Invariants of countable Boolean algebras
and Ketonen’s theorem

These are notes which are intended to give details in Ershov’s system of invariants of countable Boolean algebras, and in Pierce’s treatment of Ketonen’s theorem. In the first part, concerning the invariants, we also use the treatment of Pinus. We assume an elementary knowledge of Boolean algebras; see e.g. the book of Koppelberg.

I. Invariants of countable Boolean algebras

1. Elementary theory

A distributive lattice with 0 \((A, +, \cdot, 0)\) is relatively complemented iff \(\forall a, b \in A[a \leq b \rightarrow \exists c \in A[a + c = b \text{ and } a \cdot c = 0]]\). ABID is the class of all relatively complemented distributive lattices with 0. “ABID” abbreviates “abstract ideal”.

**Proposition 1.1.** If \(A\) is a ABID and \(a, b \in A\) with \(a \leq b\), then there is a unique \(c \in A\) such that \(a + c = b\) and \(a \cdot c = 0\).

**Proof.** Suppose that \(c, d \in A\) with \(a + c = b\), \(a \cdot c = 0\), \(a + d = b\), \(a \cdot d = 0\). Then \(c = c \cdot (a + c) = c \cdot b = c \cdot (a + d) = c \cdot a + c \cdot d = c \cdot d \leq d\). Similarly \(d \leq c\). \(\square\)

If \(A\) is a ABID and \(a, b \in A\), then \(a \cdot b \leq a\), and we let \(a \backslash b\) be the unique \(c\) such that \(a \cdot b + c = a\) and \(a \cdot b \cdot c = 0\). Note that \(\forall b \in A[0 \backslash b = 0]\). If \(A = (A, +, \cdot, 0)\) is a ABID, then we define \(A' = (A, +, \cdot, 0, \backslash)\).

**Proposition 1.2.** If \(A\) and \(B\) are ABIDs and \(f\) is a homomorphism from \(A\) to \(B\), then \(\forall a, b \in A[f(a \backslash b) = f(a) \backslash f(b)]\).

**Proof.**

\[
\begin{align*}
    f(a) \cdot f(b) + f(a \backslash b) &= f(a \cdot b + a \backslash b) = f(a) \quad \text{and} \\
    f(a) \cdot f(b) \cdot f(a \backslash b) &= f(a \cdot b \cdot (a \backslash b)) = f(0) = 0.
\end{align*}
\]

Thus if \(f\) is a homomorphism from a ABID \(A\) into a ABID \(B\), then \(f\) is a homomorphism from \(A'\) into \(B'\).

**Proposition 1.3.** If \(A\) is a ABID, \(B\) is a BA, and \((A, +_A, \cdot_A)\) is a subalgebra of \((B, +_B, \cdot_B)\), then \(\forall a, b \in A[a \backslash b = a \cdot (\neg b)]\).

**Proof.** Assume the hypotheses, and suppose that \(a, b \in A\). Then

\[
\begin{align*}
    a \backslash b &= (a \backslash b) \cdot (a \cdot b + a \backslash b) = (a \backslash b) \cdot a = (a \backslash b) \cdot (a \cdot b + a \cdot (\neg b)) \\
    &= (a \backslash b) \cdot a \cdot (\neg b) \leq a \cdot (\neg b) = a \cdot (\neg b) \cdot a \\
    &= a \cdot (\neg b) \cdot (a \cdot b + a \backslash b) = a \cdot (\neg b) \cdot (a \backslash b) \leq a \backslash b. \quad \square
\end{align*}
\]

**Proposition 1.4.** If \(A\) is a ABID, then the following are equivalent:
(i) \( \exists e \in A \forall a \in A[a \leq e] \).
(ii) There is an \( e \in A \) such that \( (A, +, \cdot, -, 0, e) \) is a BA, where \( \forall a \in A[-a = e \setminus a] \).

**Proof.** (i)⇒(ii): Assume (i), with \( e \) as indicated. Define \( - \) as in (ii). Then \( a + -a = a + (e \setminus a) = e \cdot a + (e \setminus a) = e \) and \( a \cdot -a = a \cdot (e \setminus a) = e \cdot a \cdot (e \setminus a) = 0 \).

(ii)⇒(i): clear. \( \square \)

**Proposition 1.5.** Every finitely generated ABID is a BA.

**Proof.** If \( A \) is a ABID generated by a finite set \( F \), then every element of \( A \) is \( \leq \sum F \).

**Proposition 1.6.** Every BA \( A \) is a maximal ideal in another BA \( B \), where \( \forall a, b \in A[a + A b = a + B b \text{ and } a \cdot A b = c \cdot B b] \), while \( \forall a \in A[-A a = 1_A \cdot B - B a] \).

**Proof.** Define \( f(a) = (0, a) \) for all \( a \in A \). Then \( f \) is an injection of \( A \) into \( 2 \times A \). Let \( X \) be such that \( A \cap X = \emptyset \) and \( |X| = |A| \). Let \( g \) be a bijection from \( X \) onto \( \{(1, a) : a \in A\} \).
Define \( B = A \cup X \) and define operations on \( B \) so that \( f \cup g \) is an isomorphism of \( B \) with \( 2 \times A \). Then for all \( a, b \in A \),

\[
(f \cup g)(a + B b) = (f \cup g)(a) + (f \cup g)(b) = f(a) + f(b)
= (0, a) + (0, b) = (0, a + A b) = (f \cup g)(a + A b);
\]

hence \( a + B b = a + A b \). Similarly \( a \cdot B b = a \cdot A b \). Further, if \( a \in A \), then \( f \cup g)(1_A \cdot B - B a) = (0, 1_A) \cdot -(0, a) = (0, -A a) = (f \cup g)(-A a) \) and hence \( 1_A \cdot B - B a = -A a \). Next, suppose that \( a, b \in B \) and \( a \leq B b \in A \). Then \( a \cdot B b = a \), so \( (f \cup g)(a) = (f \cup g)(a \cdot B b) = (f \cup g)(a) \cdot (0, b) \);

hence \( (f \cup g)(a) = (0, x) \) for some \( x \), so \( a \in A \). It follows that \( A \) is an ideal in \( B \). For any \( b \in B \), if \( b \notin A \) then \( b \in X \), hence \( (f \cup g)(-B b) = -(f \cup g)(b) = -g(b) = -(1, b) = (0, -b) \);

hence \( -b \in A \). So \( A \) is a maximal ideal in \( B \).

An ideal in a ABID \( A \) is a subset \( I \subseteq A \) such that \( 0 \in I, I \) is closed under \( + \), and \( \forall a \in A \forall b \in I[a \leq b \in I \rightarrow a \in I] \). \( I \) is prime iff \( \forall a, b \in A[a \cdot b \in I \rightarrow a \in I \text{ or } b \in I] \).

**Proposition 1.7.** If \( I \) is an ideal in a ABID \( A \), then \( (I, +, \cdot, 0) \) is a ABID.

**Proof.** Given \( a, b \in I \) with \( a \leq b \), we have \( b \setminus a \in I \).

**Proposition 1.8.** If \( A \) is a ABID, \( a, b \in A \), and \( a \neq b \), then there is a prime ideal \( I \) containing one of \( a, b \) but not the other.

**Proof.** Wlog \( a \not\leq b \). Then \( \{x \in A : x \leq b\} \) is a member of \( \mathcal{A} \) \( \text{def } \{I : I \text{ is an ideal on } A \text{ with } b \in I \text{ and } \forall u \geq a[u \notin I]\} \). By Zorn’s lemma let \( I \) be a maximal member of \( \mathcal{A} \). Suppose that \( x \cdot y \in I \), but \( x, y \notin I \). Then the ideal generated by \( I \cup \{x\} \) is not in \( \mathcal{A} \), so there exist \( x' \in I \) and \( u \geq a \) such that \( u \leq x + x' \). Similarly there are \( y' \in I \) and \( v \geq a \) such that \( v \leq y + y' \). Then \( a \leq u \cdot v \leq (x + x') \cdot (y + y') = x \cdot y + x \cdot y' + x' \cdot y + x' \cdot y' \in I \), contradiction. \( \square \)
Proposition 1.9. Any ABID $A$ is isomorphic to a ABID $B$ of the form

$$B = (B, \cup, \cap, \emptyset).$$

Moreover, for any $a, b \in B$, $a \setminus_B b$ is the usual set-theoretic relative complement $a \setminus b$. $B$ consists of subsets of some set $X$, and each $I \subseteq X$ is a member of some $b \in B$.

Proof. Let $A$ be any ABID. Let $X$ be the set of all proper prime ideals in $A$. For each $a \in A$ let $f(a) = \{I \subseteq X : a \notin I\}$. Then

$$I \subseteq f(a + b) \iff a + b \notin I$$
$$I \subseteq f(a \cdot b) \iff a \cdot b \notin I$$
$$I \subseteq f(0) \iff 0 \notin I$$

Moreover,

$$(*) \forall a, b \in A \forall I \subseteq X[a \setminus b \notin I \iff a \notin I \text{ and } b \in I].$$

In fact, suppose that $a, b \in A$ and $I \subseteq X$. First suppose that $a \setminus b \notin I$. Clearly then $a \notin I$. Now $(a \setminus b) \cdot b = 0 \in I$ implies that $b \in I$. Second, suppose that $a \notin I$, $b \in I$, and $a \setminus b \in I$. Then $a = a \cdot b + (a \setminus b) \in I$, contradiction. So $(*)$ holds. By $(*)$, $I \subseteq f(a \setminus b)$ iff $I \subseteq f(a) \setminus f(b)$.

Now $f$ is one-one. For, suppose that $a, b \in A$ and $a \neq b$. By Proposition 1.8, wlog there is a prime ideal $I$ such that $a \in I$ and $b \notin I$. Hence $I \notin f(a)$ but $I \subseteq f(b)$.

If $I \subseteq X$, then $I$ is proper, so there is an $a \in A \setminus I$. Then $I \subseteq f(a)$. \qed

Proposition 1.10. If $A$ is a ABID which is not a BA, then $A$ is a maximal ideal in some BA $B$, with $\forall a, b \in A[a +_A b = a +_B b, a \cdot_A b = a \cdot_B b, \text{ and } a \setminus_A b = a \setminus_B b]$. Moreover, $B = A \cup \{-x : x \in A\}$ and $A \cap \{-x : x \in A\} = \emptyset$.

Proof. By Proposition 1.9 we may assume that $A = (A, \cup, \cap, \emptyset)$, such that $\forall a, b \in A[a \setminus b$ is the set-theoretic relative complement]. Moreover, $A$ consists of subsets of some set $X$, and each $x \in X$ is a member of some $a \in A$. Now let $B$ be the Boolean algebra of subsets of $X$ generated by $A$.

1.10(1) $B = A \cup \{X \setminus a : a \in A\}$.

In fact, it suffices to note that the set on the right is closed under $\cdot$. For $a, b \in A$ we have $a \cdot b \in A$, $a \cap (X \setminus b) = a \setminus b \in A$. Finally, $(X \setminus a) \cap (X \setminus b) = X \setminus (a \cup b)$. So 1.10(1) holds.

1.10(2) $A \cap \{X \setminus a : a \in A\} = \emptyset$.

In fact, if $a \in A$ and $X \setminus a \in A$ then $X \in A$, contradiction.

1.10(3) $\forall a, b \in B[b \in A \text{ and } a \subseteq b \rightarrow a \in A]$.
For, suppose that \(a, b \in B, b \in A, a \subseteq b, \) and \(a \notin A\). By 1.10(1) there is a \(c \in A\) such that \(a = X \setminus c\). Then \(X \setminus c = (X \setminus c) \cap b = b \setminus c \in A\), contradicting 1.10(2).

Now \(A\) is an ideal in \(B\). For, obviously \(\emptyset \in A\). If \(a, b \in A\), then clearly \(a \cup b \in A\). If \(a, b \in B, b \in A,\) and \(a \subseteq b,\) and \(a \notin A\), then by (1), \(X \setminus a \in A,\) and hence \(X = b \cup (X \setminus a) \in A,\) contradiction. So \(A\) is an ideal in \(B\).

By 1.10(1), \(A\) is maximal.

Elementary arithmetic in an ABID can be carried out using the part of Propositions 1.6 and 1.10 that says that \(a \setminus b = a \cdot -b\) in the associated BA.

For any ABID \(A\) and any \(a, b \in A\), let \(a \Delta b = (a \setminus b) + (b \setminus a)\). The proof of the following proposition illustrates the use of Propositions 1.6 and 1.10.

**Proposition 1.11.** \(a \Delta (b \Delta c) = (a \Delta b) \Delta c\).

**Proof.**

\[
 a \Delta (b \Delta c) = (a \setminus ((b \setminus c) + (c \setminus b))) + (((b \setminus c) + (c \setminus b)) \setminus a) \\
= a \cdot (-b \cdot -c + c \cdot -b) + (b \cdot -c + c \cdot -b) \cdot -a \\
= a \cdot (-b + c) \cdot (-c + b) + b \cdot -c + c \cdot -a + c \cdot -b \cdot -a \\
= a \cdot -b \cdot -c + a \cdot c \cdot b + b \cdot -c + c \cdot -a + c \cdot -b \cdot -a \\
= a \cdot -b \cdot -c + a \cdot b \cdot c + -a \cdot b \cdot -c + -a \cdot -b \cdot c; \\
\]

\[
(a \Delta b) \Delta c = (a \cdot -b + b \cdot -a) \cdot -c + c \cdot -(a \cdot -b + b \cdot -a) \\
= a \cdot -b \cdot -c + b \cdot -a \cdot -c + c \cdot -(a + b) \cdot (-b + a) \\
= a \cdot -b \cdot -c + -b \cdot -a \cdot -c + c \cdot -a \cdot -b + c \cdot b \cdot a \\
= a \cdot -b \cdot -c + -a \cdot b \cdot -c + -a \cdot -b \cdot c + a \cdot b \cdot c. \\
\]

\[\square\]

**Proposition 1.12.** If \(R\) is a congruence relation on a ABID \(A, a, b, a', b' \in A, aRa',\) and \(bRb',\) then \((a \setminus b)R(a' \setminus b')\).

**Proof.** Assume the hypotheses. For each \(x \in A\) let \([x]\) be the equivalence class of \(x\) under \(R\). Define \(f(x) = [x]\) for all \(x \in A\). Then \(f\) is a homomorphism. Hence by Proposition 1.2, \(f(a \setminus b) = f(a) \setminus f(b) = f(a') \setminus f(b') = f(a' \setminus b')\) and the result follows. \[\square\]

**Proposition 1.13.** Let \(A\) be an ABID.

(i) If \(R\) is a congruence on \(A\), then \(I_R \overset{\text{def}}{=} \{a \in A : aR0\}\) is an ideal in \(A\).

(ii) If \(I\) is an ideal on \(A\), then \(R_I \overset{\text{def}}{=} \{(a, b) : (a \setminus b) + (b \setminus a) \in I\}\) is a congruence on \(A\).

(iii) \(R_{IR} = R\) for any congruence \(R\) on \(A\).

(iv) \(I_{RI} = I\) for any ideal \(I\) on \(A\).

**Proof.** (i): Assume that \(R\) is a congruence on \(A\). Since \(0R0, I_R\) is nonempty. Suppose that \(a \leq b \in I_R\). Thus \(bR0\). Hence \(a = (a \cdot b)R(a \cdot 0) = 0,\) so \(a \in I_R.\) Suppose that \(a, b \in I_R.\) Then \(aR0\) and \(bR0\), so \((a + b)R(0 + 0) = 0\) and \(a + b \in I_R.\)
(ii): Assume that \( I \) is an ideal on \( A \). Suppose that \( aR_{I}b \) and \( cR_{I}d \). Then
\[
(a+c)\setminus(b+d) = (a\setminus(b+d)) + (c\setminus(b+d)) = (a\setminus b) \cdot (a\setminus d) + (c\setminus b) \cdot (c\setminus d) \\
\leq (a\setminus b) + (c\setminus d) \in I,
\]
and similarly \((b+d)\setminus(a+c) \in I\), so \((a+c)R_{I}(b+d)\). Also,
\[
(a \cdot c)\setminus(b \cdot d) = ((a \cdot c)\setminus b) + ((a \cdot c)\setminus d) = ((a\setminus b) \cdot (c\setminus b)) + ((a\setminus d) \cdot (c\setminus d)) \\
\leq (a\setminus b) + (c\setminus d) \in I,
\]
and similarly \((b \cdot d)\setminus(a \cdot c) \in I\), so \((a \cdot c)R_{I}(b \cdot d)\).

This shows that \( R_{I} \) is a congruence on \( A \).

(iii): assume that \( R \) is a congruence on \( A \). Suppose that \( aR_{I_{R}}b \). Thus \(((a\setminus b) + (b\setminus a)) \in I_{R} \), so \(((a\setminus b) + (b\setminus a))R0 \). Now
\[
[(a\setminus b) + (b\setminus a)]a = [(a\setminus b)\setminus a] + [(b\setminus a)\setminus a] = (b\setminus a).
\]
Thus \( (b\setminus a) = [(a\setminus b) + (b\setminus a)]aR(0\setminus a) = 0 \), so \( a + b = ((b\setminus a) + a)Ra \). Similarly \( (a + b)Rb \), so \( aRb \).

Conversely, suppose that \( aRb \). Then \( 0 = (a\setminus a)Ra(b\setminus b) \), so \( (a\setminus b) \in I_{R} \). Similarly \( (b\setminus a) \in I_{R} \), so \( [(a\setminus b) + (b\setminus a)] \in I_{R} \). Thus \( aR_{I_{R}b} \).

(iv): assume that \( I \) is an ideal on \( A \). Suppose that \( a \in I_{R_{I}} \). Thus \( aR_{I}0 \), so \( a = [(a\setminus 0) + (0\setminus a)] \in I \).

Conversely, suppose that \( a \in I \). Again, \( a = [(a\setminus 0) + (0\setminus a)] \), so \( aR_{I}0 \), hence \( a \in I_{R_{I}} \).

A filter on a ABID \( A \) is a subset \( F \) of \( A \) such that \( F \) is nonempty, \( F \) is closed under \( \cdot \), and \( \forall a, b \in A \{ b \geq a \in F \rightarrow b \in F \} \). If \( A \) is a ABID and \( a \in A \) we let \( a^{\perp} = \{ b \in A : a \cdot b = 0 \} \).

For \( F \) a filter, \( F^{\perp} = \bigcup_{a \in F} a^{\perp} \).

**Proposition 1.14.** \( a^{\perp} \) is an ideal of \( A \).

**Proposition 1.15.** If \( F \) is a filter on a ABID \( A \), then \( F^{\perp} \) is an ideal of \( A \).

**Proof.** Fix \( x \in F \). Then \( 0 \in x^{\perp} \), so \( 0 \in F^{\perp} \). Suppose that \( u, v \in F^{\perp} \). Choose \( a, b \in F \) such that \( u \in a^{\perp} \) and \( v \in b^{\perp} \). Thus \( a \cdot u = 0 = b \cdot v \). Now \( a \cdot b \in F \) and \( (u + v) \cdot a \cdot b = 0 \), so \( u + v \in (a \cdot b)^{\perp} \) and so \( u + v \in F^{\perp} \). Finally, suppose that \( a \leq b \in F^{\perp} \). Say \( y \in F \) and \( b \in y^{\perp} \). Then \( b \cdot y = 0 \), and hence \( a \cdot y = 0 \). So \( a \in y^{\perp} \) and hence \( a \in F^{\perp} \).

**Proposition 1.16.** If \( A \) is a ABID and \( F \) is a filter on \( A \), then \( R_{F^{\perp}} = \{(a, b) : \exists c \in F : ((a\setminus b) + (b\setminus a)) \cdot c = 0 \} \).
Proposition 1.17. If $A$ is a ABID and $a \in A$, then $A \upharpoonright a$ is a BA. \hfill $\blacksquare$

Proposition 1.18. If $A$ is a ABID and $a \in A$, define $f(x) = (x \cdot a, x \triangleleft a)$. Then $f$ is an isomorphism of $A$ onto $(A \upharpoonright a) \times a^\perp$.

**Proof.** If $x \in A$, then $(x \triangleleft a) \cdot a = 0$, so $x \triangleleft a \in a^\perp$. Clearly $f$ preserves the operations. If $x \leq a$ and $y \in a^\perp$, then $y = y \cdot a + y \triangleleft a = y \triangleleft a$ and so $f(x + y) = (x, y)$. Thus $f$ maps onto $(A \upharpoonright a) \times a^\perp$. Suppose that $f(x) = f(y)$. Thus $x \cdot a = y \cdot a$ and $x \triangleleft a = y \triangleleft a$. So $x = x \cdot a + x \triangleleft a = y \cdot a + y \triangleleft a = y$. So $f$ is one-one.

Let $A$ be a ABID, $a \in A$, and $I$ an ideal of $A$. Then we let $[a]_I$ be the equivalence class of $a$ under the ideal $I$; thus $[a]_I = \{a \triangleleft b : b \in I\}$.

Proposition 1.19. Suppose that $X$ generates a ABID $A$ and $f : X \to B$ with $B$ a ABID. Then $f$ extends to a homomorphism $f^+ : A \to B$ iff

$$\forall m \in \omega \setminus \{0\} \forall n \in \omega \forall a \in {}^m X \forall b \in {}^n X \left[\left(\prod_{i < m} a_i\right) \setminus \left(\sum_{j < n} b_j\right) = 0 \implies \left(\prod_{i < m} f(a_i)\right) \setminus \left(\sum_{j < n} f(b_j)\right) = 0\right].$$

**Proof.** $\Rightarrow$: Clear. $\Leftarrow$: assume the indicated condition. Now $A = \bigcup \{\langle F \rangle : F \in [X]^{<\omega}\}$, so it suffices to find for each $F \in [X]^{<\omega}$ a homomorphism $g_F : \langle F \rangle \to B$ extending $f$. Now $\langle F \rangle$ is a BA, with unit $\sum F$. Moreover, $-x$ for any $x \in \langle F \rangle$ is $(\sum F) \setminus x$. Now $f \upharpoonright F$ maps into the BA $B \upharpoonright (\sum f[F])$. We claim that

$$\forall m, n \in \omega \forall a \in {}^m X \forall b \in {}^n X \left[\prod_{i < m} a_i \cdot \prod_{j < n} (-b_j) = 0 \implies \prod_{i < m} f(a_i) \cdot \prod_{j < n} (-f(b_j)) = 0\right].$$

Here $m = 0$ is possible, with $\prod_{i < 0} a_i = \sum F$; similarly $\prod_{i < 0} f(a_i) = \sum f[F]$. To prove (*) it suffices to note that $\prod_{j < n} (-b_j) = \prod_{j < n} ((\sum F) \setminus b_j) = (\sum F) \setminus \sum_{j < n} b_j$, and similarly $\prod_{j < n} (-f(b_j)) = (\sum f[F]) \setminus \sum_{j < n} f(b_j)$. So (*) follows from the indicated condition.

Now by (*) there is a homomorphism $g_F$ from the BA $\langle F \rangle$ into the BA $B \upharpoonright (\sum f[F])$ which extends $f$. Clearly $g_F$ is as desired. \hfill $\blacksquare$

Proposition 1.20. If $A, B$ are ABIDs, $L \subseteq A$ is a chain which generates $A$, and $f : L \to B$, then $f$ extends to a homomorphism from $A$ into $B$ iff $f$ is increasing.

**Proof.** By Proposition 1.19. \hfill $\blacksquare$

Proposition 1.21. If $A, B$ are ABIDs, $L \subseteq A$ is a chain which generates $A$, and $f : L \to B$, then $f$ extends to an isomorphism from $A$ into $B$ iff $\forall a, a' \in L [a \leq a' \iff f(a) \leq f(a')]$.

**Proof.** By Proposition 1.19. \hfill $\blacksquare$

Proposition 1.22. If $L$ is a linear order which generates ABIDs $A, B$, then there is an isomorphism of $A$ onto $B$ which is the identity on $L$. \hfill $\blacksquare$
2. Examples

For any linear order \( L \) which has a least element, \( \text{Intalg}(L) \) is the Boolean algebra of subsets of \( L \) generated by the half-open intervals \([a, b)\) with \( a < b \) in \( L \). This notion is carefully analysed in the Boolean algebra handbook. Now we define the \textit{interval algebra} over \( L \), as a ABID, to be

\[
\text{Intalg}_d(L) = \{[a_0, b_0) \cup \ldots \cup [a_{m-1}, b_{m-1}) : m \in \omega, a_0 < b_0 < a_1 < \cdots < b_{m-1}\}
\]

Here we allow \( m = 0 \); so \( \emptyset \in \text{Intalg}_d(L) \). Note that \( \infty \) is not adjoined, so \( \text{Intalg}_d(L) \) is in general not a BA. Thus, for example, \( \text{Intalg}(\omega) \) and \( \text{Intalg}_d(\omega) \) are different as sets.

