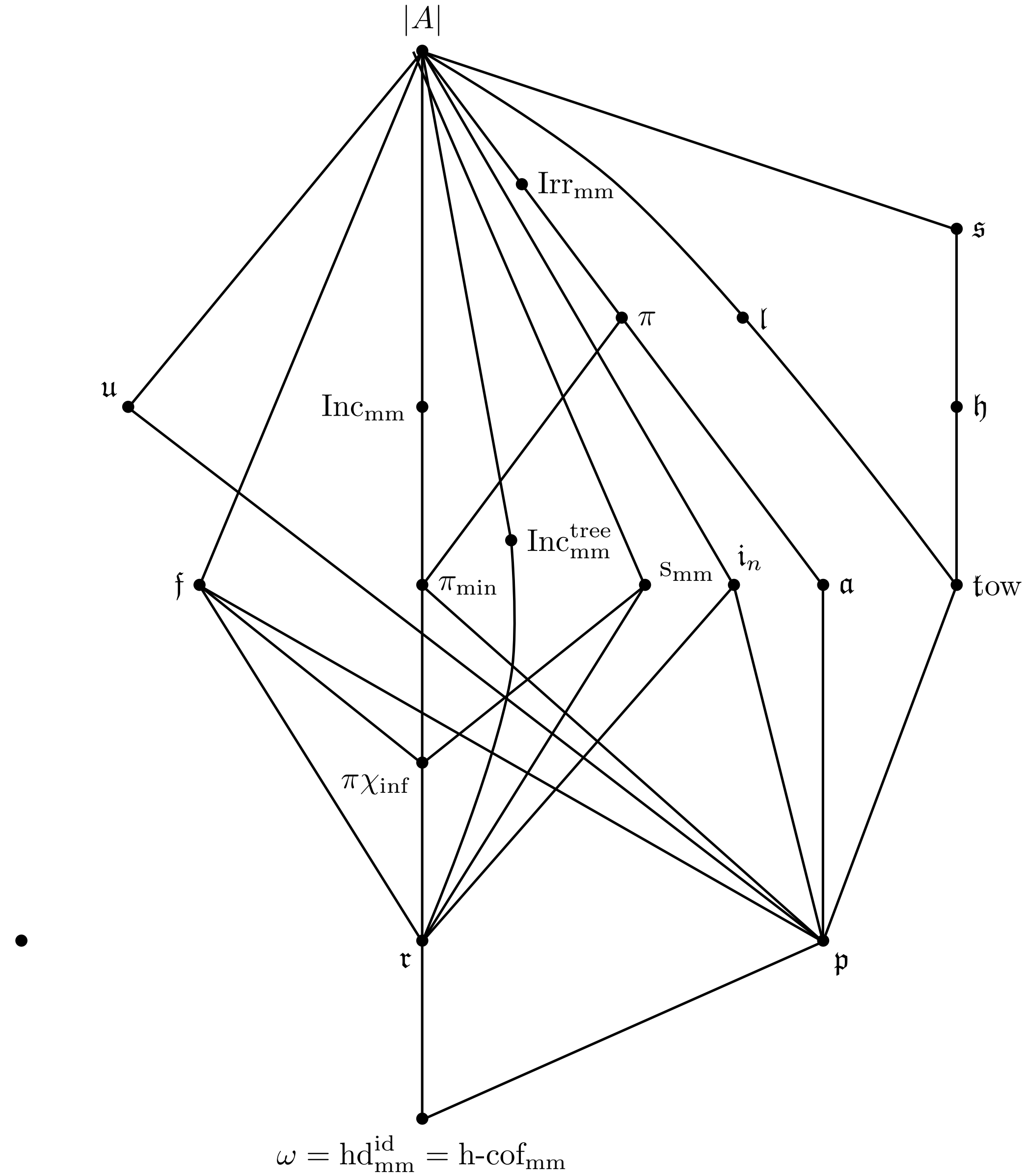
Tree algebras

4. Atomless tree algebras

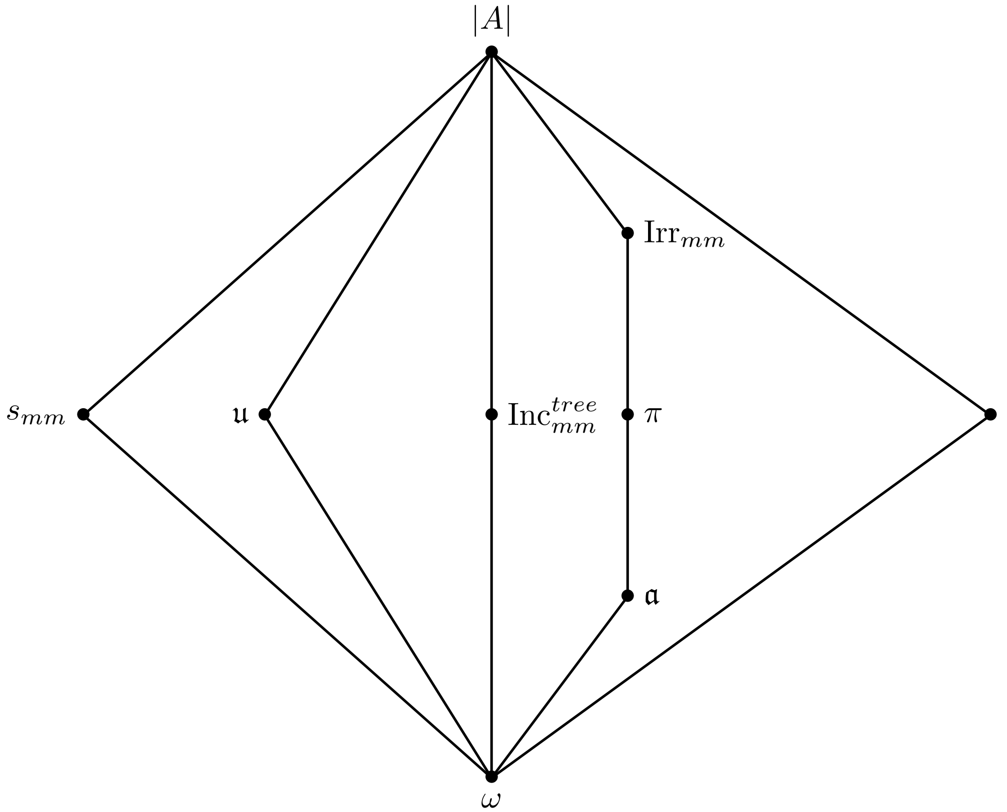
See the diagram on the following page. For $\mathrm{Card}_{H-}(A) = \omega$ see Koppelberg, Monk [92].