**Proposition 2.1.** \( \text{Intalg}_d(L) \) is closed under \( \cup, \cap, \setminus \). Hence \( \text{Intalg}_d(L) \) is a ABID.

**Proof.** For closure under \( \cap \) it suffices to prove that

\[
(*) \quad [c, d) \cap ([a_0, b_0) \cup \ldots \cup [a_{m-1}, b_{m-1})] \in \text{Intalg}_d(L)
\]

whenever \( c < d \) and \( a_0 < b_0 < a_1 < \cdots < b_{m-1} \). To prove this, for each \( i < m \) let \( s_i = \max(c, a_i) \) and \( t_i = \min(d, b_i) \). Thus \([c, d) \cap [a_i, b_i) = [s_i, t_i), \) which is nonempty iff \( s_i < t_i \). Let \( M = \{i < m : s_i < t_i\} \). If \( i, j \in M \) and \( i < j \), then \( s_i < t_i \leq b_i < a_j \leq s_j \).

If \( M = \emptyset \), then \( (*) \) is empty. If \( M \neq \emptyset \) and we enumerate \( M \) in increasing order as \( k(0) < \cdots < k(n-1) \), then \( (*) \) is \([s_k(0), t_k(0)] \cup \ldots \cup [s_k(n-1), t_k(n-1)]\). This shows closure under \( \cap \).

For closure under \( \cup \) it suffices to prove that

\[
(**) \quad [c, d) \cup [a_0, b_0) \cup \ldots \cup [a_{m-1}, b_{m-1}) \in \text{Intalg}_d(L)
\]

whenever \( c < d \) and \( a_0 < b_0 < a_1 < \cdots < b_{m-1} \). To prove this we consider several cases.

**Case 1.** \( b_{m-1} < c \). Then \( (**) \) is \([a_0, b_0) \cup \ldots \cup [a_{m-1}, b_{m-1}) \cup [c, d)\).

**Case 2.** \( b_{m-1} = c \). Then \( (**) \) is \([a_0, b_0) \cup \ldots \cup [a_{m-2}, b_{m-2}) \cup [a_{m-1}, d)\).

**Case 3.** \( c < a_0 \).

Subcase 3.1. \( d < a_0 \). Then \( (**) \) is clear.

Subcase 3.2. \( i < m, a_i \leq d \leq b_i \). Then \( (**) \) is \([c, b_i) \cup [a_{i+1}, b_{i+1}) \cup \cdots \cup [a_{m-1}, b_{m-1})\).

Subcase 3.3. \( i < m-1, b_i < d < a_{i+1} \). Then \( (**) \) is \([c, d) \cup [a_{i+1}, b_{i+1}) \cup \cdots \cup [a_{m-1}, b_{m-1})\).

Subcase 3.4. \( b_{m-1} < d \). Then \( (**) \) is \([c, d)\).

**Case 4.** \( a_0 \leq c < b_{m-1} < d \).

Subcase 4.1. \( a_i \leq c \leq b_i \). Then \( (**) \) is \([a_0, b_0) \cup \ldots \cup [a_i, d)\).

Subcase 4.2. \( i < m-1 \) and \( b_i < c < a_{i+1} \). Then \( (**) \) is \([a_0, b_0) \cup \ldots \cup [a_i, b_i) \cup [c, d)\).

**Case 5.** \( a_0 \leq c < b_{m-1} \) and \( d \leq b_{m-1} \). Let

\[
x = [c, b_{m-1}) \cup [a_0, b_0) \cup \ldots \cup [a_{m-1}, b_{m-1})
\]

\[
y = [a_0, d) \cup [a_0, b_0) \cup \ldots \cup [a_{m-1}, b_{m-1})
\]

7
Note that \([c, b_{m-1}] \cap [a_0, d) = [c, d)\]. Hence \(x \cap y\) is equal to \((**\)). Hence, because \(\text{Intalg}_d(L)\) is closed under \(\cap\), it suffices to show that \(x, y \in \text{Intalg}_d(L)\). First we consider \(x\).

**Subcase 5.1.** \(a_i \leq c < b_i\). Then \(x = [a_0, b_0) \cup \ldots \cup [a_i, b_{m-1})\).

**Subcase 5.2.** \(i < m-1\) and \(b_i < c \leq a_{i+1}\). Then \(x = [a_0, b_0) \cup \ldots \cup [a_i, b_i) \cup [c, b_{m-1})\).

Thus \(x \in \text{Intalg}_d(L)\). Now we consider \(y\).

**Subcase 5.3.** \(a_i \leq d \leq b_i\). Then \(y = [a_0, b_i) \cup [a_{i+1}, b_{i+1}) \cup \ldots \cup [a_{m-1}, b_{m-1})\).

**Subcase 5.4.** \(i < m-1\) and \(b_i < d < a_{i+1}\). Then \(y = [a_0, d) \cup [a_{i+1}, b_{i+1}) \cup \ldots \cup [a_{m-1}, b_{m-1})\).

Thus \(y \in \text{Intalg}_d(L)\).

This completes the proof of closure under \(\cup\).

To show closure under \(\setminus\), note that for \(\forall x \in A[(0\setminus x) = 0]\), and if \(m \neq 0\) then

\[
([a_0, b_0) \cup \ldots \cup [a_{m-1}, b_{m-1}) \setminus ([c_0, d_0) \cup \ldots \cup [c_{n-1}, d_{n-1})]) = \bigcup_{i < m} ([a_i, b_i) \setminus ([c_0, d_0) \cup \ldots \cup [c_{n-1}, d_{n-1}))),
\]

and for all \(i < m\),

\[
[a_i, b_i) \setminus ([c_0, d_0) \cup \ldots \cup [c_{n-1}, d_{n-1}]) = \bigcap_{j < n} ([a_i, b_i) \setminus [c_j, d_j]).
\]

Hence by the above it suffices to note that for \(i < m\) and \(j < n\) the element \([a_i, b_i) \setminus [c_j, d_j]\) is in \(\text{Intalg}_d(L)\). We show this by cases.

**Case 1.** \(b_i \leq c_j\). Then \([a_i, b_i) \setminus [c_j, d_j] = [a_i, b_i]\).

**Case 2.** \(a_i \leq c_j < b_i\).

**Subcase 2.1.** \(b_i \leq d_j\). Then \([a_i, b_i) \setminus [c_j, d_j] = [a_i, c_j]\).

**Subcase 2.2.** \(d_j < b_i\). Then \([a_i, b_i) \setminus [c_j, d_j] = [a_i, c_j) \cup [d_j, b_i]\).

**Case 3.** \(c_j < a_i\).

**Subcase 3.1.** \(b_i \leq d_j\). Then \([a_i, b_i) \setminus [c_j, d_j] = \emptyset\).

**Subcase 3.2.** \(a_i < d_j < b_i\). Then \([a_i, b_i) \setminus [c_j, d_j] = [d_j, b_i]\).

**Subcase 3.3.** \(d_j \leq a_i\). Then \([a_i, b_i) \setminus [c_j, d_j] = [a_i, b_i]\). \(\square\)

### 3. Vaught’s theorem

Let \(A\) and \(B\) be ABIDs. A **V-correspondence** between \(A\) and \(B\) is a relation \(R \subseteq A \times B\) such that the following conditions hold:

(V1) \(\forall a \in A[(a, 0) \in R \iff a = 0]\).

(V2) \(\forall b \in B[(0, b) \in R \iff b = 0]\).

(V3) \(\forall (a, b) \in R \exists c \in A \exists d_0, d_1 \in B[(a \cdot c, b \cdot d_0), (a \cap c, b \cdot d_0), (a + c, b + d_1), (c \setminus a, d_1 \setminus b) \in R]\).

(V4) \(\forall (a, b) \in R \exists d \in B \exists c_0, c_1 \in A[(a \cdot c_0, b \cdot d), (a \setminus c_0, b \cdot d), (a + c_1, b + d), (c_1 \setminus a, d \setminus b) \in R]\).

**Proposition 3.1.** If \(R\) is a V-correspondence between countably infinite ABIDs \(A, B\), then \(A\) and \(B\) are isomorphic.
Proof. Assume the hypotheses. Let \( c_0, c_1, \ldots \) enumerate all elements of \( A \) and \( d_0 = 0, d_1, \ldots \) all elements of \( B \). We now define finite linearly ordered sets \( L_0 \subseteq L_1 \subseteq \cdots \subseteq A \) and strictly increasing functions \( g_0 : L_0 \to B, g_1 : L_1 \to B, \ldots \) such that for each \( n \in \omega \) the following conditions hold:

3.1(1) If \( L_n = \{a_0, \ldots, a_k\} \) with \( a_0 < \cdots < a_k \) and \( \forall i \leq k \{b_i = g_n(a_i)\} \), then \((a_k, b_k) \in R\) and \( \forall i \leq k \{(a_{i+1}\backslash a_i, b_{i+1}\backslash b_i) \in R\}\).

3.1(2) \( g_{n+1} \upharpoonright L_n = g_n \).

3.1(3) \( c_0, \ldots, c_n \in \langle L_n \rangle \), where \( \langle L_n \rangle \) is the subalgebra of \((A, \cdot, \cdot, \backslash, 0)\) generated by \( L_n \).

3.1(4) \( d_0, \ldots, d_n \in \langle g_n[L_n] \rangle \).

We define \( L_0 = \{0\} \) and \( g_0 = \{(0, 0)\} \). Clearly 3.1(1), 3.1(3) and 3.1(4) hold for \( n = 0 \). Now suppose that \( L_m \) and \( g_m \) have been defined for all \( m \leq n \) so that 3.1(1)–3.1(4) hold. Say \( L_n = \{a_0, \ldots, a_k\} \) with \( a_0 < \cdots < a_k \) and \( \forall i \leq k \{b_i = g_n(a_i)\} \). For \( i \leq k \) let \( a'_{2i} = a_i \).

For \( i < k \) let \( a'_{2i+1} = c_{n+1} \cdot a_{i+1} + a_i \). Let \( a'_{2k+1} = c_{n+1} + a_k \).

3.1(5) \( a'_0 \leq \cdots \leq a'_{2k+1} \).

In fact, for \( i < k \) we have \( a'_{2i} = a_i \leq a'_{2i+1} \leq a_{i+1} = a'_{2i+2} \), and \( a'_{2k} = a_k \leq a'_{2k+1} \).

3.1(6) \( \forall i < k \{a'_{2i+1} \backslash a'_{2i} = c_{n+1} \cdot \{a_{i+1} \\backslash a_i\}\} \).

In fact, if \( i < k \) then \( a'_{2i+1} \backslash a'_{2i} = (c_{n+1} \cdot a_{i+1} + a_i) \backslash (c_{n+1} \cdot a_i) = c_{n+1} \cdot \{a_{i+1} \\backslash a_i\} \).

3.1(7) \( \forall i < k \{a'_{2i+2} \backslash a'_{2i+1} = (a_{i+1} \backslash a_i) \backslash c_{n+1}\} \).

In fact, if \( i < k \) then \( a'_{2i+2} \backslash a'_{2i+1} = a_{i+1} \backslash ((c_{n+1} \cdot a_{i+1}) + a_i) = (a_{i+1} \backslash (c_{n+1} \cdot a_{i+1})) \cdot (a_{i+1} \backslash a_i) = (a_{i+1} \backslash a_i) \cdot (a_{i+1} \backslash c_{n+1}) \).

3.1(8) \( a'_{2k+1} \backslash a'_{2k} = c_{n+1} \cdot a_k \).

For, \( a'_{2k+1} \backslash a'_{2k} = (c_{n+1} + a_k) \backslash a_k = c_{n+1} \cdot a_k \).

Now by 3.1(1), if \( i < k \), we have \((a_{i+1} \backslash a_i, b_{i+1} \backslash b_i) \in R\). Hence by (V3) there exist \( d'_i \in B \) for \( i < k \) such that

\[
((a_{i+1} \backslash a_i) \cdot c_{n+1}, (b_{i+1} \backslash b_i) \cdot d'_i) \in R \quad \text{and} \quad ((a_{i+1} \backslash a_i) \cdot c_{n+1}, (b_{i+1} \backslash b_i) \cdot d'_i) \in R.
\]

Also, \((a_k, b_k) \in R\), so there is a \( d'_k \in B \) such that

\[
(a_k + c_{n+1}, b_k + d'_k) \in R \quad \text{and} \quad (c_{n+1} \backslash a_k, d'_k \backslash b_k) \in R.
\]

Now for all \( i \leq k \) let \( b'_{2i} = b_i \). For all \( i < k \) let \( b'_{2i+1} = b_i + ((b_{i+1} \backslash b_i) \cdot d'_i) \), and let \( b'_{2k+1} = b_k + d'_k \).

3.1(9) \( b'_0 \leq \cdots \leq b'_{2k+1} \).

To prove this, first note that for any \( i \leq k \) we have \( b_i = g_n(a_i) \leq g_n(a_{i+1}) = b_{i+1} \).

Hence if \( i < k \), then \( b'_{2i} = b_i \leq b_i + ((b_{i+1} \backslash b_i) \cdot d'_i) = b'_{2i+1} \leq b_{i+1} = b'_{2i+2} \). Also, \( b'_{2k-1} = b'_{2(k-1)+1} = b_{k-1} + ((b_k \backslash b_{k-1}) \cdot d'_{k-1}) \leq b_k = b'_{2k} \leq b'_{2k+1} \).
3.1(10) $\forall i < k [b'_{2i+1} \setminus b'_{2i} = (b_{i+1} \setminus b_i) \cdot d'_i]$.  

In fact, if $i < k$ then $b'_{2i+1} \setminus b'_{2i} = ((b_{i+1} \setminus b_i) \cdot d'_i) \setminus b_i = ((b_{i+1} \setminus b_i) \cdot d'_i) = (b_{i+1} \setminus b_i) \cdot (d'_i \setminus b_i) = (b_{i+1} \setminus b_i) \cdot d'_i$.

3.1(11) $\forall i < k [b'_{2i+2} \setminus b'_{2i+1} = (b_{i+1} \setminus b_i) \setminus d'_i]$.  

For, if $i < k$, then  

$$b'_{2i+2} \setminus b'_{2i+1} = b_{i+1} \setminus ((b_{i+1} \setminus b_i) \cdot d'_i) = (b_{i+1} \setminus b_i) \cdot (b_{i+1} \setminus ((b_{i+1} \setminus b_i) \cdot d'_i)) = (b_{i+1} \setminus b_i) \cdot ((b_{i+1} \setminus b_i) \cdot (b_{i+1} \setminus d'_i)) = (b_{i+1} \setminus b_i) \cdot (b_{i+1} \setminus b_i + (b_{i+1} \setminus d'_i)) = (b_{i+1} \setminus b_i) \setminus d'_i.$$  

3.1(12) $b'_{2k+1} \setminus b'_{2k} = d'_k \setminus b_k$.  

For, $b'_{2k+1} \setminus b'_{2k} = (b_k + d'_k) \setminus b_k = d'_k \setminus b_k$.

3.1(13) $\forall j < 2k + 1 [(a'_{j+1} \setminus a'_j, b'_{j+1} \setminus b'_j) \in R]$.  

In fact, for $i < k$ we have  

$$a'_{2i+1} \setminus a'_{2i} = c_{n+1} \cdot (a_{i+1} \setminus a_i) \text{ by 3.1(6)},$$  

$$b'_{2i+1} \setminus b'_{2i} = (b_{i+1} \setminus b_i) \cdot d'_i \text{ by 3.1(10)}.$$  

So $(a'_{2i+1} \setminus a'_{2i}, b'_{2i+1} \setminus b'_{2i}) \in R$ by the definition of $d'_i$.  

Also, for $i < k$ we have  

$$a'_{2i+2} \setminus a'_{2i+1} = (a_{i+1} \setminus a_i) \setminus c_{n+1} \text{ by 3.1(7)};$$  

$$b'_{2i+2} \setminus b'_{2i+1} = (b_{i+1} \setminus b_i) \setminus d'_i \text{ by 3.1(11)}.$$  

So $(a'_{2i+2} \setminus a'_{2i+1}, b'_{2i+2} \setminus b'_{2i+1}) \in R$ by the definition of $d'_i$.  

Finally,  

$$a'_{2k+1} \setminus a'_{2k} = c_{n+1} \setminus a_k \text{ by 3.1(8)};$$  

$$b'_{2k+1} \setminus b'_{2k} = d'_k \setminus b_k \text{ by 3.1(12)}.$$  

So $(a'_{2k+1} \setminus a'_{2k}, b'_{2k+1} \setminus b'_{2k}) \in R$ by the definition of $d'_k$.  

This proves 3.1(13).

3.1(14) $(a'_{2k+1}, b'_{2k+1}) \in R$.

For, $a'_{2k+1} = (a'_{2k+1} \setminus a'_{2k}) + (a'_{2k+1} \setminus a'_{2k}) = (c_{n+1} \setminus a_k) + a_k = c_{n+1} + a_k$ by 3.1(8), and $b'_{2k+1} = b_k + d'_k$. Hence $(a'_{2k+1}, b'_{2k+1}) \in R$ by the definition of $d'_k$.

3.1(15) $\forall j < 2k + 1 [a'_j = a'_{j+1} \leftrightarrow b'_j = b'_{j+1}]$.  

10
In fact, suppose that \( j < 2k + 1 \) and \( a'_{j} = a'_{j+1} \). Then \( a'_{j+1} \setminus a'_{j} = 0 \), and so by 3.1(13) and (V2), \( b'_{j+1} \setminus b'_{j} = 0 \). So \( b'_{j+1} = b'_{j+1} \cdot b'_{j} = b'_{j} \). The other direction is similar, using 3.1(13) and (V1).

3.1(16) \( \forall i < k [ a'_{2i} < a'_{2i+1} \) or \( a'_{2i+1} < a'_{2i+2} ] \).

This follows from 3.1(1) and 3.1(5), since \( a'_{2i} = a_{i} < a_{i+1} = a'_{2i+2} \).

Now we let \( a'_0 < \cdots < a''_n \) be \( a'_0, \ldots, a'_{2k+1} \) in increasing order, and set \( L'_{n+1} = \{ a'_0, \ldots, a''_n \} \). Further, if \( a''_i = a'_j \) we let \( b''_i = b'_j \). This is unambiguous by 3.1(15) and 3.1(16). Let \( g'_{n+1}(a''_i) = b''_i \). Now \( a''_0 = a'_{2k+1} \), so

3.1(17) \( (a''_i, b''_i) \in R \)

by 3.1(14). Suppose that \( j < l \). Say \( a''_j = a'_i \) with \( a'_i < a'_{i+1} \). Then \( a''_{j+1} = a'_{i+1} \), so

3.1(18) \( (a''_{j+1}, a''_i, b''_{j+1} \setminus b''_i) \in R \)

by 3.1(13).

Now if \( i \leq k \) then \( a'_{2i} = a_{i} \) and \( b'_{2i} = b_{i} \). For \( a''_j = a'_2i \) we have \( g'_{n+1}(a''_j) = b''_j = b'_{2i} = g_{n}(a'_{2i}) = g_{n}(a'_i) \).

3.1(19) \( \forall i < k [ i > 0 \rightarrow a_{i} = (a_{i-1} \setminus a_{i-2}) + \cdots + (a_{1} \setminus a_{0}) ] \).

We prove 3.1(19) by induction on \( i \). It is true for \( i = 1 \) since \( a_{0} = 0 \). Assume it for \( i < k \).

Then \( a_{i+1} = (a_{i+1} \cdot a_{i}) + (a_{i+1} \setminus a_{i}) = (a_{i+1} \setminus a_{i}) + (a_{i} \setminus a_{i-1}) + \cdots + (a_{1} \setminus a_{0}) \).

So 3.1(19) holds.

Now for \( k = 0 \) we have \( a_{0} = 0, a'_{0} = 0, a'_{1} = c_{1}, c_{1} = a'_{1} \in L'_{1} \). If \( k > 0 \), then

\[
\begin{align*}
c_{n+1} &= c_{n+1} \cdot a_{k} + c_{n+1} \setminus a_{k} = \sum_{i < k} (c_{n+1} \cdot (a_{i+1} \setminus a_{i}) + c_{n+1} \setminus a_{k}) \quad \text{by 3.1(19)}
\end{align*}
\]

\[
= \sum_{i < k} (a'_{2i+1} \setminus a'_{2i}) + a'_{2k+1} \setminus a'_{2k} \quad \text{by 3.1(6) and 3.1(8)}
\in L'_{n+1}.
\]

Now for \( i \leq l \) let \( b''_{2i} = b''_l \) and for \( i < l \) let \( b''_{2i+1} = (d_{n+1} \cdot b''_{i+1}) \setminus b''_i \). Let \( b''_{2l+1} = d_{n+1} + b''_l \).

3.1(20) \( b''_0 \leq \cdots \leq b''_{2l+1} \).

In fact, for \( i < l \) we have \( b''_{2i} = b''_{l} \leq b''_{2i+1} \leq b''_{2i+2} \), and \( b''_{2l} = b''_{l} \leq b''_{2l+1} \).

3.1(21) \( \forall i < l [ b''_{2i+1} \setminus b''_{2i} = d_{n+1} \cdot (b''_{i+1} \setminus b''_i) ] \).

In fact, if \( i < l \) then \( b''_{2i+1} \setminus b''_{2i} = (d_{n+1} \cdot b''_{i+1} + b''_l) \setminus b''_l = (d_{n+1} \cdot b''_{i+1}) \setminus b''_l = d_{n+1} \cdot (b''_{i+1} \setminus b''_l) \).

3.1(22) \( \forall i < l [ b''_{2i+2} \setminus b''_{2i+1} = (b''_{i+1} \setminus b''_l) \setminus d_{n+1} ] \).

In fact, if \( i < l \) then \( b''_{2i+2} \setminus b''_{2i+1} = (d_{n+1} \cdot b''_{i+1} + b''_l) \setminus b''_l = (d_{n+1} \cdot b''_{i+1}) \cdot (b''_{i+1} \setminus b''_l) = (b''_{i+1} \setminus d_{n+1}) \cdot (b''_{i+1} \setminus b''_l) = (b''_{i+1} \setminus b''_l) \setminus d_{n+1} \).

3.1(23) \( b''_{2i+1} \setminus b''_{2i} \in d_{n+1} \).

For, \( b''_{2i+1} \setminus b''_{2i} = (d_{n+1} + b''_l) \setminus b''_l \).
Now by 3.1(18), if \(i < l\), we have \((a''_{i+1} \setminus a''_i, b''_{i+1} \setminus b''_i) \in R\). Hence by (V4) there exist \(e_i \in A\) for \(i < l\) such that

\[
((a''_{i+1} \setminus a''_i) \cdot e_i, (b''_{i+1} \setminus b''_i) \cdot d_{n+1}) \in R \quad \text{and} \quad ((a''_{i+1} \setminus a''_i) \setminus e_i, (b''_{i+1} \setminus b''_i) \setminus d_{n+1}) \in R.
\]

Also, by 3.1(17) \((a''_l, b''_l) \in R\), so by (V4) there is a \(e_l \in A\) such that

\[
(a''_l + e_l, b''_l + d_{n+1}), (e_l \setminus a''_l, d_{n+1} \setminus b''_l) \in R.
\]

Now for all \(i \leq l\) let \(a'''_{2i} = a''_i\). For all \(i < l\) let \(a'''_{2i+1} = a''_i + ((a''_{i+1} \setminus a''_i) \cdot e_i)\), and let \(a'''_{2l+1} = a''_l + e_l\).

3.1(24) \(a'''_0 \leq \cdots \leq a'''_{2l+1}\).

For, first note that for any \(i \leq l\) we have \(a''_i = g_{l+1}^{-1}(b''_i) \leq g_{l+1}^{-1}(b''_{i+1}) = a''_{i+1}\). Hence if \(i < l\), then \(a'''_{2i} = a''_i \leq a''_i + ((a''_{i+1} \setminus a''_i) \cdot e_i) = a'''_{2i+1} = a''_{2i+1} \leq a''_{2i+2}\). Also, \(a'''_{2l-1} = a''_{2(l-1)+1} = a''_{l-1} + (a''_l \setminus a''_{l-1}) \cdot e_{l-1} \leq a''_l = a''_{2l} \leq a''_{2l+1}\).

3.1(25) \(\forall i < l[a'''_{2i+1} \setminus a'''_{2i}] = (a''_{i+1} \setminus a''_i) \cdot e_i\).

In fact, if \(i < l\) then \(a'''_{2i+1} \setminus a'''_{2i} = ((a''_{i+1} \setminus a''_i) \cdot e_i) \cdot a''_i = ((a''_{i+1} \setminus a''_i) \setminus e_i) \cdot (a''_i \setminus a''_{i+1}) \cdot e_i = (a''_{i+1} \setminus a''_i) \cdot e_i\).

3.1(26) \(\forall i < l[a'''_{2i+2} \setminus a'''_{2i+1}] = (a''_{i+1} \setminus a''_i) \cdot e_i\).

For, if \(i < l\), then \(a'''_{2i+2} \setminus a'''_{2i+1} = a''_{i+1} \setminus ((a''_{i+1} \setminus a''_i) \cdot e_i) = (a''_{i+1} \setminus a''_i) \cdot (a''_{i+1} \setminus a''_i) \cdot e_i) = (a''_{i+1} \setminus a''_i) \cdot (a''_{i+1} \setminus e_i) = (a''_{i+1} \setminus a''_i) \cdot e_i\).

3.1(27) \(a'''_{2i+1} \setminus a'''_{2i} = e_i \setminus a''_i\).

For, \(a'''_{2i+1} \setminus a'''_{2i} = (a''_i + e_i) \setminus a''_i = e_i \setminus a''_i\).

3.1(28) \(\forall j < 2l + 1[(a''_{j+1} \setminus a''_j, b''_{j+1} \setminus b''_j) \in R]\).

In fact, for \(i < l\) we have

\[
a'''_{2i+1} \setminus a'''_{2i} = (a''_{i+1} \setminus a''_i) \cdot e_i \quad \text{by 3.1(25)},
\]
\[
b''_{2i+1} \setminus b''_i = d_{n+1} \cdot (b''_i \setminus b''_i) \quad \text{by 3.1(21)}.
\]

So \((a''_{2i+1} \setminus a''_{2i}, b''_{2i+1} \setminus b''_i) \in R\) by the definition of \(e_i\).

Also, for \(i < l\) we have

\[
a'''_{2i+2} \setminus a'''_{2i+1} = (a''_{i+1} \setminus a''_i) \cdot e_i \quad \text{by 3.1(26)};
\]
\[
b''_{2i+2} \setminus b''_{2i+1} = (b''_i \setminus b''_i) \setminus d_{n+1} \quad \text{by 3.1(22)}.
\]

So \((a''_{2i+2} \setminus a''_{2i+1}, b''_{2i+2} \setminus b''_{2i+1}) \in R\) by the definition of \(e_i\).

Finally,

\[
a''_{2i+1} \setminus a''_{2i} = e_i \setminus a''_i \quad \text{by 3.1(27)};
\]
\[
b''_{2i+1} \setminus b''_i = d_{n+1} \setminus b_i \quad \text{by 3.1(23)}.
\]
So \((a''_{2l+1} \setminus a''_{2l}, b''_{2l+1} \setminus b''_{2l}) \in R\) by the definition of \(e_l\).
This proves 3.1(28).

3.1(29) \((a''_{2l+1}, b''_{2l+1}) \in R\).

For, \(a''_{2l+1} = (a''_{2l+1} \setminus a''_{2l}) + (a''_{2l+1} \cdot a''_{2l}) = (e_l \setminus a''_l) + a''_l = e_l + a''_l\) by 3.1(27), and \(b''_{2l+1} = b''_l + d_{n+1}\). Hence \((a''_{2l+1}, b''_{2l+1}) \in R\) by the definition of \(e_l\).

3.1(30) \(\forall j < 2l + 1 \left[ a''_{j+1} = a''_{j+1} \leftrightarrow b''_j = b''_{j+1} \right]\).

In fact, suppose that \(j < 2l + 1\) and \(a''_j = a''_{j+1}\). Then \(a''_{j+1} \setminus a''_j = 0\), and so by 3.1(28) and (V2), \(b''_j \setminus b''_{j+1} = 0\). So \(b''_{j+1} = b''_j \cdot b''_{j+1} = b''_{j+1}\). The other direction is similar, using 3.1(28) and (V1).

3.1(31) \(\forall i \leq l \left[ a''_{2i} < a''_{2i+1} \right]\) or \(a''_{2i+1} < a''_{2i+2}\).

This follows since \(a''_{2i} = a''_i < a''_{i+1} = a''_{2i+2}\), using the definition of \(a''\) following 3.1(16).

Now we let \(a''_i = a''_{i+1} \in \{a''_0, \ldots, a''_{2l+1}\}\) and set \(L_{n+1} = \{a''_0, \ldots, a''_{2l+1}\}\). This is unambiguous by 3.1(30). Let \(g_{n+1}(a''_i) = b''_i\). Now \(a''_s = a''_{2l+1}\), so \((a''_s, b''_s) \in R\) by 3.1(29). Suppose that \(j < s\). Say \(a''_j = a''_i\) with \(a''_i < a''_{i+1}\). Then \(a''_{i+1} = a''_{i+1}\), so \((a''_i, b''_i, b''_{i+1} \setminus b''_{i+1}) \in R\) by 3.1(28).

3.1(32) \(\forall i \leq l \left[ 0 < b''_i = (b''_i \setminus b''_{i-1}) + (b''_{i-1} \setminus b''_{i-2}) + \cdots + (b''_0) \right]\).

We prove 3.1(32) by induction on \(i\). It is true for \(i = 1\) since \(b''_0 = 0\). Assume it for \(i < s\). Then \(b''_{i+1} = (b''_{i+1} \cdot b''_i) + (b''_{i+1} \setminus b''_i) = (b''_{i+1} \setminus b''_i) + b''_i = (b''_{i+1} \setminus b''_i) + (b''_i \setminus b''_{i-1}) + \cdots + (b''_i \setminus b''_0)\). So 3.1(32) holds.

Now \(s > 0\), and

\[
\sum_{i < l} (b''_{2i+1} \setminus b''_{2i}) + b''_{2i+1} \setminus b''_{2i} \quad \text{by 3.1(21) and 3.1(23)}
\]

\(\in L_{n+1}\).

3.1(33) \(L_n \subseteq L_{n+1}\).

In fact, let \(i \leq k\), so that \(a_i\) is a typical element of \(L_n\). Then \(a'_2 = a_i\). Say \(j < s\) and \(a''_j = a''_{j+1}\). Then \(a''_{j+1} = a''_j\). There is a \(u < s\) such that \(a''_u = a''_{j+1}\). Then \(a''_{j+1} \in L_{n+1}\), proving 3.1(33).

3.1(34) \(g_n \subseteq g_{n+1}\).

For, let \(i, j, u\) be as in the proof of 3.1(33). Then \(g_{n+1}(a_i) = g_{n+1}(a''_{i+1}) = b''_{i+1}\). Now \(a''_{i+1} = a''_u\), so \(b''_{i} = b''_{2i+1} = b''_j\). Since \(a''_j = a''_{2i}\), we have \(b''_j = b''_{2i} = a''_j = g_n(a_i)\). This proves 3.1(34).

We have now verified 3.1(1)–3.1(4) for \(n + 1\). It follows that \(\bigcup_{n \in \omega} g_n\) is strictly increasing from a generating chain of \(A\) to a generating chain of \(B\). By Proposition 1.22 this union extends to an isomorphism from \(A\) onto \(B\).
Let $A$ and $B$ be BAs. A *weak $V$-relation between $A$ and $B$* is a relation $R \subseteq A \times B$ such that the following conditions hold:

(V5) $(0, b) \in R$ iff $b = 0$.

(V6) If $(a, b) \in R$ and $c \in A$, then there exist $d_0, d_1 \in B$ such that $(a \cdot c, b \cdot d_0), (a \cdot c \cdot b, d_0), (a + c, b + d_1), (c \cdot a, d_1 \cdot b) \in R$.

**Proposition 3.2.** Suppose that $A$ and $B$ are BAs, $A$ is countable, and $R$ is a weak $V$-relation between $A$ and $B$. Then there is a homomorphism $f : A \to B$ such that for every $a \in A$ there exist $m \in \omega \setminus \{0\}$ and $b \in m\{A$ such that

(i) $\forall i < m[(b_i, f(b_i)) \in R]$.

(ii) $b$ is pairwise disjoint and $\sum_{i \in m} b_i = a$.

**Proof.** Assume the hypotheses. Let $0 = c_0, e_1, \ldots$ enumerate the members of $A$ without repetitions. Let $L_0 = \{0\}$ and $f_0 = \{(0, 0)\}$. Suppose that $L_0 \subseteq \ldots \subseteq L_n$, and $f_0, \ldots, f_n$ have been defined, so that each $L_i$ is a finite linearly ordered subset of $A$ and each $f_i$ is an increasing function from $L_i$ into $B$. Say $L_n = \{e_0, \ldots, e_k\}$, with $e_0 < \cdots < e_k$. For all $i < k$ let $f_n(e_i) = g_i$. We assume $\forall i < k[(e_i, g_i) \in R]$ and $\forall i < k[(e_{i+1}, g_{i+1}) \in R]$. For all $i < k$ let $e_i' = e_i$. For all $i < k$ let $e_i' = c_{n+1} \cdot e_i + e_i + e_i'$. Let $e_i = e_i' = e_i + e_i'$. If $j = 2i + 1$, then $e_i' = e_{i+1}' = c_{n+1} \cdot e_i + e_i'$. If $j = 2i$, then $e_i' = e_{i+1}' = c_{n+1} \cdot e_i + e_i'$. If $j = 2i + 1$, then $e_i' = e_{i+1}' = c_{n+1} \cdot e_i + e_i'$. If $j = 2i$, then $e_i' = e_{i+1}' = c_{n+1} \cdot e_i + e_i'$. If $j = 2i + 1$, then $e_i' = e_{i+1}' = c_{n+1} \cdot e_i + e_i'$. If $j = 2i$, then $e_i' = e_{i+1}' = c_{n+1} \cdot e_i + e_i'$.

Also we have for all $i < k$

\[
e_i' = (e_i' \cdot e_i + e_i') \backslash e_i
\]

\[
= c_{n+1} \cdot (e_i' \cdot e_i); \\
e_i' \cdot e_i' = e_i' \cdot (c_{n+1} + e_i') \\
= e_i' \cdot (c_{n+1} + e_i'); \\
= e_i' \cdot (c_{n+1} + e_i'); \\
= e_i' \cdot (c_{n+1} + e_i'); \\
= (e_i' \cdot e_i') \backslash e_i.
\]

Also,

\[
e_i' = (c_{n+1} + e_i') \backslash e_i = c_{n+1} \cdot e_i'.
\]

Now by (V6), for each $i < k$, with $a, b, c$ replaced by $e_i' \backslash e_i$, $g_{i+1} \backslash g_i$, $c_{n+1}$, we obtain $h_i' \in B$ such that

\[
((e_i' \cdot e_i) \cdot c_{n+1} ; (g_{i+1} \backslash g_i) \cdot h_i') \cdot ((e_i' \cdot e_i) \cdot c_{n+1} ; (g_{i+1} \backslash g_i) \cdot h_i') \in R.
\]
and with \(a, b, c\) replaced by \(e_k, g_k, c_{n+1}\) we get \(h'_k \in B\) such that
\[
(e_k + c_{n+1}, g_k + h'_k), (c_{n+1}|e_k, h'_k \setminus g_k) \in R.
\]

For all \(i \leq k\) let \(g'_{2i} = g_i\), and for all \(i < k\) let \(g'_{2i+1} = g_i + ((g_{i+1}\setminus g_i) \cdot h'_i)\). Also, let \(g'_{2k+1} = g_k + h'_k\). Then

3.2(2) \(g'_0 \leq g'_1 \leq \cdots \leq g'_{2k+1}\).

In fact, we prove 3.2(2) by showing that \(g'_0 \leq \cdots \leq g'_j\) for all \(j \leq 2k + 1\) by induction on \(j\). This is trivial for \(j = 0\). Assume that it holds for \(j < 2k + 1\). If \(j = 2i\) and \(i < k\), then \(g'_{2i} = g_{2i} = g_i \leq g'_{2i+1}\). Further, \(g'_{2k} = g_k \leq g'_{2k+1}\). If \(j = 2i + 1 < 2k + 1\), then \(g'_{2i+1} = g_{2i+1} = g_i + ((g_{i+1}\setminus g_i) \cdot h'_i) \leq g_{i+1} = g'_{2i+2} = g'_{j+1}\).

Now for any \(i < k\),
\[
\begin{align*}
g'_{2i+1} \setminus g'_{2i} &= (g_i + ((g_{i+1}\setminus g_i) \cdot h'_i)) \setminus g_i \\
&= (g_{i+1}\setminus g_i) \cdot h'_i; \\
g'_{2i+2} \setminus g'_{2i+1} &= g_{i+1} \setminus (g_i + ((g_{i+1}\setminus g_i) \cdot h'_i)) \\
&= g_{i+1} - (g_i + g_{i+1} \cdot (-g_i) \cdot h'_i) \\
&= g_{i+1} \cdot (-g_i) \cdot h'_i \\
&= g_{i+1} \cdot (-g_i) \cdot h'_i \\
&= (g_{i+1}\setminus g_i) \cdot h'_i.
\end{align*}
\]

Also,
\[
g'_{2k+1} \setminus g'_{2k} = (g_k + h'_k) \setminus g_k = h'_k \setminus g_k.
\]

It follows that \(\forall i < 2k + 1[(e'_{i+1}\setminus e'_i, g'_{i+1}\setminus g'_i) \in R]\). Also, \((e'_{2k+1}, g'_{2k+1}) \in R\).

3.2(3) If \(i < 2k + 1\) and \(e'_i = e'_{i+1}\), then \(g'_i = g'_{i+1}\).

In fact, suppose that \(e'_i = e'_{i+1}\). Then \(e'_{i+1} \setminus e'_i = 0\). and so by (V5), \(g'_{i+1}\setminus g'_i = 0\) and hence \(g'_i = g'_{i+1}\).

Now let \(L_{n+1} = \{e'_0, \ldots , e'_{2k+1}\}\) and \(f_{n+1}(e'_i) = g'_i\) for all \(i < 2k + 1\).

3.2(4) \(\forall i \leq k[i > 0 \rightarrow e_i = (e_i | e_{i-1}) + (e_{i-1}| e_{i-2}) + \cdots + (e_1 | e_0)]\).

We prove 3.2(4) by induction on \(i\). It is true for \(i = 1\) since \(e_0 = 0\). Assume it for \(i < k\). Then \(e_{i+1} = (e_{i+1} | e_i) + (e_{i+1} | e_i) = (e_{i+1} | e_i) + e_i = (e_{i+1} | e_i) + (e_i | e_{i-1}) + \cdots + (e_1 | e_0)\).

So 3.2(4) holds.

Now
\[
c_{n+1} = c_{n+1} \cdot e_k + (c_{n+1} \setminus e_k) = c_{n+1} \sum_{i=0}^{k-1} (e_{i+1} | e_i) + (c_{n+1} \setminus e_k) = \sum_{i=0}^{k-1} (e'_{2i+1} \setminus e'_{2i}) + (e'_{2k+1} \setminus e'_{2k}).
\]

This proves that \(c_{n+1}\) is in the subalgebra of \(A\) generated by \(L_{n+1}\) and also at the final stage gives (i) and (ii).  \(\square\)
If $A$ and $B$ are BAs, a subset $R$ of $A \times B$ is a $V$-relation between $A$ and $B$ iff the following conditions hold:

(V7) $1R1$
(V8) $0R0$
(V9) $\forall x[xR0 \to x = 0]$
(V10) $\forall y[0Ry \to y = 0]$
(V11) $\forall x,y,z[xR(y+z) \text{ and } y \cdot z = 0 \to \exists u,v[x = u + v \text{ and } u \cdot v = 0 \text{ and } uRy \text{ and } vRz]]$
(V12) $\forall x,y,z[(x+y)Rz \text{ and } x \cdot y = 0 \to \exists u,v[z = u + v \text{ and } u \cdot v = 0 \text{ and } xRu \text{ and } yRv]].$

Theorem 3.3. If $R$ is a $V$-relation between denumerable BAs $A$ and $B$, then there is an isomorphism $f$ from $A$ onto $B$ such that for any $x \in A$ there exist $n \in \omega$ and $y \in ^n A$ such that $\forall i,j \in n[i \neq j \to y_i \cdot y_j = 0]$ and $x = \sum _{i<n} y_i$ and $y_i R f(y_i)$ for all $i \leq n$.

Proof. Let $A = \{a_i : i \in \omega\}$ and $B = \{b_i : i \in \omega\}$. We now define $c \in ^\omega A$ and $d \in ^\omega B$ by recursion. Let $c_0 = 1$ and $d_0 = 1$. Now suppose that $c_j$ and $d_j$ have been defined for all $j \leq 2i$ so that the following conditions hold:

3.3(1) $\{(c_j,d_j) : j \leq 2i\}$ extends to an isomorphism $f$ from $\langle \{c_j : j \leq 2i\} \rangle$ into $B$.

3.3(2) For each $x \in \langle \{c_j : j \leq 2i\} \rangle$ there exist $n \in \omega$ and $y \in ^n \langle \{c_j : j \leq 2i\} \rangle$ such that $\forall i,j < n[i \neq j \to y_i \cdot y_j = 0]$ and $x = \sum _{i<n} y_i$ and $y_i R f(y_i)$ for all $k < n$.

Clearly 3.3(1) and 3.3(2) hold for $i = 0$. Now we let $c_{2i+1} = a_i$. Let the atoms of $\langle \{c_j : j \leq 2i\} \rangle$ be $e_0, \ldots, e_n$ without repetitions. Take any $k$ with $k \leq n$. Then there exist $m \in \omega$ and $y \in ^m \langle \{c_j : j \leq 2i\} \rangle$ such that $\forall i,j < m[i \neq j \to y_i \cdot y_j = 0]$ and $e_k = \sum _{l<m} y_l$ and $y_l R f(y_l)$ for all $l < m$. Since $e_k$ is an atom, there is an $l < m$ such that $y_l = e_k$. So $e_k R f(e_k)$. Thus

3.3(3) $\forall k \leq n[e_k R f(e_k)].$

Now $e_k = e_k \cdot a_i + e_k \cdot -a_i$, so by (V12) there exist $u_k, v_k$ such that $u_k \cdot v_k = 0$, $f(e_k) = u_k + v_k$, $(e_k \cdot a_i)Ru_k$, and $(e_k \cdot -a_i)RV_k$. Let $d_{2i+1} = \sum _{k<n} u_k$. To show that $f \cup \{(c_{2i+1},d_{2i+1})\}$ extends to an isomorphism of $\langle \{c_j : j \leq 2i + 1\} \rangle$ into $B$, by Sikorski’s extension criterion it suffices to take any $x \in \langle \{c_j : j \leq 2i\} \rangle$ and show that $(x \cdot c_{2i+1} = 0 \leftrightarrow f(x) \cdot d_{2i+1} = 0)$ and $(x \cdot -c_{2i+1} = 0 \leftrightarrow f(x) \cdot -d_{2i+1} = 0)$. Write $x = \sum _{k \in M} e_k$.

Suppose that $x \cdot c_{2i+1} = 0$. Thus $x \cdot a_i = 0$, so $\forall k \in M[e_k \cdot a_i = 0]$. Since $\forall k \leq n[e_k \cdot a_i]Ru_k$, it follows by (V10) that $\forall k \in M[u_k = 0]$. Hence

$$f(x) \cdot d_{2i+1} = \sum _{k \notin M} (f(x) \cdot u_k) \leq \sum _{k \notin M} (f(x) \cdot f(e_k)) = \sum _{k \notin M} f(x \cdot e_k) = 0.$$  

Conversely, suppose that $x \cdot c_{2i+1} \neq 0$. Say $k \in M$ and $e_k \cdot a_i \neq 0$. Since $(e_k \cdot a_i)Ru_k$, it follows by (V9) that $u_k \neq 0$. Then $f(x) \cdot d_{2i+1} \geq f(e_k) \cdot d_{2i+1} \geq u_k \neq 0$.

Suppose that $x \cdot -c_{2i+1} = 0$. Thus $x \cdot -a_i = 0$, so $\forall k \in M[e_k \cdot -a_i = 0]$. Since $\forall k \leq n[(e_k \cdot -a_i)RV_k]$, it follows by (V10) that $\forall k \in M[v_k = 0]$, hence $\forall k \in M[u_k = f(e_k)]$,  

16
hence $\forall k \in M[-u_k = f(-e_k)]$. Hence

$$f(x) \cdot -d_{2i+1} = f(x) \cdot \prod_{k \leq n} -u_k \leq f(x) \cdot \prod_{k \in M} -u_k = f(x) \cdot \prod_{k \in M} f(-e_k)$$

$$= f \left( \sum_{k \in M} e_k \right) \cdot \prod_{k \in M} f(-e_k) = 0.$$ 

Conversely suppose that $x \cdot -c_{2i+1} \neq 0$. Say $k \in M$ and $e_k \cdot -a_i \neq 0$. So by (V9), $v_k \neq 0$. Then $f(x) \cdot -d_{2i+1} \geq f(e_k) \cdot -d_{2i+1} \geq v_k \neq 0$.

Thus we have shown

3.3(4) $\{(c_j, d_j) : j \leq 2i + 1\}$ extends to an isomorphism $f'$ from $\langle\{c_j : j \leq 2i + 1\}\rangle$ into $B$; and $f'$ extends $f$.

Now we claim

3.3(5) For each $x \in \langle\{c_j : j \leq 2i + 1\}\rangle$ there exist $m \in \omega$ and $y \in \sum_{k \leq m} \langle\{c_j : j \leq 2i + 1\}\rangle$ such that $\forall i, j < m[i \neq j \rightarrow y_i \cdot y_j = 0]$ and $x = \sum_{k < m} y_k$ and $y_k R f'(y_k)$ for all $k < m$.

For, suppose that $x \in \langle\{c_j : j \leq 2i + 1\}\rangle$. Then there are $x', x'' \in \langle\{c_j : j \leq 2i\}\rangle$ such that $x = x' \cdot c_{2i+1} + x'' \cdot -c_{2i+1}$. Say $x' = \sum_{k \in M} e_k$ and $x'' = \sum_{k \in N} e_k$. Then

$$x = x' \cdot c_{2i+1} + x'' \cdot -c_{2i+1} = \sum_{k \in M \cap N} e_k + \sum_{k \in M \setminus N} (e_k \cdot c_{2i+1}) + \sum_{k \in N \setminus M} (e_k \cdot -c_{2i+1}).$$

For $k \in M \cap N$ we have $e_k R f(e_k) = f'(e_k)$. For $k \in M \setminus N$ we have

$$(e_k \cdot c_{2i+1}) = (e_k \cdot a_i) R u_k = f(e_k) \cdot d_{2i+1} = f(e_k) \cdot f'(c_{2i+1}) = f'(e_k) \cdot c_{2i+1}.$$ 

For $k \in N \setminus M$ we have

$$(e_k \cdot -c_{2i+1}) = (e_k \cdot -a_i) R v_k = f(e_k) \cdot -d_{2i+1} = f(e_k) \cdot f'(-c_{2i+1}) = f'(e_k) \cdot -c_{2i+1}. $$

This proves 3.3(5).

Now we let $d_{2i+2} = b_i$. The atoms of $\langle\{d_j : j \leq 2i + 1\}\rangle$ be $e_0, \ldots, e_n$ without repetitions. Take any $k$ with $k \leq n$. Then there exist $m \in \omega$ and $y \in \sum_{k \leq m} \langle\{c_j : j \leq 2i + 1\}\rangle$ such that $\forall i, j \in n[i \neq j \rightarrow y_i \cdot y_j = 0]$ and $f^{-1}(e_k) = \sum_{l < m} y_l$ and $y_l R f'(y_l)$ for all $l < m$. Since $e_k$ is an atom, there is an $l < m$ such that $y_l = f'^{-1}(e_k)$. So $f'^{-1}(e_k) R e_k$. Thus

3.3(6) $\forall k \leq n[f'^{-1}(e_k) R e_k].$

Now $e_k = e_k \cdot b_i + e_k \cdot -b_i$, so by (V11) there exist $u_k, v_k$ such that $u_k \cdot v_k = 0$, $f'^{-1}(e_k) = u_k + v_k$, $u_k R (e_k \cdot b_i)$, and $v_k R (e_k \cdot -b_i)$. Let $c_{2i+2} = \sum_{k \leq n} u_k$. To show that $f' \cup \{(c_{2i+2}, d_{2i+2})\}$ extends to an isomorphism of $\langle\{c_j : j \leq 2i + 2\}\rangle$ into $B$, by Sikorski’s extension criterion it suffices to take any $x \in \langle\{c_j : j \leq 2i + 1\}\rangle$ and show that $(x \cdot c_{2i+2} = 0 \leftrightarrow f'(x) \cdot d_{2i+2} = 0)$ and $(x \cdot -c_{2i+2} = 0 \leftrightarrow f'(x) \cdot -d_{2i+2} = 0)$. Write $f'(x) = \sum_{k \in M} e_k$. 