5. Pseudo-tree algebras

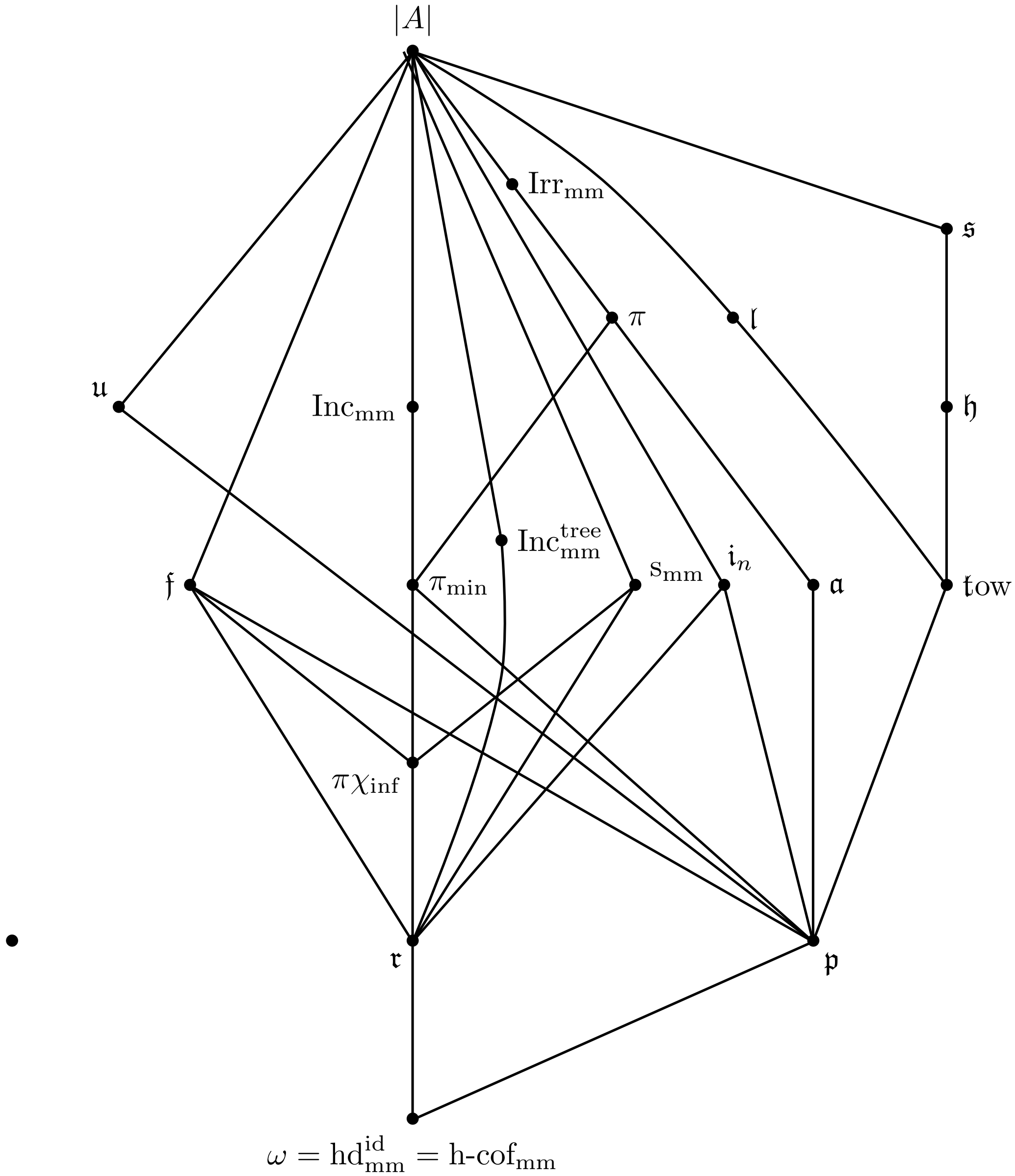
See the diagram below.



Atomless tree algebras



Pseudo-tree algebras



Atomless pseudo-tree algebras

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