17
Suppose that $f'(x) \cdot d_{2i+2} = 0$. Thus $f'(x) \cdot b_i = 0$, so $\forall k \in M[e_k \cdot b_i = 0]$. Since $u_k R(e_k \cdot b_i)$, it follows from (V9) that $\forall k \in M[u_k = 0]$. Hence

$$x \cdot c_{2i+2} = \sum_{k \in M} (x \cdot u_k) \leq \sum_{k \in M} (f^{-1}(e_k) \cdot f'(x) \cdot e_k) = \sum_{k \in M} f^{-1}(f'(x) \cdot e_k) = 0.$$

Conversely, suppose that $f'(x) \cdot c_{2i+2} \neq 0$. Say $k \in M$ and $e_k \cdot d_{2i+2} \neq 0$. Hence $x \cdot c_{2i+2} = \sum_{i \in M} f^{-1}(f'(x) \cdot e_k) \cdot c_{2i+2} \neq 0$. Thus by (V9), $u_k \neq 0$. Hence $x \cdot c_{2i+2} = \sum_{i \in M} f^{-1}(f'(x) \cdot e_k) \cdot c_{2i+2} = u_k \neq 0$.

Suppose that $f'(x) \cdot d_{2i+2} = 0$. Thus $f'(x) \cdot b_i = 0$, so $\forall k \in M[e_k \cdot b_i = 0]$. Hence by (V9), $\forall k \in M[u_k = 0]$. Hence $f'(x) = \sum_{k \in M} e_k = \sum_{k \in M} f^{-1}(f'(x) \cdot e_k)$. Hence $x = \sum_{k \in M} u_k$. So

$$x \cdot c_{2i+2} = x \cdot \prod_{i \leq n} u_k = \sum_{k \in M} u_k \cdot \prod_{i \leq n} -u_k = 0.$$

Conversely, suppose that $f'(x) \cdot d_{2i+2} = 0$. Say $k \in M$ and $e_k \cdot b_i \neq 0$. Then by (V10) $u_k \neq 0$. Now $x \cdot c_{2i+2} \geq f^{-1}(e_k) \cdot v_k = v_k \neq 0$.

Thus we have shown

3.3(7) \{(c_j, d_j) : j \leq 2i + 2\} extends to an isomorphism $f''$ from $\langle \{c_j : j \leq 2i + 2\} \rangle$ into $B$; and $f''$ extends $f'$.

Now we claim

3.3(8) For each $x \in \langle \{c_j : j \leq 2i + 2\} \rangle$ there exist $m \in \omega$ and $y \in m \langle \{c_j : j \leq 2i + 2\} \rangle$ such that $\forall i, j : i \neq j \rightarrow y_i \cdot y_j = 0$ and $x = \sum_{k \leq m} y_k$ and $y_k R f(y_k)$ for all $k < m$.

For, suppose that $x \in \langle \{c_j : j \leq 2i + 2\} \rangle$. Then $f''(x) \in \langle \{d_j : j \leq 2i + 2\} \rangle$, so there exist $x', x'' \in \langle \{d_j : j \leq 2i + 1\} \rangle$ such that $f''(x) = x' \cdot d_{2i+2} + x'' \cdot -d_{2i+2}$. Say $x' = \sum_{k \in M} e_k$ and $x'' = \sum_{k \in N} e_k$. Then

$$f''(x) = x' \cdot d_{2i+2} + x'' \cdot -d_{2i+2} = \sum_{k \in M \cap N} e_k + \sum_{k \in M \setminus N} (e_k \cdot d_{2i+2}) + \sum_{k \in N \setminus M} (e_k \cdot -d_{2i+2}).$$

For $k \in M \cap N$ we have $f^{-1}(e_k) R e_k$. For $k \in M \setminus N$ we have

$$(e_k \cdot d_{2i+2}) = (e_k \cdot b_i) R^{-1} u_k = f^{-1}(e_k) \cdot c_{2i+2} = f''^{-1}(e_k \cdot d_{2i+2}),$$

For $k \in N \setminus M$ we have

$$(e_k \cdot -d_{2i+2}) = (e_k \cdot b_i) R^{-1} u_k = f^{-1}(e_k) \cdot c_{2i+2} = f''^{-1}(e_k \cdot -d_{2i+2}).$$

This proves 3.3(8). \hfill \Box

**Proposition 3.4.** Suppose that $R$ is a relation between countable BAs such that the following conditions hold:

(i) If ARB with $|B| = 1$ then $|A| = 1$. 

18
(ii) If $ABR$ with $|A| = 1$, then $|B| = 1$.

(iii) If $ABR$ and $A \cong A_0 \times A_1$ then there exist $B_0, B_1$ such that $B \cong B_0 \times B_1$, $A_0 RB_0$, and $A_1 RB_1$.

(iv) If $ABR$ and $B \cong B_0 \times B_1$ then there exist $A_0, A_1$ such that $A \cong A_0 \times A_1$, $A_0 RB_0$, and $A_1 RB_1$.

Then $ABR$ implies that $A \cong B$.

Proof. Assume the hypotheses, and suppose that $ABR$. Let $S = \{(a, b) : a \in A, b \in B, (A \upharpoonright a)R(B \upharpoonright b)\}$. Then $S$ is a $V$-relation between $A$ and $B$, so $A \cong B$ by Theorem 3.3. \qed

4. Superatomic ABIDs

An element $a$ of a ABID $A$ is an atom iff $a \neq 0$ and $\forall b \in A[b \cdot a = 0 \text{ or } b \cdot a = a]$. $A$ is atomless iff it has no atoms, and atomic iff $\forall b \in A[b \neq 0 \rightarrow \exists a[a \text{ is an atom and } a \leq b]]$. $A$ is superatomic iff every homomorphic image of $A$ is atomic. A ABID is trivial iff it has only one element. Note that the trivial ABID is atomless, atomic, and superatomic. Now for any ABID $A$ we define

$$I_0(A) = \{0\};$$
$$I_{\alpha+1}(A) = \{a \in A : [a]_{I_\alpha(A)} \text{ is a finite sum of atoms of } A/I_\alpha(A)\}$$
$$I_\lambda(A) = \bigcup_{\alpha<\lambda} I_\alpha(A) \quad \text{for } \lambda \text{ limit.}$$

Note that $I_\alpha \subseteq I_{\alpha+1}$, since $\forall a \in I_\alpha([a]_{I_\alpha(A)} = 0$ and 0 is a finite sum of atoms of $A/I_\alpha(A)$.

Proposition 4.1. Let $A$ be a ABID, and $\alpha$ and $\gamma$ ordinals.

(i) $I_\gamma(A/I_\alpha(A)) = \{[a]_{I_\alpha(A)} : a \in I_{\alpha+\gamma}(A)\}$.

(ii) There is an isomorphism $f_\gamma$ from $(A/I_\alpha(A))/I_\gamma(A/I_\alpha(A))$ onto $A/I_{\alpha+\gamma}(A)$ such that $\forall a \in A[f_\gamma([a]_{I_\alpha(A)})I_\gamma(A/I_\alpha(A)) = [a]_{I_{\alpha+\gamma}(A)}]$.

Proof. Induction on $\gamma$. It is clear for $\gamma = 0$. Assume it for $\gamma$. To prove (i) for $\gamma+1$, let $[a]_{I_\alpha(A)} \in I_{\gamma+1}(A/I_\alpha(A))$. Then exist an $m \in \omega$ and for each $i < m$ atoms $[[c_i]_{I_\alpha(A)}]_{I_\gamma(A/I_\alpha(A))}$ of $(A/I_\alpha(A))/I_\gamma(A/I_\alpha(A))$ such that

\[
[[a]_{I_\alpha(A)}]_{I_\gamma(A/I_\alpha(A))} = \sum_{i<m} [[c_i]_{I_\alpha(A)}]_{I_\gamma(A/I_\alpha(A))}.
\]

By (ii) for $\gamma$, for each $i < m$, $[c_i]_{I_{\alpha+\gamma}(A)}$ is an atom of $A/I_{\alpha+\gamma}(A)$, and by applying $f_\gamma$ to (*) we have $[a]_{I_{\alpha+\gamma}(A)} = \sum_{i<m} [c_i]_{I_{\alpha+\gamma}(A)}$. Thus $a \in I_{\alpha+\gamma+1}(A)$. This proves that $I_{\gamma+1}(A/I_\alpha(A)) \subseteq \{[a]_{I_\alpha(A)} : a \in I_{\alpha+\gamma+1}(A)\}$. The other inclusion is proved similarly, so (i) holds for $\gamma+1$. For (ii) for $\gamma+1$,

\[
[[a]_{I_\alpha(A)}]_{I_{\gamma+1}(A/I_\alpha(A))} = [[a']_{I_\alpha(A)}]_{I_{\gamma+1}(A/I_\alpha(A))}
\]

iff $[a]_{I_\alpha(A)} \Delta [a']_{I_\alpha(A)} \in I_{\gamma+1}(A/I_\alpha(A))$
iff \([a\triangle a']_{I_\alpha(A)} \in I_{\gamma+1}(A/I_\alpha(A))\)
iff \(\exists b \in I_{\alpha+\gamma+1}(A)[[a\triangle a']_{I_\alpha(A)} = [b]_{I_\alpha(A)}]\) by (i) for \(\gamma + 1\),
iff \(\exists b \in I_{\alpha+\gamma+1}(A)[(a\triangle a') \triangle b \in I_\alpha(A)]\)
iff \(a\triangle a' \in I_{\alpha+\gamma+1}(A)\) iff \([a]_{I_{\alpha+\gamma+1}(A)} = [a']_{I_{\alpha+\gamma+1}(A)}\).

Now (ii) for \(\gamma + 1\) follows.

The case with \(\gamma\) limit is straightforward.

\[\Box\]

**Proposition 4.2.** Let \(A\) be a ABID and \(\alpha\) and \(\beta\) ordinals with \(\beta \leq \alpha\). For \(a \in I_\alpha(A)\) we denote by \([a]_{I_\alpha(A)}'\) the equivalence class of \(a\) in \(I_\alpha(A)\), while \([a]_{I_\alpha(A)}\) is the equivalence class of \(a\) in \(A\). Then there is an isomorphism \(f\) from \(I_\alpha(A)/I_\beta(A)\) into \(A/I_\beta(A)\) such that \(\forall a \in I_\alpha(A) \left[f([a]_{I_\beta(A)}) = [a]_{I_\beta(A)}\right]\). Moreover, for any \(a \in I_\alpha(A)\), \([a]_{I_\beta(A)}'\) is an atom of \(I_\alpha(A)/I_\beta(A)\) iff \([a]_{I_\beta(A)}\) is an atom of \(A/I_\beta(A)\).

**Proof.** The first part of the proposition is clear. Now suppose that \(a \in I_\alpha(A)\) and \([a]_{I_\beta(A)}'\) is an atom of \(I_\alpha(A)/I_\beta(A)\). Suppose that \(b \in A\) and \([b]_{I_\beta(A)} \leq [a]_{I_\beta(A)}\). Then \(b\backslash a \in I_\beta(A)\). So \(b = b \cdot a + b\backslash a \in I_\alpha(A)\). Hence \(b \in I_\beta(A)\) or \((a\backslash b) \in I_\beta(A)\). So \([a]_{I_\beta(A)}\) is an atom of \(A/I_\beta(A)\).

Conversely, suppose that \([a]_{I_\beta(A)}\) is an atom of \(A/I_\beta(A)\). Suppose that \(b \in I_\alpha(A)\) and \([b]_{I_\beta(A)}' \leq [a]_{I_\beta(A)}'\). Then \((b\backslash a) \in I_\beta(A)\), so \([b]_{I_\beta(A)} \leq [a]_{I_\beta(A)}\). So \(b \in I_\beta(A)\) or \((a\backslash b) \in I_\beta(A)\). Hence \([a]_{I_\beta(A)}'\) is an atom of \(I_\beta(A)/I_\beta(A)\).

\[\Box\]

**Proposition 4.3.** \(\forall \beta \leq \alpha [I_\beta(I_\alpha(A)) = I_\beta(A)]\).

**Proof.** Induction on \(\beta\). It is clear for \(\beta = 0\). Now assume it for \(\beta\). Suppose that \(x \in I_{\beta+1}(I_\alpha(A))\). Say

\([x]_{I_\beta(I_\alpha(A))} = \sum_{i<m} [y_i]_{I_\beta(I_\alpha(A))}\)

with each \([y_i]_{I_\beta(I_\alpha(A))}\) an atom of \(I_\alpha(A)/I_\beta(I_\alpha(A))\). Thus by the inductive hypothesis,

\([x]_{I_\beta(A)} = \sum_{i<m} [y_i]_{I_\beta(A)}\),

and by Proposition 4.2, each \([y_i]_{I_\beta(A)}\) an atom of \(A/I_\beta(A)\). Thus \(x \Delta \sum_{i<m} y_i \in I_\beta(A)\), so \(x \in I_{\beta+1}(A)\). The converse is similar.

The limit step is clear.

The **atomic rank** of \(A\) is the least ordinal \(\text{ar}(A) = \alpha\) such that \(I_\alpha(A) = I_{\alpha+1}(A)\).

**Proposition 4.4.** If \(A\) is an atomic ABID and \(A\) is a maximal ideal in a BA \(B\), then \(B\) is atomic.

**Proof.** Suppose that \(b \in B\) and \(B \upharpoonright b\) is nontrivial and atomless. Let \(u, v\) be nonzero and disjoint with \(u + v \leq b\). Then \(u, v \notin A\). In fact, suppose that \(u \in A\). Let \(a\) be an atom of \(A\) with \(a \leq u\). Then \(a \leq b\), so there is a nonzero \(w \in B\) with \(w < a\). But \(w \in A\) since \(A\)
is an ideal, contradicting $a$ being an atom of $A$. Similarly $v \notin A$. say $u = -a$ and $v = -c$ with $a, c \in A$. Then $0 = u \cdot v = (-a) \cdot (-c)$ and hence $a + c = 1$, contradiction. \hfill \Box

**Proposition 4.5.** If $A$ is a ABID, $A$ is a maximal ideal in a BA $B$, and $f : B \rightarrow C$ is a surjective homomorphism with $C$ nontrivial, then $f[A] = C$ or $f[A]$ is a maximal ideal in $C$.

**Proof.** Assume the hypotheses. Clearly $f[A]$ is closed under $+$. If $c \leq f(a)$ with $c \in C$ and $a \in A$, choose $x \in B$ such that $c = f(x)$. Then $f(x \cdot a) = f(x) \cdot f(a) = c \cdot f(a) = c$. Since $x \cdot a \in A$, this shows that $c \in f[A]$. Thus $f[A]$ is an ideal of $C$. Now $\forall x \in B[x \in A$ or $-x \in A]$. If $c \in C$, choose $x \in B$ such that $f(x) = c$. Then $c \in f[A]$ or $-c \in f[A]$; the conclusion follows. \hfill \Box

**Proposition 4.6.** If $A$ is a superatomic ABID and $A$ is a maximal ideal in a BA $B$, then $B$ is superatomic.

**Proof.** Assume the hypotheses, and let $f : B \rightarrow C$ be a surjective homomorphism, with $C$ nontrivial. By Proposition 4.5 we have two cases.

Case 1. $f[A] = C$. Then $C$ is atomic, since $A$ is superatomic.

Case 2. $f[A]$ is a maximal ideal in $C$. Then $f[A]$ is atomic since $A$ is superatomic, and then $C$ is atomic by Proposition 4.4. \hfill \Box

**Proposition 4.7.** If $A$ is an atomless ABID which is not a BA, $B$ is a BA, and $(A, +, \cdot)$ is a subalgebra of $(B, +, \cdot)$, then the subalgebra $A'$ of $B$ generated by $A$ has at most one atom.

**Proof.** First note that $A' = A \cup \{-c : c \in A\}$. For, if $a, b \in A$, then

$$a \cdot_B b = a \cdot_A b \in A;$$

$$a \cdot_B (-b) = a \cdot_A b \in A \quad \text{by Proposition 1.3};$$

$$(-b a) \cdot_B (-b) = -b(a +_A b).$$

Now suppose that $A'$ has two distinct atoms $x, y$. Then there exist $a, b \in A$ such that $x = -a$ and $y = -b$. Then $(-a) \cdot (-b) = 0$, so $a + b = 1$, contradiction. \hfill \Box

**Proposition 4.8.** For any ABID $A$ the following conditions are equivalent:

(i) $A$ is superatomic.

(ii) No nontrivial homomorphic image of $A$ is atomless.

(iii) Every subalgebra of $A$ is atomic.

(iv) Every subalgebra of $A$ has an atom.

(v) $\neg \exists a \in A q, r \in \mathbb{Q} [q < r \rightarrow a_q < a_r].$

(vi) $I_{\text{ar}(A)}(A) = A.$

**Proof.** (i) $\Rightarrow$ (ii): obvious.

(ii) $\Rightarrow$ (i): Assume that (i) fails; so there is a homomorphic image $B$ of $A$ which is not atomic. Hence there is a $b \in B$ such that $B \upharpoonright b$ is atomless and nontrivial. Clearly $B \upharpoonright b$ is a homomorphic image of $A$, so (ii) fails.
(i)⇒(iii): Suppose that $A$ is a superatomic ABID and $B$ is a non-atomic subalgebra of $A$. Thus $A$ is not a BA. Say $b \in B\setminus\{0\}$ with no atoms of $B$ below $b$. So $B \upharpoonright b$ is atomless. By Proposition 1.10, $A$ is a maximal ideal in some BA $C$. By Proposition 4.6, $C$ is superatomic. Hence $C \upharpoonright b$ is superatomic. But $B \upharpoonright b$ is an atomless subalgebra of $C \upharpoonright b$, contradiction.

(iii)⇒(iv): obvious.

(iv)⇒(v): Assume that (iv) holds but (v) fails; so we get $a \in \mathcal{Q}A$ such that $\forall q,r \in \mathcal{Q}[q < r \rightarrow a_q < a_r]$. Let $B$ be the subalgebra of $(A, +, \cdot, 0)$ generated by $\text{rng}(a)$.

4.8(1) $B$ consists of all finite sums of elements of $\{a_q : q \in \mathcal{Q}\} \cup \{a_qa_r : q, r \in \mathcal{Q}, r < q\}$. In fact, let $S$ be the set of all such finite sums. Obviously $S$ contains $\text{rng}(a)$ and is closed under $\cdot$. To see that it is closed under $\cdot$ it suffices to note that

$$a_q \cdot a_r = a_{\min(q,r)};$$

if $t < r$, then $a_q \cdot (a_r \downharpoonright a_t) = \begin{cases} a_r \downharpoonright a_t & \text{if } t < r \leq q, \\ a_q \downharpoonright a_t & \text{if } t < q < r, \\ 0 & \text{if } q \leq t < r. \end{cases}$

if $r < q$ and $t < s$, then

$$(a_r \downharpoonright a_r) \cdot (a_s \downharpoonright a_t) = \begin{cases} a_{\min(q,s)} \downharpoonright a_{\max(r,t)} & \text{if } \max(r,t) < \min(q,s), \\ 0 & \text{if } \min(q,s) \leq \max(r,t). \end{cases}$$

For closure under $\setminus$, note that

$$a_q \downharpoonright a_r = \begin{cases} a_q \downharpoonright a_r & \text{if } r < q, \\ 0 & \text{if } q \leq r; \end{cases}$$

if $t < r$, then $a_q \downharpoonright (a_r \downharpoonright a_t) = (a_q \downharpoonright a_r) + a_{\min(q,t)}$;

if $t < r$, then $(a_r \downharpoonright a_t) \downharpoonright a_q = \begin{cases} a_r \downharpoonright a_{\max(q,t)} & \text{if } q < r, \\ 0 & \text{if } r \leq q. \end{cases}$

if $r < q$ and $t < s$, then $(a_q \downharpoonright a_r) \setminus (a_s \downharpoonright a_t) = (a_q \downharpoonright a_{\max(r,s)}) + (a_{\min(q,t)} \downharpoonright a_r)$.

Thus 4.8(1) holds.

Now let $x$ be an atom of $B$. By (1) we may assume that one of the following two cases holds:

**Case 1.** $x = a_q$ for some $q \in \mathcal{Q}$. Let $r$ be a rational less than $q$. Then $0 < a_r < a_q$, contradiction.

**Case 2.** $x = a_q \downharpoonright a_r$ for some $q, r \in \mathcal{Q}$ with $r < q$. Choose $t \in \mathcal{Q}$ such that $r < t < q$. Then $0 < a_q \downharpoonright a_t < a_q \downharpoonright a_r$, contradiction. To see that $a_q \downharpoonright a_t < a_q \downharpoonright a_r$, note that $a_r < a_t$, hence $a_q \downharpoonright a_t \leq a_q \downharpoonright a_r$. Also, $(a_q \downharpoonright a_r) \downharpoonright (a_q \downharpoonright a_t) = a_t \downharpoonright a_r$, so $a_q \downharpoonright a_t < a_q \downharpoonright a_r$.

(v)⇒(i): Assume that (i) fails; so there is a homomorphism $f$ of $A$ onto a ABID $B$ which is not atomic. Then there is a nonzero $b \in B$ such that $B \upharpoonright b$ is atomless. For any $a \in A$ let $g(a) = f(a) \cap b$. Then $g$ is a homomorphism from $A$ onto $B \upharpoonright b$.

We now construct $b \in \mathcal{Q}(B \upharpoonright b)$. Let $\langle q_i : i \in \omega \rangle$ be a one-one enumeration of $\mathcal{Q}$. We define $c \in \mathcal{Q}(B \upharpoonright b)\setminus\{0\}$ by recursion. Let $c_0 \in (B \upharpoonright b)\setminus\{0\}$ be arbitrary. Now suppose that $c_0, \ldots, c_{m-1}$ have been constructed so that the following condition holds:

22
4.8(2) If \( p \) is the permutation of \( m \) such that \( q_p(0) < \cdots < q_p(m-1) \), then \( c_p(0) < \cdots < c_p(m-1) \).

To define \( c_m \) we consider three cases.

Case 1. \( \forall i < m[q_m < q_i] \). Let \( c_m \) be such that \( \forall i < m[c_m < c_i] \).

Case 2. \( \forall i < m[q_i < q_m] \). Let \( c_m \) be such that \( \forall i < m[c_i < c_m] \).

Case 3. \( \exists i < m - 1[q_p(i) < q_m < q_p(i+1)] \). Take \( d \) such that \( 0 < d < c_p(i+1) \setminus c_p(i) \) and let \( c_m = c_p + d \).

Clearly in any case 4.8(2) holds for \( m + 1 \).

Now for any \( r \in \mathbb{Q} \) choose \( i \in \omega \) such that \( r = q_i \), and let \( b_r = c_i \). Suppose that \( r, s \in \mathbb{Q} \) and \( r < s \). Say \( r = q_i \) and \( s = q_j \). Let \( m = \max(i, j) + 1 \). Let \( p \) be the permutation of \( m \) such that \( q_p(0) < \cdots < q_p(m-1) \). Say \( i = p(u) \) and \( j = p(v) \). Then \( r < s \) implies that \( q_i < q_j \), hence \( q_p(u) < q_p(v) \), hence \( u < v \), hence \( c_p(u) < c_p(v) \), hence \( c_i < c_j \), hence \( b_r < b_s \).

Now for each \( r \in \mathbb{Q} \) let \( d_r \in A \) be such that \( g(d_r) = b_r \). We now define \( e_q \) for each \( i \in \omega \) by recursion. Let \( e_{q_0} = d_{q_0} \). Suppose that \( e_{q_0}, \ldots, e_{q_m} \) have been defined so that the following conditions hold:

4.8(3) \( \forall i \leq m[g(e_{q_i}) = b_{q_i}] \).

4.8(4) \( \forall i, j \leq m[b_{q_i} < b_{q_j} \rightarrow e_{q_i} < e_{q_j}] \).

Then we define

\[
e_{q_{m+1}} = \left( d_{q_{m+1}}, \prod_i \{e_{q_i} : i \leq m, b_{q_{m+1}} < b_{q_i}\}\right) + \sum \{e_{q_i} \setminus d_{q_{m+1}} : i \leq m, b_{q_i} < b_{q_{m+1}}\}.
\]

Clearly 4.8(3) holds for \( m + 1 \). Now suppose that \( i \leq m \). If \( b_{q_i} < b_{q_{m+1}} \), then \( \forall j \leq m[b_{q_{m+1}} < b_{q_j} \rightarrow e_{q_i} < e_{q_j}] \), and so \( e_{q_i} \leq e_{q_{m+1}} \). By 4.8(3), \( e_{q_i} < e_{q_{m+1}} \). If \( b_{q_{m+1}} < b_{q_i} \), then \( \forall j \leq m[b_{q_j} < b_{q_{m+1}} \rightarrow e_{q_j} < e_{q_i}] \), and so \( e_{q_{m+1}} \leq e_{q_i} \). By 4.8(3), \( e_{q_{m+1}} < e_{q_i} \).

Thus 4.8(3) and 4.8(4) hold for all \( m \), giving \( -(v) \).

(ii)\(\Rightarrow\)(vi): Assume (ii). If \( I_{A \alpha}(A) \neq A \), then \( A/I_{A \alpha}(A) \) is nontrivial and atomless, contradicting (ii).

(vi)\(\Rightarrow\)(iv): Assume that (iv) fails. Let \( B \) be a nontrivial atomless subalgebra of \( A \). We claim

4.8(5) \( B \cap I_{A \alpha}(A) = \{0\} \) for all \( \alpha \).

Of course this will show that (vi) fails. We prove 4.8(5) by induction on \( \alpha \). It is obvious for \( \alpha = 0 \). Assume it for \( \alpha \), but suppose that \( 0 \notin b \in B \cap I_{\alpha+1}(A) \). Then there exist \( c_0, \ldots, c_m \) such that each \( [c_i]I_{\alpha}(A) \) is an atom in \( A/I_{\alpha}(A) \) and \( [b]I_{\alpha}(A) = [c_0]I_{\alpha}(A) + \cdots + [c_m]I_{\alpha}(A) \). There are disjoint nonzero \( d_0, \ldots, d_{m+1} \in B \) with each \( d_i < b \). For each \( i \leq m + 1 \) there is a nonempty subset \( M_i \) of \( m + 1 \) such that \( [d_i]I_{\alpha}(A) = \sum_{j \in M_i} [c_j]I_{\alpha}(A) \). The sets \( M_i \) are pairwise disjoint, contradiction. The induction step is clear, so 4.8(5) holds.

\[\square\]

Proposition 4.9. If \( A \) is a ABID and \( \alpha \) is an ordinal, then \( I_{\alpha}(A) \) is superatomic.
**Proposition 4.11.** If $A$ and $B$ are ABIDs and $\alpha$ is an ordinal, then:

1. $I_\alpha(A \times B) = I_\alpha(A) \times I_\alpha(B)$.
2. There is an isomorphism of $(A \times B)/I_\alpha(A \times B)$ onto $(A/I_\alpha(A)) \times (B/I_\alpha(B))$ such that $\forall a \in A \forall b \in B[f([a,b])I_\alpha(A\times B)] = ([a]I_\alpha(A), [b]I_\alpha(B))$.

**Proof.** Induction on $\alpha$. It is clear for $\alpha = 0$. Now assume it for $\alpha$. For (i) for $\alpha + 1$, first suppose that $(a, b) \in I_{\alpha+1}(A \times B)$. Say

$$[(a, b)]I_\alpha(A \times B) = [(c_0, d_0)]I_\alpha(A \times B) + \cdots + [(c_{m-1}, d_{m-1})]I_\alpha(A \times B)$$

with each $[(c_i, d_i)]I_\alpha(A \times B)$ an atom of $(A \times B)/I_\alpha(A \times B)$. We apply (ii) for $\alpha$: for each $i < m$, $f([(c_i, d_i)]I_\alpha(A \times B)) = ([c_i]I_\alpha(A), [d_i]I_\alpha(B))$. It follows that we can write $m = M \cup N$ with $M$ and $N$ disjoint, $\forall i \in M [[c_i]I_\alpha(A) = 0$ and $[c_i]I_\alpha(A)$ is an atom of $A/I_\alpha(A)]$ and $\forall i \in N [[c_i]I_\alpha(A) = 0$ and $[d_i]I_\alpha(B)$ is an atom of $B/I_\alpha(B)]$. Hence $[a]I_\alpha(A) = \sum_{i \in M} [c_i]I_\alpha(A)$, and so $a \in I_{\alpha+1}(A)$. Similarly $b \in I_{\alpha+1}(B)$. This proves that $I_{\alpha+1}(A \times B) \subseteq I_{\alpha+1}(A) \times I_{\alpha+1}(B)$. The other inclusion is proved similarly, so (i) holds for $\alpha + 1$. (ii) for $\alpha + 1$ is clear.

The limit step is clear.

**Proposition 4.12.** For any ABIDs $A, B$, $\text{ar}(A \times B) = \max\{\text{ar}(A), \text{ar}(B)\}$.

**Proof.** Let $\gamma = \text{ar}(A \times B)$. Then by Proposition 4.11,

$$I_\gamma(A) \times I_\gamma(B) = I_\gamma(A \times B) = I_{\gamma+1}(A \times B) = I_{\gamma+1}(A) \times I_{\gamma+1}(B).$$

Hence $I_\gamma(A) = I_{\gamma+1}(A)$ and $I_\gamma(B) = I_{\gamma+1}(B)$. So $\max\{\text{ar}(A), \text{ar}(B)\} \leq \gamma$. On the other hand, let $\alpha = \text{ar}(A)$ and $\beta = \text{ar}(B)$. Let $\delta = \max(\alpha, \beta)$. Then by Proposition 4.11,

$$I_\delta(A \times B) = I_\delta(A) \times I_\delta(B) = I_{\delta+1}(A) \times I_{\delta+1}(B) = I_{\delta+1}(A \times B).$$
Hence the assertion of the proposition follows. □

**Proposition 4.13.** If $A$ and $B$ are superatomic ABIDS, then so is $A \times B$.

**Proof.** By Proposition 4.12, $\ar(A \times B) = \max(\ar(A), \ar(B))$. By Theorem 4.8, $I_{\ar(A)}(A) = A$ and $I_{\ar(B)}(B) = B$. Hence by Proposition 4.11,

$$I_{\ar(A \times B)}(A \times B) = I_{\ar(A \times B)}(A) \times I_{\ar(A \times B)}(B) = A \times B.$$ 

So by Theorem 4.8, $A \times B$ is superatomic. □

**Proposition 4.14.** Suppose that $A_1$ and $A_2$ are superatomic ABIDS, with $\tau_s(A_1) = (\alpha_1, \beta_1, n_1)$ and $\tau_s(A_2) = (\alpha_2, \beta_2, n_2)$. Define $\gamma = \max(\alpha_1, \alpha_2)$, $\delta = \max(\beta_1, \beta_2)$ and

$$m = \begin{cases} n_1 & \text{if } \alpha_2 < \alpha_1, \\ n_2 & \text{if } \alpha_1 < \alpha_2, \\ n_1 + n_2 & \text{if } \alpha_1 = \alpha_2. \end{cases}$$

Then $\tau_s(A_1 \times A_2) = (\gamma, \delta, m)$.

**Proof.** First we show that

4.14(1) $\alpha_s(A_1 \times A_2) = \max(\alpha_s(A_1), \alpha_s(A_2))$.

For, let $\gamma = \alpha_s(A_1 \times A_2)$. Then $(A_1 \times A_2)/I_\gamma(A_1 \times A_2)$ has a greatest element. By Proposition 4.11, $(A_1/I_\gamma(A_1)) \times (A_2/I_\gamma(A_2))$ has a greatest element $(x, y)$. Clearly $x$ is the greatest element of $A_1/I_\gamma(A_1)$ and $y$ is the greatest element of $A_2/I_\gamma(A_2)$. Hence $\max(\alpha_s(A_1), \alpha_s(A_2)) \leq \alpha_s(A_1 \times A_2)$. The other inequality is clear also. So 4.14(1) holds.

Now we consider some cases.

Case 1. $\alpha_s(A_1) < \ar(A_1)$ and $\alpha_s(A_2) < \ar(A_2)$. Then $\ar(A_1) = \alpha^*(A_1) + 1$ and $\ar(A_2) = \alpha^*(A_2) + 1$. Also, $\alpha_s(A_1 \times A_2) = \max(\alpha_s(A_1), \alpha_s(A_2)) < \max(\ar(A_1), \ar(A_2)) = \ar(A_1 \times A_2)$. Hence $\ar(A_1 \times A_2) = \alpha^*(A_1 \times A_2) + 1$ and $n(A_1 \times A_2)$ is the number of atoms in $(A_1 \times A_1)/I_{\alpha^*(A_1 \times A_2)}(A_1 \times A_2)$. Also, $\max(\alpha^*(A_1), \alpha^*(A_2)) = \alpha^*(A_1 \times A_2)$, since $\ar(A_1 \times A_2) = \max(\ar(A_1), \ar(A_2))$.

Subcase 1.1. $\ar(A_1) < \ar(A_2)$. Then $(A_1 \times A_2)/I_{\alpha^*(A_2)}(A_1 \times A_2) \cong A_2/I_{\alpha^*(A_2)}(A_2)$. Hence $n(A_1 \times A_2) = n(A_2)$, as desired.

Subcase 1.2. $\ar(A_2) < \ar(A_1)$. This is symmetric to Subcase 1.1.

Subcase 1.3. $\ar(A_1) = \ar(A_2)$. Then

$$(A_1 \times A_2)/I_{\alpha^*(A_2)}(A_1 \times A_2) \cong (A_1/I_{\alpha^*(A_1)}(A_1)) \times (A_2/I_{\alpha^*(A_2)}(A_2)).$$

Hence $n(A_1 \times A_2) = n(A_1) \cdot n(A_2)$, as desired.

Case 2. $\alpha_s(A_1) = \ar(A_1)$ and $\ar(A_2) < \ar(A_1)$. Now $\alpha_s(A_2) \leq \ar(A_2) < \ar(A_1) = \alpha_s(A_1)$, so $\alpha_s(A_1 \times A_2) = \alpha_s(A_1) = \ar(A_1) = \max(\ar(A_1), \ar(A_2)) = \ar(A_1 \times A_2)$. Hence $n(A_1 \times A_2) = 0$. Also, $\alpha^*(A_1 \times A_2) = \ar(A_1 \times A_2) = \ar(A_1) = \alpha^*(A_1)$. Moreover, $\alpha^*(A_2) \leq \ar(A_2) < \ar(A_1) = \alpha^*(A_1)$. Hence $\alpha^*(A_1 \times A_2) = \max(\alpha^*(A_1), \alpha^*(A_2))$. Since $\alpha^*(A_2) < \alpha^*(A_1)$ and $n(A_1) = 0$, this is as desired.

25
Proposition 4.17. \( \alpha_* (A_1) = \text{ar}(A_1) \) and \( \text{ar}(A_2) = \text{ar}(A_1) \). Then \( \text{ar}(A_1 \times A_2) = \text{ar}(A_1) \) and \( \alpha_* (A_1 \times A_2) = \alpha_* (A_1) = \text{ar}(A_1 \times A_2) \). So \( \alpha^* (A_1) = \text{ar}(A_1) \). Since \( \alpha_* (A_1 \times A_2) = \text{ar}(A_1 \times A_2) \), we have \( \alpha^* (A_1 \times A_2) = \text{ar}(A_1 \times A_2) = \text{ar}(A_1) = \alpha^* (A_1) = \max(\alpha^* (A_1), \alpha^* (A_2)) \). Note that \( n(A_1) = n(A_1 \times A_2) = 0 \). Now if \( \alpha_* (A_2) < \text{ar}(A_2) \) then the condition on \( m \) holds. If \( \alpha_* (A_2) = \text{ar}(A_2) \), then \( n(A_2) = 0 \) and again the condition on \( m \) holds.

Case 4. \( \alpha_* (A_1) = \text{ar}(A_1) \) and \( \alpha_* (A_2) < \text{ar}(A_2) \).

Subcase 4.1. \( \alpha_* (A_2) < \text{ar}(A_2) \). Say \( \alpha^* (A_2) = \varepsilon \) with \( \text{ar}(A_2) = \varepsilon + 1 \). Then \( \text{ar}(A_1) \leq \varepsilon \), so \( (A_1 \times A_2)/I_\varepsilon (A_1 \times A_2) \cong A_2/I_\varepsilon (A_2) \). Hence \( \alpha^* (A_1 \times A_2) = \varepsilon \). Since \( \alpha^* (A_1) \leq \text{ar}(A_1) \) we have \( \alpha^* (A_1) \leq \varepsilon \). So \( \alpha^* (A_1 \times A_2) = \max(\alpha^* (A_1), \alpha^* (A_2)) \). Now \( n(A_1) = 0 \) and \( n(A_1 \times A_2) = n(A_2) \), so the conditions on \( m \) hold.

Subcase 4.2. \( \alpha_* (A_2) = \text{ar}(A_2) \). Then \( \alpha^* (A_2) = \text{ar}(A_2) \) and the conditions in the proposition are clear.

Case 5. \( \alpha_* (A_2) = \text{ar}(A_2) \). This is symmetric to cases 2–4.

\[ \square \]

Proposition 4.15. If \( A \) is a superatomic BA, then \( \tau_* (A) = (\alpha^* (A), 0, n) \) with \( \text{ar}(A) = \alpha^* (A) + 1 \) and \( n > 0 \) is the finite number of atoms in \( A/I_{\alpha^* (A)} (A) \).

\[ \square \]

Proposition 4.16. If \( A \) is a ABID, \( a \in A \), and \( \alpha \) is an ordinal, then \( I_{\alpha} (A) \cap (A \uparrow a) = I_{\alpha} (A \uparrow a) \) and there is an isomorphism \( f_{\alpha} (a \uparrow a)/I_{\alpha} (A \uparrow a) \) onto \( (A/I_{\alpha} (A)) \uparrow [a]_{I_{\alpha} (A)} \) such that \( f_{\alpha} ([b]_{I_{\alpha} (A) [a]}) = [b]_{I_{\alpha} (A)} \) for all \( b \leq a \).

\[ \text{Proof} \] Induction on \( \alpha \). It is clear for \( \alpha = 0 \). Assume it is true for \( \alpha \), and suppose that \( b \in I_{\alpha+1} (A) \cap (A \uparrow a) \). Then there exist \( c_0, \ldots, c_m \in A \) such that each \( [c_i]_{I_{\alpha} (A)} \) is an atom of \( A/I_{\alpha} (A) \) and \( \sum c_i = I_{\alpha} (A \uparrow a) \). Since \( b \leq a \), we may assume that each \( c_i \leq a \). Then \( f_{\alpha}^{-1} [c_i]_{I_{\alpha} (A)} = [c_i]_{I_{\alpha+1} (A \uparrow a)} \) is an atom of \( (A \uparrow a)/I_{\alpha} (A \uparrow a) \) and \( [b]_{I_{\alpha} (A \uparrow a)} = [c_0]_{I_{\alpha} (A \uparrow a)} + \cdots + [c_m]_{I_{\alpha} (A \uparrow a)} \). Hence \( b \in I_{\alpha+1} (A \uparrow a) \). Thus \( I_{\alpha+1} (A) \cap (A \uparrow a) \subseteq I_{\alpha+1} (A \uparrow a) \). The other inclusion is proved similarly, so \( I_{\alpha+1} (A) \cap (A \uparrow a) = I_{\alpha+1} (A \uparrow a) \).

Now for all \( b, b' \leq a \),

\[
[b]_{I_{\alpha+1} (A \uparrow a)} = [b']_{I_{\alpha+1} (A \uparrow a)} \quad \text{iff} \quad b \bar{\Delta} b' \in I_{\alpha+1} (A \uparrow a)
\]

\[
\text{iff} \quad b \bar{\Delta} b' \in I_{\alpha+1} (A)
\]

\[
\text{iff} \quad [b]_{I_{\alpha+1} (A)} = [b']_{I_{\alpha+1} (A)}.
\]

Hence \( f_{\alpha+1} \) exists as indicated.

The limit case is clear.

\[ \square \]

Proposition 4.17. Suppose that \( A \) is a superatomic BA, \( \tau_* (A) = (\alpha, 0, m) \), and \( \beta, n, \gamma, p \) are such that \( \alpha = \max(\beta, \gamma) \) and

\[ m = \begin{cases} p & \text{if } \beta < \gamma, \\ n & \text{if } \gamma < \beta, \\ p + n & \text{if } \beta = \gamma. \end{cases} \]

Then there is an \( a \in A \) such that \( \tau_* (A \uparrow a) = (\beta, 0, n) \) and \( \tau_* (A \uparrow (-a)) = (\gamma, 0, p) \).

26
**Proof.** Assume the hypotheses.

*Case 1.* $\beta < \gamma$. Then $A/I_\beta(A)$ is infinite and superatomic, so we can choose $b_1, \ldots, b_n \in A$ such that $[b_1]_{I_\beta(A)}, \ldots, [b_n]_{I_\beta(A)}$ are distinct atoms of $A/I_\beta(A)$. We may assume that $b_1, \ldots, b_n$ are pairwise disjoint. Let $a = b_1 + \cdots + b_n$. Then by Proposition 4.16, $(A \upharpoonright a)/I_\beta(A \upharpoonright a) \cong (A/I_\beta(A)) \upharpoonright [a]_{I_\beta(A)}$, so $(A \upharpoonright a)/I_\beta(A \upharpoonright a)$ is finite with exactly $n$ atoms. Hence $\tau_s(A \upharpoonright a) = (\beta, 0, n)$. By Proposition 4.14, $\tau_s(A \upharpoonright (-a)) = (\gamma, 0, m)$.

*Case 2.* $\gamma < \beta$. This is symmetric to Case 1.

*Case 3.* $\beta = \gamma$. Now $A/I_\alpha(A)$ has exactly $m$ atoms. Let $[b_1]_{I_\alpha(A)}, \ldots, [b_n]_{I_\alpha(A)}$ be $n$ of them. We may assume that $b_1, \ldots, b_n$ are pairwise disjoint. Let $a = b_1 + \cdots + b_n$. Clearly $a$ is as desired. $\square$

**Proposition 4.18.** If two superatomic countable BAs have the same atomic type, then they are isomorphic.

**Proof.** Let $A$ and $B$ be superatomic countable BAs each with type $(\alpha, 0, m)$. We may assume that $\alpha \neq 0$; then $A$ and $B$ are infinite. Define

$$aRb \text{ iff } a \in A, b \in B, \tau_s(A \upharpoonright a) = \tau_s(B \upharpoonright b), \tau_s(A \upharpoonright (-a)) = \tau_s(B \upharpoonright (-b)).$$

We check the conditions (V1)–(V4) for a V-correspondence. (V1) and (V2) are clear. Now by symmetry it suffices to treat (V3). So, suppose that $aRb$ and $c \in A$. Say $\tau_s(A \upharpoonright a) = \tau_s(B \upharpoonright b) = (\beta, 0, n)$, $\tau_s(A \upharpoonright (a \cdot c)) = (\gamma, 0, p)$, $\tau_s(A \upharpoonright (a \cdot -c)) = (\delta, 0, q)$, and $\tau_s(A \upharpoonright (-a)) = \tau_s(B \upharpoonright (-b)) = (\varepsilon, 0, r)$. By Proposition 4.14 we have $\beta = \max(\gamma, \delta)$, and

$$n = \begin{cases} p & \text{if } \delta < \gamma, \\ q & \text{if } \gamma < \delta, \\ p+q & \text{if } \gamma = \delta. \end{cases}$$

Hence by Proposition 4.17 there is a $d_0 \leq b$ such that $\tau_s(B \upharpoonright d_0) = (\gamma, 0, p)$ and $\tau_s(B \upharpoonright (b-d_0)) = (\delta, 0, q)$. Thus $\tau_s(A \upharpoonright (a \cdot c)) = \tau_s(B \upharpoonright d_0)$ and $\tau_s(A \upharpoonright (a \cdot -c)) = \tau_s(B \upharpoonright (b-d_0))$. Now $B \upharpoonright (-d_0) \cong (B \upharpoonright -b) \times (B \upharpoonright (b \cdot -d_0))$, so by Proposition 4.14 we get

4.18(1) $\tau_s(B \upharpoonright (-d_0)) = (\varphi, 0, t)$, where $\varphi = \max(\varepsilon, \delta)$ and

$$t = \begin{cases} r & \text{if } \delta < \varepsilon, \\ q & \text{if } \varepsilon < \delta, \\ r+q & \text{if } \delta = \varepsilon. \end{cases}$$

But also $A \upharpoonright (-a+c) \cong A \upharpoonright (-a+a \cdot -c) \cong (A \upharpoonright (-a)) \times (A \upharpoonright (a \cdot -c))$ and $\tau_s(A \upharpoonright (-a)) = \tau_s(B \upharpoonright (-b))$ and $\tau_s(A \upharpoonright (a \cdot -c)) = \tau_s(B \upharpoonright (b \cdot -d_0))$, so by Proposition 4.14 and 4.18(1) we get $\tau_s(A \upharpoonright (-a + c)) = \tau_s(B \upharpoonright (-d_0))$. Thus

4.18(2) $(a \cdot c) R (b \cdot d_0)$.

Now $A \upharpoonright (-a+c) = A \upharpoonright (-a+ c \cdot a) \cong (A \upharpoonright (-a)) \times (A \upharpoonright (c \cdot a))$ and $B \upharpoonright (-b+d_0) \cong (B \upharpoonright (-b)) \times (B \upharpoonright d_0)$. Since $\tau_s(A \upharpoonright (-a)) = \tau_s(B \upharpoonright (-b))$ and $\tau_s(A \upharpoonright (c \cdot a)) = \tau_s(B \upharpoonright d_0)$, it follows from Proposition 4.14 that $\tau_s(A \upharpoonright (-a+c)) = \tau_s(-b+d_0)$. Thus
4.18(3) \((a \cdot -c)R(b \cdot -d_0)\).

Now let \(\tau_s(A \upharpoonright (-a \cdot -c)) = (\varphi, 0, s)\) and \(\tau_s(A \upharpoonright (-a \cdot c)) = (\psi, 0, t)\). Then by Proposition 4.14 we get \(\varepsilon = \max(\varphi, \psi)\) and

\[
r = \begin{cases} 
  s & \text{if } \psi < \varphi, \\
  t & \text{if } \varphi < \psi, \\
  s + t & \text{if } \varphi = \psi.
\end{cases}
\]

Since \(\tau_s(B \upharpoonright (-b)) = \tau_s(A \upharpoonright (-a))\), it follows from Proposition 4.17 that there is a \(d_1 \leq -b\) such that \(\tau_s(B \upharpoonright d_1) = \tau_s(A \upharpoonright (-a \cdot -c))\) and \(\tau_s(B \upharpoonright (-b \cdot -d_1)) = \tau_s(A \upharpoonright (-a \cdot c))\) Now \(A \upharpoonright (a+c) = A \upharpoonright (a+c-a) \cong (A \upharpoonright a) \times (A \upharpoonright (c-a))\) and \(B \upharpoonright (b+d_1) = B \upharpoonright (b+d_1-b) \cong (B \upharpoonright b) \times (B \upharpoonright (d_1-b)).\) Since \(\tau_s(A \upharpoonright a) = \tau_s(B \upharpoonright b)\) and \(\tau_s(A \upharpoonright (c-a)) = \tau_s(B \upharpoonright (d_1-b))\), it follows from Proposition 4.14 that \(\tau_s(A \upharpoonright (a+c)) = \tau_s(B \upharpoonright (b+d_1-b)).\) Thus

4.18(4) \((a + c)R(b + -d_1).\)

Now \(A \upharpoonright (-c + a) = A \upharpoonright (a + c-a) \cong (A \upharpoonright a) \times (A \upharpoonright (c-a))\) and \(B \upharpoonright (d_1 + b) = B \upharpoonright (d_1 + b - b) \cong (B \upharpoonright b) \times (B \upharpoonright (d_1-b)).\) Since \(\tau_s(A \upharpoonright a) = \tau_s(B \upharpoonright b)\) and \(\tau_s(A \upharpoonright (c-a)) = \tau_s(d_1 - b),\) it follows from Proposition 4.14 that \(\tau_s(A \upharpoonright (-c + a)) = \tau_s(B \upharpoonright (d_1 + b)).\) Thus 4.18(5) \((c \cdot -a)R(-d_1 \cdot -b).\)

Now 4.18(2)–4.18(5) verify (V3).

If \(A\) is a countable superatomic BA, we define the invariant of \(A\) to be \(\operatorname{INV}(A) = (0, \alpha^*(A), 0, n(A)).\) INV\(_0\) is the set of all invariants of countable superatomic BAs.

**Theorem A.** (i) INV\(_0\) = \{(0, \alpha, 0, n) : \alpha\ a countable ordinal, n a positive integer\} \(\cup\) \{(0, 0, 0, 0)\}.

(ii) For any countable superatomic BAs \(A, B, A \cong B\) iff \(\operatorname{INV}(A) = \operatorname{INV}(B).\)

**Proposition 4.19.** For every \(\alpha > 0\) and \(m > 0, \ \tau_s(\operatorname{Intalg}(\omega^\alpha \cdot m)) = (\alpha, 0, m).\)

**Proof.** Let \(A = \operatorname{Intalg}(\omega^\alpha \cdot m).\) We claim that for all \(\beta < \alpha,\)

4.19(1) \(\text{atoms}(A/I_\beta(A)) = \{[[\omega^\alpha \cdot i + \omega^\beta \cdot \xi, \omega^\alpha \cdot i + \omega^\beta \cdot (\xi + 1)]]_{I_\beta(A)} : i < n, \xi < \omega^\alpha\}.\)

4.19(2) \(A/I_\beta(A)\) is generated by the increasing sequence

\[
\begin{align*}
\{[0, \omega^\beta \cdot \xi]_{I_\beta(A)} : \xi < \omega^\alpha\} & \sim \\
\{[0, \omega^\alpha + \omega^\beta \cdot \xi]_{I_\beta(A)} : \xi < \omega^\alpha\} & \sim \\
\cdots & \\
\{[0, \omega^\alpha \cdot (n-1) + \omega^\beta \cdot \xi]_{I_\beta(A)} : \xi < \omega^\alpha\} & \sim
\end{align*}
\]

We prove these statements by induction on \(\beta\). They are clear for \(\beta = 0\). Assume them for \(\beta\). Then they are clear for \(\beta + 1\). Now assume that \(\gamma\) is limit \(< \omega^\alpha\) and 4.19(1) and 4.19(2) hold for all \(\beta < \gamma\). Then for any \(i < n\) and \(\xi < \omega^\alpha, [[\omega^\alpha \cdot i + \omega^\gamma \cdot \xi, \omega^\alpha \cdot i + \omega^\gamma \cdot (\xi + 1)]]_{I_\gamma(A)}\)

28
Now 4.19(1) and 4.19(2) follow for \( \gamma \). So
\[
A/I
\]

is the submonoid of \( SBA \) consisting of superatomic BAs with universe contained in \( \omega \).

**Corollary 4.20.** If \( A \) is a countable superatomic BA with \( \tau_s(A) = (\alpha, 0, m) \), then \( A \cong \text{Intalg}(\omega^\alpha \cdot m) \).

A **commutative monoid with 0** is an algebra \( (M, +, 0) \) such that + is commutative and associative, and \( \forall x \in M \) \( x + 0 = x = 0 + x \). **BA** is the set of all BAs with universe contained in in some \( \alpha \leq \omega \). For \( A, B \in \text{BA} \), \( A + B \) is a member of \( \text{BA} \) isomorphic to \( A \times B \). 0 is the one-element BA. Let \( M \) be the set \( \omega_1 \times (\omega \setminus \{0\}) \cup \{0'\} \) with + defined as follows on \( M \):

\[
(\alpha, m) + (\beta, n) = \begin{cases} 
(\alpha, m) & \text{if } \alpha > \beta, \\
(\beta, n) & \text{if } \alpha < \beta, \\
(\alpha, m + n) & \text{if } \alpha = \beta;
\end{cases}
\]

\[
(\alpha, m) + 0' = 0' + (\alpha, m) = (\alpha, m); \\
0' + 0' = 0'
\]

**SBA** is the submonoid of **BA** consisting of superatomic BAs with universe contained in \( \omega \).

**Theorem 4.21.** \( (\text{SBA}, +, 0) \cong (M, +, 0') \).

**Proposition 4.22.** If \( \alpha \) is a nonzero ordinal, \( A \) is a superatomic \( ABID \), \( \tau_s(A) = (\alpha, \alpha, 0) \), \( \beta < \alpha \), and \( n \in \omega \setminus \{0\} \), then there is an \( a \in A \) such that \( \tau_s(A \upharpoonright a) = (\beta, 0, n) \) and \( \tau_s(a^+) = (\alpha, \alpha, 0) \).

**Proof.** \( A/I\beta(A) \) is infinite and superatomic, so we can choose \( b_1, \ldots, b_n \in A \) such that \([b_1]_{I\beta(A)}, \ldots, [b_n]_{I\beta(A)} \) are distinct atoms of \( A/I\beta(A) \). We may assume that \( b_1, \ldots, b_n \) are pairwise disjoint. Let \( a = b_1 + \cdots + b_n \). Then by Proposition 4.16, \( (A \upharpoonright a)/I\beta(A \upharpoonright a) \)
Proposition 4.21. Let $d \equiv (A/I_\beta(A)) \upharpoonright [a]_{I_\beta(A)}$, so $(A \upharpoonright a)/I_\beta(A \upharpoonright a)$ is finite with exactly $n$ atoms. Hence $\tau_s(A \upharpoonright a) = (\beta, 0, n)$. Then $\tau_s(a^+) = (\alpha, \alpha, 0)$ by Propositions 1.18 and 4.14. \hfill \Box

**Proposition 4.23.** For any countable ordinal $\alpha$ and any countable superatomic ABIDs $A, B$, if $\tau_s(A) = (\alpha, \alpha, 0) = \tau_s(B)$, then $A \cong B$.

**Proof.** Assume the hypotheses. Let $R = \{(a, b) \in A \times B : a = b = 0$, or $a \neq 0 \neq b$, $A \upharpoonright a \cong B \upharpoonright b\}$. It suffices now to show that $R$ is a $V$-correspondence between $A$ and $B$. Clearly (V1) and (V2) hold. By symmetry, now, it suffices to check (V3). So suppose that $aRb$ and $c \in A$. Let $f$ be an isomorphism from $A \upharpoonright a$ onto $B \upharpoonright b$. Let $d_0 = f(a \cdot c)$. Then $A \upharpoonright (a \cdot c) \cong B \upharpoonright (b \cdot d_0)$ and $A \upharpoonright (a \setminus c) \cong B \upharpoonright (b \setminus d_0)$. So $(a \cdot c)R(b \cdot d_0)$ and $(a \setminus c)R(b \setminus d_0)$.

Next, $(c \setminus a) \in a^\perp$. By Proposition 1.18 we have $A \cong (A \upharpoonright a) \times a^\perp$. Say $\tau_s(A \upharpoonright b) = \tau_s(A \upharpoonright a) = (\beta, 0, n)$. By Proposition 4.14 we have $\beta < \alpha$. Hence by Proposition 4.14, $\tau_s(b^+) = (\alpha, \alpha, 0)$. Say $\tau_s(c \setminus a) = (\beta', 0, n')$. Then $\beta' < \alpha$ by Proposition 4.14. By Proposition 4.21 let $d_1 \in b^+$ be such that $\tau_s(b^+ \upharpoonright d_1) = (\beta', 0, n')$ and $\tau_s(d_1^+) = (\alpha, \alpha, 0)$. Then

\[ A \upharpoonright (a + c) = A \upharpoonright (a + (c \setminus a)) \cong (A \upharpoonright a) \times A \upharpoonright (c \setminus a) \cong (B \upharpoonright b) \times (B \upharpoonright d_1) \cong B \upharpoonright (b + d_1). \]

Hence $(a + c)R(b + d_1)$. Also,

\[ A \upharpoonright (c \setminus a) \cong (B \upharpoonright d_1) \cong B \upharpoonright (d_1 \setminus b). \]

Hence $(c \setminus a)R(d_1 \setminus b)$.

**Proposition 4.24.** For every $\alpha > 0$, $\tau_s(\text{Intalg}_d(\omega^\alpha)) = (\alpha, \alpha, 0)$.

**Proof.** Let $A = \text{Intalg}_d(\omega^\alpha)$. We prove the following statement by induction on $\beta$:

\[ 4.23(1) \] For all $\beta < \alpha$, $A/I_\beta(A)$ is generated by

\[ \{[[0, \omega^\beta \cdot \xi]]_{I_\beta(A)} : \xi < \omega^\alpha\}. \]

This is obvious for $\beta = 0$ and the induction steps are clear. From 4.23(1) it is clear that $A/I_\alpha(A) = \{0\}$.

**Corollary 4.25.** If $\alpha$ is a countable ordinal and $A$ is a superatomic ABID with $\tau_s(A) = (\alpha, \alpha, 0)$, then $A \cong \text{Intalg}_d(\omega^\alpha)$.

**Proposition 4.26.** Suppose that $A$ is a countable superatomic ABID, $\alpha \geq \beta > 0$ and $n \in \omega \setminus \{0\}$. Suppose that $\tau_s(A) = (\alpha, \beta, n)$. Then $A \cong \text{Intalg}(\omega^\alpha \cdot n) \times \text{Intalg}_d(\omega^\beta)$.

**Proof.** Assume the hypotheses. Let $[a_0]_{I_\alpha(A)}, \ldots, [a_{n-1}]_{I_\alpha(A)}$ be the distinct atoms of $A/I_\alpha(A)$. We may assume that $a_i \cdot a_j = 0$ for $i \neq j$. Let $b = \sum_{i < n} a_i$. Then $\tau_s(A \upharpoonright b) = (\alpha, 0, n)$. By Proposition 4.14, $\tau_s(b^+) = (\beta, \beta, 0)$. Now the desired result follows from Proposition 1.18 and Corollaries 4.20 and 4.25.

**Proposition 4.27.** If $\alpha < \beta$ and $n \in \omega$, then $\text{Intalg}(\omega^\alpha \cdot n) \times \text{Intalg}_d(\omega^\beta) \cong \text{Intalg}_d(\omega^\beta)$.
Then by Proposition 4.14, \( \tau_s(\text{Intalg}(\omega^\alpha \cdot n)) = (\alpha, 0, n) \) and \( \tau_s(\text{Intalg}_d(\omega^\beta)) = (\beta, \beta, 0) \). Hence the desired conclusion follows from Corollary 4.25.

**Proposition 4.28.** If \( A \) and \( B \) are countable superatomic ABIDs such that \( \tau_s(A) = \tau_s(B) \), then \( A \cong B \).

**Proof.** By Proposition 4.26.

**Proposition 4.29.** Suppose that \( A \) is a countable superatomic ABID, and let \( \tau_s(A) = (\alpha, \beta, n) \). Also suppose given ordinals \( \alpha_0, \alpha_1 \) and natural numbers \( n_0, n_1 \) such that the following conditions hold:

1. \( \alpha = \max(\alpha_0, \alpha_1) \).
2. \( \beta \leq \alpha_1 \).
3. \( n_0 > 0 \).
4. If \( n_1 = 0 \), then \( \alpha_1 = \beta \).
5. If \( \alpha_1 < \alpha_0 \), then \( n_0 = n \).
6. If \( \alpha_0 < \alpha_1 \), then \( n_1 = n \).
7. If \( \alpha_0 = \alpha_1 \), then \( n_0 + n_1 = n \).

Then there is an \( a \in A \) such that \( \tau_s(A \upharpoonright a) = (\alpha_0, 0, n_0) \) and \( \tau_s(a \downarrow) = (\alpha_1, \beta, n_1) \).

**Proof.** Case 1. \( n = 0 \). Then \( \alpha = \beta \). By (iii), (v), (vii), \( \alpha_0 < \alpha_1 \). Hence the ABID \( A/I_{\alpha_0}(A) \) has at least \( n_0 \) distinct atoms \( [a_i]_{I_{\alpha_0}(A)} \) for \( i < n_0 \). We may assume that \( a_i \cdot a_j = 0 \) for \( i \neq j \). Let \( b = \sum_{i < n_0} a_i \). Then \( \tau_s(A \upharpoonright b) = (\alpha_0, 0, n_0) \). By Proposition 4.17, \( \tau_s(b^\downarrow) = (\alpha_0, 0, n_0) \), as desired.

Case 2. \( n \neq 0 \). Then \( \text{ar}(A) = \alpha + 1 \), and \( A/I_{\alpha}(A) \) is finite with \( n \) atoms.

Subcase 2.1. \( \alpha_0 < \alpha \). Then \( \alpha = \alpha_1 \) and \( n_1 = n \) by (vi). Now \( A/I_{\alpha_0}(A) \) has \( n \) infinitely many atoms. Let \( [a_i]_{I_{\alpha_0}(A)} \) be an atom for \( i < n_0 \). We may assume that \( a_i \cdot a_j = 0 \) for \( i \neq j \). Let \( b = \sum_{i < n_0} a_i \). Then \( \tau_s(A \upharpoonright b) = (\alpha_0, 0, n_0) \). By Proposition 4.14, \( \tau_s(b^\downarrow) = (\alpha_1, \beta, n_1) \).

Subcase 2.2. \( \alpha_0 = \alpha \) and \( n_0 < n \). By (v), \( \alpha_1 = \alpha \). Hence by (vii), \( n_1 = n - n_0 \). Let \( [a_i]_{I_{\alpha_0}(A)} \) be an atom for \( i < n_0 \). We may assume that \( a_i \cdot a_j = 0 \) for \( i \neq j \). Let \( b = \sum_{i < n_0} a_i \). Then \( \tau_s(A \upharpoonright b) = (\alpha_0, 0, n_0) \). By Proposition 4.14, \( \tau_s(b^\downarrow) = (\alpha_1, \beta, n_1) \).

Subcase 2.3. \( \alpha_0 = \alpha \) and \( n_0 = n \).

Subsubcase 2.3.1. \( \alpha_1 = \alpha \). Then \( n_1 = 0 \) by (vii), and \( \alpha_1 = \beta \) by (iv). Let \( [a_i]_{I_{\alpha_0}(A)} \) be an atom for \( i < n_0 \). We may assume that \( a_i \cdot a_j = 0 \) for \( i \neq j \). Let \( b = \sum_{i < n_0} a_i \). Then \( \tau_s(A \upharpoonright b) = (\alpha_0, 0, n_0) \). By Proposition 4.14, \( \tau_s(b^\downarrow) = (\alpha_1, \beta, n_1) \).

Subcase 2.3.2. \( \alpha_1 < \alpha \).

Subsubcase 2.3.2.1. \( n_1 = 0 \). So \( \alpha_1 = \beta \) by (iv). Let \( f \) be an isomorphism of \( A \) onto \( \text{Intalg}(\omega^\alpha \cdot n) \times \text{Intalg}_d(\omega^\beta) \). Let \( b = f^{-1}(1, 0) \). Then \( \tau(b) = (\alpha, 0, n) \) and \( \tau(b^\downarrow) = (\beta, \beta, 0) \).

Subsubcase 2.3.2.2. \( n_1 \neq 0 \). Now \( \beta \leq \alpha_1 \), so \( A/I_{\alpha_1}(A) \) is a BA. It has infinitely many atoms since \( \alpha_1 < \alpha \). Let \( [a_i]_{I_{\alpha_1}(A)} \) be distinct atoms for \( i < n_1 \). We may assume
that \(a_i \cdot a_j = 0\) for \(i \neq j\). Let \(b = \sum_{i<n} a_i\). Then \(\tau(A \upharpoonright b) = (\alpha_1, 0, n_1)\). By Proposition 4.14, \(\tau(b^+) = (\alpha, \beta, n)\). Let \(g\) be an isomorphism from \(b^+\) onto \(\text{Intalg}(\omega^\alpha \cdot n) \times \text{Intalg}_d(\omega^\beta)\). Let \(c = g^{-1}(1, 0)\). Then \(\tau(A \upharpoonright c) = (\alpha, 0, n)\). For any \(x \in c^\perp\) let \(f(x) = (x \cdot b, x \wedge b)\). Then \(f(x) \in (A \upharpoonright b) \times \{y \in b^+: y \cdot g^{-1}(1, 0) = 0\}\). Clearly \(f\) is one-one and is a homomorphism.

If \(y \in (A \upharpoonright b)\) and \(z \in b^+\) with \(z \cdot g^{-1}(1, 0) = 0\), then \(y + z \in c^\perp\), and \(f(y + z) = (y, z)\). So \(f\) is an isomorphism, and hence \(\tau(c^+) = (\alpha_1, \beta, n_1)\), as desired. \(\square\)

**Proposition 4.30.** If \(J\) is an ideal in a ABID \(A\) and \(\alpha\) is an ordinal, then

(i) \(\forall \alpha[\alpha \in J \cap J_\alpha(A) = I_\alpha(J)]\).

(ii) \(\{a I_\alpha(J), [a]_{I_\alpha(A)}: a \in J\}\) is an isomorphism of \(J/I_\alpha(J)\) onto an ideal of \(A/I_\alpha(A)\).

**Proof.** Induction on \(\alpha\). It is clear for \(\alpha = 0\). Assume it for \(\alpha\). First we prove (i) for \(\alpha + 1\). Suppose that \(x \in J \cap J_{\alpha+1}(A)\). Then we can write \(x I_\alpha(A) = \sum_{i<m} [y_i I_\alpha(A)],\) where each \([y_i I_\alpha(A)\) is an atom of \(A/I_\alpha(A)\). Let \(y_i' = y_i \cdot x\) for all \(i < m\). Then for each \(i < m, y_i \triangle y_i' = y_i \wedge x \in I_\alpha(A),\) so \([y_i I_\alpha(A)] = [y_i']_{I_\alpha(A)}\). Note that \(\forall i < m[y_i' \in J]\). By (ii) for \(\alpha, [y_i']_{I_\alpha(J)}\) is an atom of \(J/I_\alpha(J)\). Now \(x \triangle \sum_{i<m} y_i' \in J \cap I_\alpha(A),\) so by (i) for \(\alpha, x \triangle \sum_{i<m} y_i' \in I_\alpha(J)\). It follows that \(x \in I_{\alpha+1}(J)\).

Conversely, suppose that \(x \in I_{\alpha+1}(J)\). Then we can write

\[x I_\alpha(J) = \sum_{i<m} [y_i I_\alpha(J)],\]

with each \([y_i I_\alpha(J)\) an atom of \(J/I_\alpha(J)\). Then by (ii) for \(\alpha, each [y_i I_\alpha(A)\) is an atom of \(A/I_\alpha(A)\). Also, \(x \triangle \sum_{i<m} y_i \in I_\alpha(J) \subseteq I_\alpha(A)\). So \(x \in I_{\alpha+1}(A)\). This proves (i) for \(\alpha + 1\).

To prove (ii) for \(\alpha + 1\), suppose that \(a, a' \in J\). Then

\[a I_\alpha(J) = [a']_{I_\alpha(J)} \iff a \triangle a' \in I_\alpha(J) \]

\[\iff a \triangle a' \in I_\alpha(A).\]

Then (ii) follows.

The limit case is clear. \(\square\)

**Corollary 4.31.** For any ordinals \(\alpha, \beta\), \(\text{Intalg}_d(\omega^\alpha) \times \text{Intalg}_d(\omega^\beta) \cong \text{Intalg}_d(\omega^{\max(\alpha, \beta)})\).

**Proof.** By Proposition 4.14. \(\square\)

**Corollary 4.32.** Let \(A\) be a countable superatomic ABID, \(\tau_s(A) = (\alpha, \alpha, 0), \beta < \alpha, m \in \omega \setminus \{0\}\). Then there is an \(a \in A\) such that \(\tau_s(A \upharpoonright a) = (\beta, 0, m)\) and \(\tau_s(a^+) = (\alpha, \alpha, 0)\).

**Proof.** Replace \(A, \alpha, \beta, n, \alpha_0, \alpha_1, n_0, n_1\) by \(A, \alpha, \alpha, 0, \beta, \alpha, m, 0\) in Proposition 4.29. \(\square\)

5. Countable BAs in general

**Proposition 5.1.** If \(A\) is a BA, \(J\) is a superatomic ideal of \(A\), and \(A/J\) is atomless, then \(\forall \alpha[I_\alpha(A) \subseteq J].\)

32
Let $b \in A$. Then there is a atomic type of an element $\sigma = \min(I_\beta(A))$ by Proposition 4.30. Let $u' = u \cdot y_i$. Since $u \cdot y_i \in J$, it follows that $u' \notin J$. Hence $u' \notin I_\beta(A)$, so $[u']I_\beta(A) = [y_i]I_\beta(A)$. Hence $y_i \cdot u' \in I_\beta(A)$, and so $y_i \cdot u' \notin J$, contradiction.

**Proposition 5.2.** If $A$ is a BA, $J$ is a superatomic ideal of $A$, and $A/J$ is atomless, then $I_{\text{at}(A)}(J) = J$.

**Proof.** By Proposition 4.30, $I_{\text{at}(A)+1}(J) = J \cap I_{\text{at}(A)+1}(A) = J \cap I_{\text{at}(A)}(A) = I_{\text{at}(A)}(J)$. Hence $\text{at}(J) \leq \text{at}(A)$ and $I_{\text{at}(A)}(J) = J \subseteq I_{\text{at}(A)}(A)$. By Proposition 4.30, $I_{\text{at}(A)}(A) \subseteq J$. So $I_{\text{at}(A)}(A) = J$.

For any ABID $A$ define the atomic type of an element $a \in A$ to be $\tau(a) \overset{\text{def}}{=} \tau_s(I_{\text{at}(A)}(A))$. The atomic type of an element $a$ in $A$ is $\tau(a) \overset{\text{def}}{=} \tau_s((A \upharpoonright a) \cap I_{\text{at}(A)}(A))$. If $A$ is superatomic, then $I_{\text{at}(A)}(A) = A$, and hence $\tau(a) = \tau_s(A)$. A function $f : A \to \alpha$ is additive iff $f(0) = 0$ and for all $a, b \in A$, $f(a + b) = \max\{f(a), f(b)\}$. The middle entry of $\tau(A)$ is denoted by $\sigma'(A)$, and the middle entry of $\tau(a)$ by $\sigma'(a)$. These entries are also called special. Thus $\sigma'(A)$ is the least $\beta$ such that $I_{\text{at}(A)}(A)/I_\beta(I_{\text{at}(A)}(A))$ has a greatest element, and for each $a \in A$, $\sigma'(a)$ is the least $\beta$ such that $((A \upharpoonright a) \cap I_{\text{at}(A)}(A))/I_\beta((A \upharpoonright a) \cap I_{\text{at}(A)}(A))$ has a greatest element. By convention, $\sigma'(0) = 0$. A BA $A$ is special iff it is atomic with atomic type $(\text{ar}(A), \text{ar}(A), 0)$.

**Proposition 5.3.** Suppose that $A$ is a ABID, $\tau(A) = (\alpha, \beta, n)$, $a \in A$, $\sigma'(a) = \gamma$, $\sigma'(a^\perp) = \delta$, and $(\alpha_0, \gamma, m)$ and $(\alpha_1, \delta, k)$ are such that

(i) $\alpha_0 \geq \gamma$, $\alpha_1 \geq \delta$, and $\alpha = \max(\alpha_0, \alpha_1)$.
(ii) If $\alpha_1 < \alpha_0$, then $m = n$.
(iii) If $\alpha_0 < \alpha_1$, then $k = n$.
(iv) If $\alpha_0 = \alpha_1$, then $m + k = n$.
(v) If $m = 0$ then $\alpha_0 = \gamma$.
(vi) If $k = 0$ then $\alpha_1 = \delta$.

Then there is a $b \in A$ such that $[a]I_{\text{at}(A)}(A) = [b]I_{\text{at}(A)}(A)$, $\tau(A \upharpoonright b) = (\alpha_0, \gamma, m)$, and $\tau(b^\perp) = (\alpha_1, \delta, k)$.

**Proof.** Assume the hypotheses. We may assume that $|A| > 1$. Let $A' = I_{\text{at}(A)}(A)$. Let $f$ be an isomorphism from $(A \upharpoonright a) \cap A'$ onto $\text{Intalg}(\omega^\rho \cdot p) \times \text{Intalg}_d(\omega^\gamma)$, and let $g$ be an isomorphism from $a^\perp \cap A'$ onto $\text{Intalg}(\omega^\xi \cdot q) \times \text{Intalg}_d(\omega^\delta)$. Then

$$A' \cong \text{Intalg}(\omega^\rho \cdot p) \times \text{Intalg}(\omega^\xi \cdot q) \times \text{Intalg}_d(\omega^\gamma) \times \text{Intalg}_d(\omega^\delta);$$

$$\text{Intalg}(\omega^\rho \cdot p) \times \text{Intalg}(\omega^\xi \cdot q) \cong \text{Intalg}(\omega^\alpha \cdot n);$$

$$\text{Intalg}_d(\omega^\gamma) \times \text{Intalg}_d(\omega^\delta) \cong \text{Intalg}_d(\omega^\beta).$$

Let $d = f^{-1}(1, 0) + g^{-1}(1, 0)$. Then $A' \upharpoonright d \cong \text{Intalg}(\omega^\alpha \cdot n)$. 33
In fact if so it suffices to show that y. Hence from (1) we get and τ. Finally, suppose that k = 0. Then by (vi), α₁ = δ. Also, by (iii) α₁ ≤ α₀; so α = α₀. By (ii) and (iv), m = n. Let b = a + d. Then τ(A | b) = (α,γ,m). We have b ⊥ ∩ A' = {x ∈ A' : x ⋅ (a + d) = 0} and so τ(b ⊥) = (δ,δ,0). Clearly aΔb ∈ A'.

Case 2. n = 0. Then α = β.

Subcase 2.1. m = k = 0. By (v) and (vi), a is as desired.

Subcase 2.2. m ≠ 0. Then by (ii) and (iv), α₀ < α₁. So α = α₁. By (iii), k = 0. Hence by (vi), α₁ = δ. Since α₀ ≥ γ, we have γ < α. max(γ,δ) = α, so δ = α. Hence τₙ(a ⊥ ∩ A') = (δ,δ,0). Now we apply Corollary 4.32 with A,α,β,m replaced by a ⊥ ∩ A',δ,α₀,m. This gives c ∈ a ⊥ ∩ A' such that τₙ(A' | c) = (α₀,0,m) and τₙ({x ∈ a ⊥ ∩ A' : x ⋅ c = 0}) = (δ,δ,0).

Subsubcase 2.2.1. p = 0. Then τ(A | a) = (γ,γ,0). Let b = a + c. Then (A | b) ∩ A' ≅ (A | a) ∩ A' × (A | c), and so τ(A | b) = (α₀,γ,m). Now b ⊥ ∩ A' = {x ∈ A' : x ⋅ (a + c) = 0}, so τ(b ⊥) = (δ,δ,0).

Subsubcase 2.2.2. p ≠ 0.

5.3(2) \( \varphi < \alpha \).

In fact, suppose that \( \varphi = \alpha \). Since \( \gamma < \alpha \), it follows that \( \gamma < \ar((A | a) \cap A') \) and so ar((A | a) ∩ A') = α + 1. Since ar(A') = α, this is a contradiction.
Let $b = a \setminus f^{-1}(1, 0) + c$. Then $A \upharpoonright b \cong (A \upharpoonright (a \setminus f^{-1}(1, 0))) \times (A \upharpoonright c)$. Now $\tau(A \upharpoonright (a \setminus f^{-1}(1, 0))) = (\gamma, \gamma, 0)$ and $\tau(A \upharpoonright c) = (\alpha_0, 0, m)$. Hence $\tau(A \upharpoonright b) = (\alpha_0, \gamma, m)$.

5.3(3) $b^\perp \cap A' = \{x + y : x \cdot (a + c) = 0, y \leq f^{-1}(1, 0)\}$.

In fact, $\supseteq$ is clear. Now suppose that $z \in b^\perp \cap A'$ and $z \cdot f^{-1}(1, 0) = 0$. Then

$$0 = z \cdot (a \setminus f^{-1}(1, 0) + c) = z \cdot (a \setminus f^{-1}(1, 0)) + z \cdot c = z \cdot (a \setminus f^{-1}(1, 0)) + z \cdot a \cdot f^{-1}(1, 0) + z \cdot c = z \cdot (a + c).$$

So 5.3(3) holds.

It follows that $b^\perp \cap A' \cong \{x : x \cdot (a + c) = 0\} \times (A \upharpoonright f^{-1}(1, 0))$. Now $\{x : x \cdot (a + c) = 0\} \cong \text{Intalg}_d(\omega^\delta)$ and $(A \upharpoonright f^{-1}(1, 0)) \cong \text{Intalg}(\omega^\varphi \cdot p)$. Hence by Proposition 4.27, $\tau(b^\perp) = (\delta, \delta, 0)$.

Subcase 2.3. $k \neq 0$. Then by (iii) and (iv), $\alpha_1 < \alpha_0$ and $m = 0$. Hence $\alpha_0 = \gamma$ by (v). $\delta \leq \alpha_1$ so $\delta < \gamma$. Hence $\alpha = \max(\gamma, \delta) = \gamma$. Hence $\tau(A \upharpoonright a) = (\gamma, \gamma, 0)$. Now we apply Corollary 4.32 with $A, \alpha, \beta, m$ replaced by $(A \upharpoonright a) \cap A'$, $\gamma, \gamma, 0$. This gives $c \in (A \upharpoonright a) \cap A'$ such that $\tau(A \upharpoonright c) = (\alpha_1, 0, k)$ and $\tau_s(\{x : x \leq a, x \cdot c = 0\}) = (\gamma, \gamma, 0)$.

Subcase 2.3.2. $q = 0$. Then $\tau(a^\perp) = (\delta, \delta, 0)$. Let $b = a \setminus c$. Then $\tau(A \upharpoonright b) = (\gamma, \gamma, 0)$.

5.3(4) $b^\perp = \{y + z : y \cdot a = 0, z \leq c\}$.

In fact, $\supseteq$ is clear, and if $x \in b^\perp$ and $x \cdot c = 0$, then $0 = x \cdot (a \setminus c) = x \cdot (a \setminus c) + x \cdot a \cdot c = x \cdot a$, proving $\subseteq$.

It follows that $b^\perp \cap A' \cong (a^\perp \cap A') \times (A \upharpoonright c)$. Hence $\tau(b^\perp) = (\alpha_1, \delta, k)$, as desired.

Subcase 2.3.2. $q \neq 0$.

5.3(5) $\xi < \alpha$.

This is proved as for 5.3(2).

Let $b = a \setminus c + g^{-1}(1, 0)$. Then $(A \upharpoonright b) \cap A' \cong \text{Intalg}_d(\omega^\gamma) \times \text{Intalg}(\omega^\xi \cdot q) \cong \text{Intalg}_d(\omega^\gamma)$ by Proposition 4.27. So $\tau(A \upharpoonright b) = (\gamma, \gamma, 0)$. Now $b^\perp \cap A' = \{x + y : x \cdot (a + g^{-1}(1, 0)) = 0, y \leq c\}$. Hence $b^\perp \cap A' \cong \{x : x \cdot (a + g^{-1}(1, 0)) = 0\} \times (A \upharpoonright c)$. Hence $\tau(b^\perp) = (\alpha_1, \delta, k)$.

\[\square\]

**Proposition 5.4.** If $A$ is a ABID, $a \in A$, $J$ is an ideal of $A$, and $\beta$ is an ordinal, then

(i) $I_\beta((A \upharpoonright a) \cap J) = (A \upharpoonright a) \cap I_\beta(J)$.

(ii) There is an isomorphism $f$ of $((A \upharpoonright a) \cap J)/I_\beta((A \upharpoonright a) \cap J)$ onto $(J/I_\beta(J)) \upharpoonright (a \triangle I_\beta(J))$ such that for any $x \in (A \upharpoonright a) \cap J$, $f([x]_{I_\beta((A \upharpoonright a) \cap J)}) = [x]_{I_\beta(J)}$.

**Proof.** For brevity let $B = (A \upharpoonright a) \cap J$. We prove the proposition by induction on $\beta$.

It is clear for $\beta = 0$. Now assume it for $\beta$. Suppose that $x \in I_{\beta+1}(B)$. Then there exist an $m \in \omega$ and elements $y_i \in B$ for $i < m$ such that each $[y_i]_{I_\beta(B)}$ is an atom in $B/I_\beta(B)$ and $[x]_{I_\beta(B)} = \sum_{i < m} [y_i]_{I_\beta(B)}$. Hence $x \Delta \sum_{i < m} y_i \in I_\beta(B)$, so by the inductive hypothesis $(x \Delta \sum_{i < m} y_i) \in I_\beta(J)$. Also by the inductive hypothesis each $f([y_i]_{I_\beta(B)} = [y_i]_{I_\beta(J)}$ is an atom in $J/I_\beta(J)$. Applying $f$ to the equation $[x]_{I_\beta(B)} = \sum_{i < m} [y_i]_{I_\beta(B)}$ we get $[x]_{I_\beta(J)} = \sum_{i < m} [y_i]_{I_\beta(J)}$. Hence $x \in I_{\beta+1}(J)$. Thus we have $\subseteq$ in (i) for $\beta + 1$. Reversing these steps we get $\supseteq$ also. So (i) holds for $\beta + 1$.
For (ii) for $\beta + 1$, we have for any $x, x' \in A \upharpoonright a$

$$[x]_{I_{\beta+1}(A|a)} = [x']_{I_{\beta+1}(A|a)} \quad \text{iff} \quad (x \triangle x') \in I_{\beta+1}(A \upharpoonright a) \cap J$$

$$\text{iff} \quad (x \triangle x') \in I_{\beta}(J) \quad \text{by (i) for } \beta + 1$$

$$\text{iff} \quad [x]_{I_{\beta}(J)} = [x']_{I_{\beta}(J)}.$$  

Now (ii) for $\beta + 1$ follows.

The limit case is clear. \hfill \Box

**Proposition 5.5.** For any ABID $A$ let $\alpha = \sigma'(A)$. Then $\sigma' \upharpoonright A$ is an additive function with range contained in $\alpha + 1$.

**Proof.** We have $\sigma'(0) = 0$.

5.5(1) $\forall a \in A [\sigma'(a) \leq \sigma'(A)]$.

For, suppose that $a \in A$. $I_{\alpha,(A)}(A)/I_{\alpha}(I_{\alpha}(A)(A))$ has a largest element $[b]_{I_{\alpha}(I_{\alpha}(A)(A))}$, where $b \in I_{\alpha}(A)(A)$. Now we apply Proposition 5.4, with $J = I_{\alpha}(A)(A)$. Note that $(a \cdot b) \in (A \upharpoonright a) \cap I_{\alpha}(A)(A)$. We claim that $[a \cdot b]_{I_{\alpha}(A \upharpoonright a) \cap I_{\alpha}(A)(A))}$ is the greatest element of $(I_{\alpha}(A)(A))/I_{\alpha}(A \upharpoonright a) \cap I_{\alpha}(A)(A))$. For, suppose that $c \in (A \upharpoonright a) \cap I_{\alpha}(A)(A)$. Then $[c]_{I_{\alpha}(I_{\alpha}(A)(A))} \in I_{\alpha}(A)(A)/I_{\alpha}(I_{\alpha}(A)(A))$, and hence $[c]_{I_{\alpha}(I_{\alpha}(A)(A))} \leq [b]_{I_{\alpha}(I_{\alpha}(A)(A))}$. But also $[c]_{I_{\alpha}(I_{\alpha}(A)(A))} \leq [a]_{I_{\alpha}(I_{\alpha}(A)(A))}$, so $[c]_{I_{\alpha}(I_{\alpha}(A)(A))} \leq [a \cdot b]_{I_{\alpha}(I_{\alpha}(A)(A))}$. Applying $f^{-1}$ from Proposition 5.4 we get $[c]_{I_{\alpha}(I_{\alpha}(A)(A))} \leq [a \cdot b]_{I_{\alpha}(I_{\alpha}(A)(A))}$. This proves the claim. Hence (1) holds. It follows that the range of $\sigma' \upharpoonright A$ is contained in $\alpha + 1$.

5.5(2) $\forall a, b \in A [a \leq b \rightarrow \sigma'(a) \leq \sigma'(b)].$

In fact, suppose that $a, b \in A$ and $a \leq b$. Let $\beta = \sigma'(b)$. Then $((A \upharpoonright b) \cap I_{\alpha}(A)(A))/I_{\beta}((A \upharpoonright b) \cap I_{\alpha}(A)(A))$ has a largest element $[d]_{I_{\beta}((A \upharpoonright b) \cap I_{\alpha}(A)(A))}$. By Proposition 5.4, $[d]_{I_{\beta}(I_{\alpha}(A)(A))}$ is the largest element of $(I_{\alpha}(A)(A)/I_{\beta}(I_{\alpha}(A)(A))) \upharpoonright [b]_{I_{\beta}(I_{\alpha}(A)(A))}$. We claim that $(a \cdot d) \triangle I_{\beta}((A \upharpoonright a) \cap I_{\alpha}(A)(A))$ is the greatest element of $((A \upharpoonright a) \cap I_{\alpha}(A)(A))/I_{\beta}((A \upharpoonright a) \cap I_{\alpha}(A)(A))$.

For, suppose that $c \in (A \upharpoonright a) \cap I_{\alpha}(A)(A)$. Then $[c]_{I_{\beta}(I_{\alpha}(A)(A))} \leq [b]_{I_{\beta}(I_{\alpha}(A)(A))}$, and hence $[c]_{I_{\beta}(I_{\alpha}(A)(A))} \leq [d]_{I_{\beta}(I_{\alpha}(A)(A))}$. But also $[c]_{I_{\beta}(I_{\alpha}(A)(A))} \leq [a]_{I_{\beta}(I_{\alpha}(A)(A))}$, so

$$[c]_{I_{\beta}(I_{\alpha}(A)(A))} \leq [a \cdot d]_{I_{\beta}(I_{\alpha}(A)(A))}.$$ 

Applying $f^{-1}$ from Proposition 5.4 we get $[c]_{I_{\beta}(I_{\alpha}(A)(A))} \leq [a \cdot d]_{I_{\beta}(I_{\alpha}(A)(A))}$. This proves the claim. Hence 5.5(2) holds.

Now let $a, b \in A$. By 5.5(2), $\max\{\sigma'(a), \sigma'(b)\} \leq \sigma'(a + b)$. Let $\beta = \max\{\sigma'(a), \sigma'(b)\}$. Then by Proposition 5.4, there exist a largest element $[c]_{I_{\beta}(I_{\alpha}(A)(A))}$ of

$$(I_{\alpha}(A)/I_{\beta}(I_{\alpha}(A)(A))) \upharpoonright (a \triangle I_{\beta}(I_{\alpha}(A)(A)))$$

and a largest element $[d]_{I_{\beta}(I_{\alpha}(A)(A))}$ of

$$(I_{\alpha}(A)/I_{\beta}(I_{\alpha}(A)(A))) \upharpoonright (b \triangle I_{\beta}(I_{\alpha}(A)(A))).$$
Now we claim that \([c + d]_{I_\beta(I_{ar}(A))}\) is the largest element of
\[
(I_{ar}(A))/I_\beta(I_{ar}(A)) \uparrow ((a + b)\triangle I_\beta(I_{ar}(A)) (A)).
\]
For, suppose that \([x]_{I_\beta(I_{ar}(A))} \in (I_{ar}(A))/I_\beta(I_{ar}(A)) \uparrow ((a + b)\triangle I_\beta(I_{ar}(A)) (A)).
Then
\[
[x]_{I_\beta(I_{ar}(A))} = [x \cdot a]_{I_\beta(I_{ar}(A))} + [x \cdot b]_{I_\beta(I_{ar}(A))}
\leq [c]_{I_\beta(I_{ar}(A))} + [d]_{I_\beta(I_{ar}(A))}
= [c + d]_{I_\beta(I_{ar}(A))}.
\]

A countable BA \(A\) is normal iff \(\tau(A) = (\ar(A), \ar(A), 0)\) and \(A/I_{ar}(A) \cong \Fr(\omega)\).

**Proposition 5.6.** Suppose that \(A\) is a countable BA, \(d, d' \in A\), and \([d]_{I_{ar}(A)} = [d']_{I_{ar}(A)}\). Then \(\alpha_*(A \uparrow d) = \alpha_*(A \uparrow d')\).

**Proof.** Note that if \(a \in I_{ar}(A)\) then \(\alpha_*(A \uparrow a) = 0\). Since \(d \cdot -d' \in I_{ar}(A)\) and \(d' \cdot -d \in I_{ar}(A)\), we have
\[
\alpha_*(A \uparrow d) = \max(\alpha_*(A \uparrow (d \cdot d')), \alpha_*(A \uparrow (d \cdot -d'))) = \alpha_*(A \uparrow (d \cdot d')),
\]
and similarly \(\alpha_*(A \uparrow d') = \alpha_*(A \uparrow (d \cdot d'))\).

**Proposition 5.7.** Suppose that \(A\) is normal and \(g\) is an isomorphism from \(A/I_{ar}(A)\) onto \(\Fr(\omega)\). Then there is a function \(r_{Ag} : \Fr(\omega) \to (\alpha_*(A) + 1)\) such that for all \(d \in A\), \(r_{Ag}(g([d]_{I_{ar}(A)})) = \alpha_*(A \uparrow d)\). Moreover, if \(a, b \in \Fr(\omega)\) and \(a \cdot b = 0\), then \(r_{Ag}(a + b) = \max(r_{Ag}(a), r_{Ag}(b))\).

**Proof.** \(r_{Ag}\) is well-defined, since if \(g([d]_{I_{ar}(A)}(A)) = g([d']_{I_{ar}(A)}(A))\) then \([d]_{I_{ar}(A)}(A)) = g([d']_{I_{ar}(A)}(A))\) and so by Proposition 5.6, \(\alpha_*(A \uparrow d) = \alpha_*(A \uparrow d')\). \(r_{Ag}\) maps into \(\alpha_*(A) + 1\) by Proposition 5.5. If \(a, b \in \Fr(\omega)\) and \(a \cdot b = 0\), choose \(d, d' \in A\) such that \(g([d]_{I_{ar}(A)}(A)) = a\) and \(g([d']_{I_{ar}(A)}(A)) = b\). We may assume that \(d \cdot d' = 0\). Then by Proposition 4.14,
\[
r_{Ag}(a + b) = g([d + d']_{I_{ar}(A)}(A)) = \alpha_*(A \uparrow (d + d'))
\]
\[
= \max(\alpha_*(A \uparrow d), \alpha_*(A \uparrow d')) = \max(r_{Ag}(a), r_{Ag}(b)).
\]

**Theorem 5.8.** Suppose that \(A_1\) and \(A_2\) are normal BA with the same atomic type, and \(g_1 : A_1/I_{ar}(A_1) \to \Fr(\omega)\) and \(g_2 : A_2/I_{ar}(A_2) \to \Fr(\omega)\) are isomorphisms. Then the following are equivalent:

(i) \(A_1 \cong A_2\).

(ii) There is an automorphism \(k\) of \(\Fr(\omega)\) such that \(r_{A_1g_1} = r_{A_2g_2} \circ k\).

**Proof.** (i)⇒(ii): Suppose that \(s\) is an isomorphism of \(A_1\) onto \(A_2\). Now there is an isomorphism \(s^+\) of \(A_1/I_{ar}(A_1)\) onto \(A_2/I_{ar}(A_2)\) such that \(\forall a \in A_1[s^+([a]_{I_{ar}(A_1)}(A_1)) =

37
Let \( k = g_2 \circ s^+ \circ g_1^{-1} \). So \( k \) is an automorphism of \( \Fr(\omega) \). For any \( a \in \Fr(\omega) \) let \( d \in A_1 \) be such that \( g_1([d]_{I_{ar(A_1)}(A_1)}) = a \). Then

\[
\begin{align*}
\tau(A_2 \upharpoonright d) &= \tau(A_2 \upharpoonright (-b))
\end{align*}
\]

We verify that this is a \( V \)-correspondence. If \((0, b) \in R\), then \( \tau(A_1 \upharpoonright 0) = \tau(A_2 \upharpoonright b) \), so \( b = 0 \). Conversely, \( b = 0 \) clearly implies that \((0, b) \in R\). So (V1) holds. (V2) is similar.

Now by symmetry it suffices to prove (V3). So suppose that \((a, b) \in R\) and \( c \in A_1 \).

(a) There is a \( d_0 \leq b \) such that \( g_2([d_0]_{I_{ar(B)}(B)}) = k(g_1([a \cdot c]_{I_{ar(A_1)}(A_1)}) \), \( \tau(A_1 \upharpoonright (a \cdot c)) = \tau(A_2 \upharpoonright d_0) \), \( \tau(A_1 \upharpoonright (a + c)) = \tau(A_2 \upharpoonright (b + d_0)) \), and \( \tau(A_1 \upharpoonright (-a + c)) = \tau(A_2 \upharpoonright (-b + d_0)) \).

Assuming that (a) holds, we have

\[
\begin{align*}
g_2([b \setminus d_0]_{I_{ar(A_2)}(A_2)}) &= g_2([b]_{I_{ar(A_2)}(A_2)} \setminus [d_0]_{I_{ar(A_2)}(A_2)}) \\
&= k(g_1([a]_{I_{ar(A_1)}(A_1)}) \setminus k(g_1([a \cdot c]_{I_{ar(A_1)}(A_1)}) \\
&= k(g_1([a \setminus (a \cdot c)])_{I_{ar(A_1)}(A_1)}) = k(g_1([a \cdot c]_{I_{ar(A_1)}(A_1)}))
\end{align*}
\]

Hence \( (a, c, b \cdot d_0) \in R \) and \( (a \setminus c, b \cdot d_0) \in R \).

To prove (a), let \( x = k(g_1([a \cdot c]_{I_{ar(A_1)}(A_1)}) \). Then \( r_{A_2 g_2}(k(g_1([a \cdot c]_{I_{ar(A_1)}(A_1)}))) = r_{A_2 g_2}(g_2([b]_{I_{ar(A_2)}(A_2)})) \) and

\[
\begin{align*}
r_{A_1 g_1}(k([a \cdot c]_{I_{ar(A_1)}(A_1)}) &= r_{A_2 g_2}(k(g_1([a \cdot c]_{I_{ar(A_1)}(A_1)}))) \\
&= r_{A_2 g_2}(k(g_1([a \cdot c]_{I_{ar(A_1)}(A_1)})) \setminus k(g_1([a \cdot c]_{I_{ar(A_1)}(A_1)}))) \\
&= r_{A_2 g_2}(g_2([b]_{I_{ar(A_2)}(A_2)} \setminus x)).
\end{align*}
\]

Now since \( k(g_1([a]_{I_{ar(A_1)}(A_1)}))) = g_2([b]_{I_{ar(A_2)}(A_2)}) \), there is a \( y \leq b \) such that

\[
\begin{align*}
g_2([y]_{I_{ar(A_2)}(A_2)}) &= k(g_1([a \cdot c]_{I_{ar(A_1)}(A_1)})).
\end{align*}
\]

Now let \( \tau(A_1 \upharpoonright a) = (\alpha, \beta, n) \), \( \tau(A_1 \upharpoonright (a \cdot c)) = (\alpha_0, \gamma, m) \), and \( \tau(A_1 \upharpoonright (a \cdot c)) = (\alpha_1, \delta, k) \). Now \( a_*(A_1 \upharpoonright y) = r_{A_2 g_2}(g_2([y]_{I_{ar(A_2)}(A_2)}) = r_{A_2 g_2}(k(g_1([a \cdot c]_{I_{ar(A_1)}(A_1)}))) = r_{A_1 g_1}(g_1([a \cdot c]_{I_{ar(A_1)}(A_1)})))
\]

38
\[
\alpha_*(A_2 \upharpoonright (b \cdot y)) = r_{A_2, \gamma_2}(g_2([b \cdot y]_{\text{arr}(A_2)}(A_2)))
\]
\[
= r_{A_2, \gamma_2}(g_2([b]_{\text{arr}(A_2)}(A_2)) \cdot g_2([y]_{\text{arr}(A_2)}(A_2)))
\]
\[
= r_{A_2, \gamma_2}(k(a, [a]_{\text{arr}(A_1)}(A_1)) \cdot k(c, [c]_{\text{arr}(A_1)}(A_1)))
\]
\[
= r_{A_2, \gamma_2}(k(a, [a]_{\text{arr}(A_1)}(A_1)))
\]
\[
= r_{A_1, \gamma_1}(g_1([a]_{\text{arr}(A_1)}(A_1))) = \alpha_*(a \cdot c) = \delta.
\]

Now we apply Proposition 5.3 with \(A, a\) replaced by \(A_2 \upharpoonright b, y\). This gives \(d_0 \leq b\) such that 
\[
[y]_{\text{arr}(A_2)\upharpoonright b} = [d_0]_{\text{arr}(A_2)\upharpoonright b},
\]
\(\tau(A_2 \upharpoonright d_0) = (\alpha_0, \gamma, m) = \tau(A_1 \upharpoonright (a \cdot c))\), and 
\(\tau(A_2 \upharpoonright (b \cdot d_0)) = (\alpha_1, \delta, k) = \tau(A_1 \upharpoonright (a \cdot c))\). This gives the first part of (a).

Now let \(\tau(A_1 \upharpoonright (-a)) = (\alpha_2, \varphi, q)\). Then by Proposition 4.14, 
\(\tau((A_1 \upharpoonright (-a)) \times (A_1 \upharpoonright (a \cdot c))) = (\alpha_3, \psi, r)\), where 
\(\alpha_3 = \max(\alpha_1, \alpha_2),\) \(\psi = \max(\varphi, \delta)\), and 
\[
r = \begin{cases} 
q & \text{if } \alpha_1 < \alpha_2, \\
k & \text{if } \alpha_2 < \alpha_1, \\
q + k & \text{if } \alpha_1 = \alpha_2.
\end{cases}
\]

Since \(\tau(A_2 \upharpoonright (-b)) = \tau(A_1 \upharpoonright (-a))\) and \(\tau(A_2 \upharpoonright (b \cdot d_0)) = \tau(A_1 \upharpoonright (a \cdot c))\), it follows that 
\(\tau((A_1 \upharpoonright (-a)) \times (A_1 \upharpoonright (a \cdot c))) = \tau((A_2 \upharpoonright (-b)) \times (A_2 \upharpoonright (b \cdot d_0))).\)

Now 
\((A_1 \upharpoonright (-a)) \times (A_1 \upharpoonright (a \cdot c)) \cong A_1 \upharpoonright (-a + a \cdot c) = A_1 \upharpoonright (-a + c)\)

and 
\((A_2 \upharpoonright (-b)) \times (A_2 \upharpoonright (b \cdot d_0)) \cong A_2 \upharpoonright (-b + b \cdot d_0) = A_2 \upharpoonright (-b + d_0).\)

Hence \(\tau(A_1 \upharpoonright (-a + c)) = \tau(A_2 \upharpoonright (-b + d_0))\). Now by Proposition 4.14, 
\(\tau((A_1 \upharpoonright (-a)) \times (A_1 \upharpoonright (a \cdot c))) = (\alpha_4, \rho, u)\) where 
\(\alpha_4 = \max(\alpha_0, \alpha_2),\) \(\rho = \max(\gamma, \varphi)\), and 
\[
u = \begin{cases} 
q & \text{if } \alpha_0 < \alpha_2, \\
m & \text{if } \alpha_2 < \alpha_0, \\
m + q & \text{if } \alpha_0 = \alpha_2.
\end{cases}
\]

Since \(\tau(A_1 \upharpoonright (-a)) = \tau(A_2 \upharpoonright (-b))\) and \(\tau(A_1 \upharpoonright (a \cdot c)) = \tau(A_2 \upharpoonright d_0)\), it follows that 
\(\tau((A_1 \upharpoonright (-a)) \times (A_1 \upharpoonright (a \cdot c))) = \tau((A_2 \upharpoonright (-b)) \times (A_2 \upharpoonright d_0))).\)

Now 
\((A_1 \upharpoonright (-a)) \times (A_1 \upharpoonright (a \cdot c)) \cong A_1 \upharpoonright (-a + a \cdot c) = A_1 \upharpoonright (-a + c)\)

and 
\((A_2 \upharpoonright (-b)) \times (A_2 \upharpoonright d_0) \cong A_2 \upharpoonright (-b + d_0).\)

Hence \(\tau(A_1 \upharpoonright (-a + c)) = \tau(A_2 \upharpoonright (-b + d_0))\).
This proves (a).

(b) There is a $d_1 \leq -b$ such that $g_2([d_1]I_{ar(A_2)}(A_2)) = k(g_1([c\backslash a]I_{ar}(A_1)(A_1)))$, $\tau(A \upharpoonright (a+c)) = \tau(B \upharpoonright (b+d_1))$, $\tau(A \upharpoonright ((-a) \cdot (-c))) = \tau(B \upharpoonright ((-b) \cdot (-d_1)))$, $\tau(A \upharpoonright (c\backslash a)) = \tau(B \upharpoonright (d_1 \backslash b))$, and $\tau(A \upharpoonright (a+c)) = \tau(B \upharpoonright (b-d_1))$.

Assuming that (b) holds, we have

$$g_2([b+d_1]I_{ar(A_2)}(A_2)) = g_2([b]I_{ar(A_2)}(A_2) + g_2([d_1]I_{ar(A_2)}(A_2)) = k(g_1([a]I_{ar(A_2)}(A_2)) + k(g_1([c\backslash a]I_{ar(A_2)}(A_2)))$$

$$= k(g_1([a+c]I_{ar(A_2)}(A_2)))$$

Hence $(a+c, b+1) \in R$ and $(c\backslash a, d_1 \backslash b) \in R$.

To prove (b), let $x = k(g_1([c\backslash a]I_{ar(A_1)}(A_1)))$. Then

$$r_{A_1g_1}(g_1([c\backslash a]I_{ar(A_1)}(A_1))) = r_{A_2g_2}(k(g_1([c\backslash a]I_{ar(A_1)}(A_1))) = r_{A_2g_2}(x).$$

Now since $k(g_1([a]I_{ar(A_1)}(A_1))) = g_2([-b]I_{ar(A_2)}(A_2))$, there is a $y \leq -b$ such that $g_2([y]I_{ar(A_2)}(A_2))) = k(g_1(A \upharpoonright ((-a) \cdot (-c))]I_{ar(A_1)}(A_1)))$. Say $\tau(A \upharpoonright (-a)) = (\alpha, \beta, n)$, $\tau(A \upharpoonright (-a) \cdot (-c)) = (\alpha_0, \gamma, m)$, and $\tau(A \upharpoonright (c\backslash a)) = (\alpha_1, \delta, k)$. Then

$$\sigma'(y) = r_{A_2g_2}(g_2([y]I_{ar(A_2)}(A_2))) = r_{A_2g_2}(k(g_1(A_1 \upharpoonright ((-a) \cdot (-c)]I_{ar(A_1)}(A_1))))$$

$$= r_{A_1g_1}(g_1(A_1 \upharpoonright ((-a) \cdot (-c)]I_{ar(A_1)}(A_1))) = \sigma'(-a \cdot (-c)) = \gamma.$$

Also,

$$\sigma'(-b \cdot (-y)) = r_{A_2g_2}(g_2([-b] \cdot (-y)]I_{ar(A_2)}(A_2))$$

$$= r_{A_2g_2}(g_2([-b]I_{ar(A_2)}(A_2)) \cdot g_2([-y]I_{ar(A_2)}(A_2)))$$

$$= r_{A_2g_2}(k(g_1([-a]I_{ar(A_1)}(A_1))) \cdot k(g_1([c\backslash a]I_{ar(A_1)}(A_1))))$$

$$= r_{A_1g_1}(g_1([c\backslash a]I_{ar(A_1)}(A_1)))$$

$$= \sigma'(c\backslash a) = \delta.$$

Now we apply Proposition 5.3 with $A, a$ replaced by $A_2 \upharpoonright (-b), y$. This gives $d_1 \leq -b$ such that $d_1I_{ar(A_2)}(-b)](A \upharpoonright (-b)) = yI_{ar(A_2)}(-b)](A \upharpoonright (-b))$, $\tau(A_2 \upharpoonright d_1) = (\alpha_0, \gamma, m)$, and $\tau(A_2 \upharpoonright ((-b) \backslash d_1)) = (\alpha_1, \delta, k)$. As in the proof of (a), $g_2([d_1]I_{ar(A_2)}(A_2)) = k(g_1([c\backslash a]I_{ar(A_1)}(A_1)))$.

Then $\tau(A_2 \upharpoonright ((-b) \cdot d_1) = \tau(A_2 \upharpoonright d_1) = (\alpha_0, \gamma, m) = \tau(A_1 \upharpoonright ((-a) \cdot (-c))$, and $\tau(A_2 \upharpoonright ((-b) \cdot (-d_1)) = (\alpha_1, \delta, k) = \tau(A_1 \upharpoonright (c\backslash a)))$. Now let $\tau(A_1 \upharpoonright a) = (\alpha_2, \psi, p)$. Then by Proposition 4.14,

$$\tau((A_1 \upharpoonright a) \times (A_1 \upharpoonright (c\backslash a))) = (\alpha_3, \psi, q),$$

40
where $\alpha_3 = \max(\alpha_2, \alpha_1)$, $\psi = \max(\varphi, \delta)$, and

$$q = \begin{cases} p & \text{if } \alpha_1 < \alpha_2, \\ k & \text{if } \alpha_2 < \alpha_1, \\ p + k & \text{if } \alpha_1 = \alpha_2. \end{cases}$$

It follows that

$$\tau((A_1 \uparrow a) \times (A_1 \uparrow (c \setminus a))) = \tau((A_2 \uparrow b) \times (A_2 \uparrow ((-d_1) \cdot (-b)))).$$

Now $A_1 \uparrow (a + c) \cong (A_1 \uparrow a) \times (A_1 \uparrow (c \setminus a))$ and $A_2 \uparrow (b - d_1)$, so $\tau(A_1 \uparrow (a + c)) = \tau(A_2 \uparrow (b - d_1)).$

Next, $\tau((A_1 \uparrow a) \times (A_1 \uparrow ((-a) \cdot (-c)))) = (\alpha_4, \rho, s)$, where $\alpha_4 = \max(\alpha_2, \alpha_0)$, $\rho = \max(\gamma, \varphi)$, and

$$s = \begin{cases} m & \text{if } \alpha_2 < \alpha_0, \\ p & \text{if } \alpha_0 < \alpha_2, \\ m + p & \text{if } \alpha_0 = \alpha_2. \end{cases}$$

It follows that

$$\tau((A_1 \uparrow a) \times (A_1 \uparrow ((-a) \cdot (-c)))) = \tau((A_2 \uparrow b) \times (A_2 \uparrow ((-b) \cdot d_1))).$$

Now $A_1 \uparrow (a - c) \cong (A_1 \uparrow a) \times (A_1 \uparrow ((-a) \cdot (-c)))$ and $A_2 \uparrow (b + d_1) \cong (A_2 \uparrow b) \times (A_2 \uparrow ((-b) \cdot d_1))$. Hence $\tau(a - c) = \tau(b + d_1)$.

If $A$ is a normal BA with atomic type $\alpha$, and $g$ is an isomorphism of $A/I_{ar(A)}(A)$ onto $\operatorname{Fr}(\omega)$, then $\operatorname{INV}(A) = (1, \alpha, \{r_{Ag} \circ h : h \in \operatorname{Aut}(\operatorname{Fr}(\omega))\})$.

**Theorem B.** If $A$ and $B$ are normal BA$s$, then $A \cong B$ iff $\operatorname{INV}(A) = \operatorname{INV}(B)$.

**Proof.** Let $\operatorname{INV}(A) = (1, \alpha, \{r_{Ag} \circ h : h \in \operatorname{Aut}(\operatorname{Fr}(\omega))\})$ and $\operatorname{INV}(B) = (1, \beta, \{r_{Bk} \circ h : h \in \operatorname{Aut}(\operatorname{Fr}(\omega))\})$. $\Rightarrow$: Suppose that $h$ is an isomorphism of $A$ onto $B$. Then obviously $\alpha = \beta$. By Theorem 5.8 there is an automorphism $l$ of $\operatorname{Fr}(\omega)$ such that $r_{Ag} = r_{Bk} \circ l$. Then

$$(*) \{r_{Bk} \circ h : h \in \operatorname{Aut}(\operatorname{Fr}(\omega))\} = \{r_{Ag} \circ h : h \in \operatorname{Aut}(\operatorname{Fr}(\omega))\}.$$  

In fact, if $h \in \operatorname{Aut}(\operatorname{Fr}(\omega))$, then $r_{Bk} \circ h = r_{Bk} \circ l \circ l^{-1} \circ h = r_{Ag} \circ l^{-1} \circ h$. So $\subseteq$ holds in $(*)$. The other inclusion follows similarly.

$\Leftarrow$: Suppose that $\operatorname{INV}(A) = \operatorname{INV}(B)$. In particular, $\alpha = \beta$. We have $r_{Ag} \in \{r_{Bk} \circ h : h \in \operatorname{Aut}(\operatorname{Fr}(\omega))\}$. So there is an $h \in \operatorname{Aut}(\operatorname{Fr}(\omega))$ such that $r_{Ag} = r_{Bk} \circ h$. By Proposition 5.8, $A \cong B$.

We also define $\operatorname{INV}_1 = \{(1, \alpha, \{r \circ h : h \in \operatorname{Aut}(\operatorname{Fr}(\omega)), r : \operatorname{Fr}(\omega) \rightarrow \alpha + 1, r(1) = \alpha\})\}$. Thus if $A$ is a normal BA, then $\operatorname{INV}(A) \in \operatorname{INV}_1$. The next portion of these notes is aimed at showing that for every $\alpha \in \operatorname{INV}_1$ there is a normal BA $A$ such that $\operatorname{INV}(A) = \alpha$.

If $\langle B_n : n \in \omega \rangle$ is a system of BA$s$, we define

$$\sum_{n \in \omega} B_n = \left\{ f \in \prod_{n \in \omega} B_n : \{i \in \omega : f_i \neq 0\} \text{ is finite} \right\}$$

41
Proposition 5.9. \( \sum_{n \in \omega} B_n \) is an ideal in \( \prod_{n \in \omega} B_n \). \( \square \)

Proposition 5.10. Let \( B \overset{\text{def}}{=} \langle B_n : n \in \omega \rangle \) be a system of BAs. For each \( x \in \prod_{n \in \omega} B_n \) let \( h(x) = \{ y \in \sum_{n \in \omega} B_n : y \leq x \} \). For \( X, Y \in \text{rng}(h) \) let \( X \cdot' Y = X \cap Y \) and \( X +' Y = \{ u + v : u \in X \text{ and } v \in Y \} \). For \( X \in \text{rng}(h) \) let \( -'X = \{ u \in \sum_{n \in \omega} B_n : \forall x \in X \ [u \cdot x = 0] \} \).

Then

(i) \( B' \overset{\text{def}}{=} (\text{rng}(h), +', ', \{0\}, \sum_{n \in \omega} B_n) \) is a BA.

(ii) \( h \) is an isomorphism of \( \prod_{n \in \omega} B_n \) onto \( B' \).

(iii) \( \forall x, y \in \prod_{n \in \omega} B_n [h(x) \cap -' h(y) = \{0\} \cup \{ w \in \sum_{n \in \omega} B_n [w \leq x \text{ and } w \cdot y = 0] \} \).

**Proof.** (iii) is clear. Now it suffices to prove (ii). Suppose that \( x, y \in \prod_{n \in \omega} B_n \), and \( f \in \sum_{n \in \omega} B_n \). Then

\[
f \in h(x) \cdot' h(y) \iff (f \leq x \text{ and } f \leq y) \iff f \leq x \cdot y \iff f \in h(x \cdot y).
\]

\[
f \in h(x) +' h(y) \quad \text{iff} \quad \exists u \in h(x) \exists v \in h(y) [f = u + v]
\]

\[
\quad \text{iff} \quad \exists u, v [u \leq x \text{ and } v \leq y \text{ and } f = u + v]
\]

\[
\quad \text{iff} \quad f \leq x + y
\]

\[
\quad \text{iff} \quad f \in h(x + y).
\]

\[
f \in h(-x) \quad \text{iff} \quad f \leq -x
\]

\[
\quad \text{iff} \quad \forall g \in h(x) [f \cdot g = 0]
\]

\[
\quad \text{iff} \quad f \in -' h(x).
\]

Clearly \( h(0) = \{0\} \) and \( h(1) = \sum_{n \in \omega} B_n \).

Finally, suppose that \( x, y \in \prod_{n \in \omega} B_n \) and \( x \neq y \); say \( x \cdot -y \neq 0 \). Choose \( f \in h(x \cdot -y) \) with \( f \neq 0 \). Then \( f \in h(x) \) and \( f \leq -y \), hence \( f \in -' h(y) \). So \( f \in h(x) \cdot' -' h(y) \). Hence \( h(x) \cdot' -' h(y) \neq \{0\} \), and so \( h(x) \neq h(y) \). \( \square \)

Proposition 5.11. If \( A \) is a special ABID and \( \sigma'(A) = \alpha + 1 \), then there is a system \( \langle B_n : n \in \omega \rangle \) of superatomic BAs each of atomic type \( (\alpha, 0, 1) \) such that \( A \cong \sum_{n \in \omega} B_n \).

**Proof.** We know that \( A \cong \text{Intalg}_d(\omega^{\alpha + 1}) \). For each \( n \in \omega \) let \( B_n = \text{Intalg}([\omega^\alpha \cdot n, \omega^\alpha \cdot (n + 1)]) \). Clearly each \( B_n \) has atomic type \( (\alpha, 0, 1) \). Now for each \( \xi < \omega^{\alpha + 1} \) and \( i \in \omega \) define

\[
(f([0, \xi]))_i = \begin{cases} 
[\omega^\alpha \cdot i, \omega^\alpha \cdot (i + 1)] & \text{if } \omega^\alpha \cdot (i + 1) \leq \xi, \\
[\omega^\alpha \cdot i, \xi] & \text{if } \omega^\alpha \cdot (i + 1) \leq \xi < \omega^\alpha \cdot (i + 1), \\
0 & \text{if } \xi < \omega^\alpha \cdot i.
\end{cases}
\]

Clearly \( f \) maps \( \{ [0, \xi] : \xi < \omega^{\alpha + 1} \} \) into \( \sum_{n \in \omega} B_n \), and \( x \leq y \iff f(x) \leq f(y) \). Since the domain of \( f \) generates \( \text{Intalg}_d(\omega^{\alpha + 1}) \) and the range of \( f \) generates \( \sum_{n \in \omega} B_n \), by Proposition 1.21 \( f \) extends to an isomorphism of \( \text{Intalg}_d(\omega^{\alpha + 1}) \) onto \( \sum_{n \in \omega} B_n \). \( \square \)
Proposition 5.12. If $A$ is a special ABID, $\sigma'(A)$ is a limit ordinal, and $\langle \alpha_m : n \in \omega \rangle$ is a strictly increasing sequence of ordinals with supremum $\sigma'(A)$, then there is a system $\langle B_n : n \in \omega \rangle$ of superatomic BAs such that each $B_n$ has atomic type $(\alpha_{n+1}, 0, 1)$ and $A \cong \sum_{n \in \omega} B_n$.

Proof. Let $\beta = \sigma'(A)$. Thus $A \cong \text{Intalg}(\omega^\beta)$. For each $n \in \omega$ let

$$B_n = \text{Intalg}(\omega^{\alpha_n}, \omega^{\alpha_{n+1}}).$$

Clearly each $B_n$ has atomic type $(\alpha_{n+1}, 0, 1)$. $A \cong \sum_{n \in \omega} B_n$ as in the proof of Proposition 5.11.

If $B = \langle B_n : n \in \omega \rangle$ is a sequence of BAs and $\beta$ is an ordinal, we define

$$^*I^B_\beta = \left\{ f \in \prod_{n \in \omega} B_n : \forall n \in \omega [f(n) \in I_\beta(B_n)] \right\}.$$

Proposition 5.13. If $\langle B_n : n \in \omega \rangle$ is a system of ABIDs and $\beta$ is an ordinal, then

(i) $I_\beta(\sum_{n \in \omega} B_n) = \sum_{n \in \omega} I_\beta(B_n)$.
(ii) There is an onto isomorphism

$$f : \left( \sum_{n \in \omega} B_n \right) / I_\beta \left( \sum_{n \in \omega} B_n \right) \to \sum_{n \in \omega} (B_n / I_\beta(B_n))$$

such that for any $x \in \sum_{n \in \omega} B_n$ and $m \in \omega$, $(f([x]_I_\beta(\sum_{n \in \omega} B_n)))^m = [x_m]_I_\beta(B_m)$.

Proof. Induction on $\beta$. It is clear for $\beta = 0$. Suppose that $x \in I_{\beta+1}(\sum_{n \in \omega} B_n)$. Then we can write

$$[x]_I_\beta(\sum_{n \in \omega} B_n) = \sum_{i < m} [y_i]_I_\beta(\sum_{n \in \omega} B_n),$$

where each $[y_i]_I_\beta(\sum_{n \in \omega} B_n)$ is an atom of $(\sum_{n \in \omega} B_n) / I_\beta(\sum_{n \in \omega} B_n)$. Hence

$$x \triangle \sum_{i < m} y_i \in I_\beta \left( \sum_{n \in \omega} B_n \right) = \sum_{n \in \omega} I_\beta(B_n).$$

Now for each $n \in \omega$, $(x \triangle \sum_{i < m} y_i)_n \in I_\beta(B_n)$ and for each $i < m$, $[y_{in}]_I_\beta(B_n)$ is 0 or an atom. Thus $[x_n]_I_\beta(B_n) = \sum_{i < m} [y_{in}]_I_\beta(B_n)$. So $x_n \in I_{\beta+1}(B_n)$. This proves $\subseteq$ in (i) for $\beta + 1$.

Now suppose that $x \in \sum_{n \in \omega} I_{\beta+1}(B_n)$. Let $F = \{ n \in \omega : x_n \neq 0 \}$. For each $n \in F$ we can write

$$[x_n]_I_\beta(B_n) = \sum_{j < m_n} ([y_{jn}]_I_\beta(B_n)).$$

43
where each \([y_{jn}]_\beta(B_n)\) is an atom in \(B_n/I_\beta(B_n)\). For each \(n \in F\) and \(j < m_n\) define 
\[y'_{jn} \in \sum_{m \in \omega} B_m\] by 
\[
(y'_{jn})_m = \begin{cases} y_{jn} & \text{if } m = n, \\ 0 & \text{otherwise}. \end{cases}
\]

Then \(\langle[y'_{jn}]_\beta(B_k) : k \in \omega\rangle\) is an atom in \(\sum_{k \in \omega} (B_k/I_\beta(B_k))\). Hence by the inductive hypothesis for (ii), \([y'_{jn}]_\beta(\sum_{n \in \omega} B_n)\) is an atom in \((\sum_{n \in \omega} B_n)/I_\beta(\sum_{n \in \omega} B_n)\). Now for any \(p \in F\),

\[
\left[ \left( \sum_{q \in F} \sum_{j < m_q} y'_{jq} \right) \right]_p I_\beta(B_p) = \sum_{j < m_p} [y_{jp}]_\beta(B_p) = [x_p]_\beta(B_p)
\]

For \(p \in \omega \setminus F\) we have \((\sum_{q \in F} \sum_{j < m_q} y'_{jq})_p = 0 = x_p\). Hence for all \(p \in \omega\),

\[
\left[ \left( \sum_{q \in F} \sum_{j < m_q} y'_{jq} \right) \right]_p I_\beta(B_p) = [x_p]_\beta(B_p)
\]

Hence by (ii) for \(\beta\), for all \(p \in \omega\),

\[
\left[ x \right]_\beta(\sum_{n \in \omega} B_n) = \left( \left( \sum_{q \in F} \sum_{j < m_q} y'_{jq} \right) \right)_p = \left( \sum_{q \in F} \sum_{j < m_q} [y'_{jq}]_\beta(\sum_{n \in \omega} B_n) \right)_p.
\]

Hence

\[
[x]_\beta(\sum_{n \in \omega} B_n) = \sum_{q \in F} \sum_{j < m_q} [y'_{jq}]_\beta(\sum_{n \in \omega} B_n).
\]

This shows that \(x \in I_\beta(\sum_{n \in \omega} B_n)\). Hence (i) for \(\beta + 1\) holds.

Now for (ii), if \(x, x' \in \sum_{n \in \omega} B_n\), then

\[
x_{I_{\beta+1}}(\sum_{n \in \omega} B_n) = x'_{I_{\beta+1}}(\sum_{n \in \omega} B_n) \iff (x \triangle x') \in I_{\beta+1}(\sum_{n \in \omega} B_n)
\]

\[
\iff (x \triangle x') \in \sum_{n \in \omega} I_{\beta+1}(B_n)
\]

\[
\iff \forall n \in \omega \left[ (x \triangle x')_n \in I_{\beta+1}(B_n) \right]
\]

\[
\iff \forall n \in \omega \left[ [x_n]_{I_{\beta+1}(B_n)} = [x'_n]_{I_{\beta+1}(B_n)} \right].
\]

44
Hence (ii) for \( \beta + 1 \) follows.

The limit case is clear. \( \square \)

**Proposition 5.14.** If \( A \) is a special ABID and \( \langle B_n : n \in \omega \rangle \) is as in Propositions 5.11 or 5.12, then \( \prod_{n \in \omega} B_n \subseteq \ast I_{\sigma(A)}^B \).

**Proof.** Let \( \beta = \sigma'(A) \). Then \( \beta = \sigma'(\sum_{n \in \omega} B_n) \), and \( \sum_{n \in \omega} B_n = I_\beta(\sum_{n \in \omega} B_n) = \sum_{n \in \omega} I_\beta(B_n) \) using Proposition 5.13. Hence \( \forall n \in \omega[B_n = I_\beta(B_n)] \). So if \( f \in \prod_{n \in \omega} B_n \) then \( \forall n \in \omega[f(n) \in I_\beta(B_n)] \), so that \( f \in \ast I_\beta^B \). \( \square \)

**Proposition 5.15.** Let \( \beta \) an ordinal and \( B \overset{\text{def}}{=} \langle B_n : n \in \omega \rangle \) of BAs. Let \( h \) be as in Proposition 5.10, and suppose that \( x \in \prod_{n \in \omega} B_n \). Then \( h(x) \subseteq I_\beta(\sum_{n \in \omega} B_n) \) if \( x \in \ast I_\beta^B \).

**Proof.** First suppose that \( h(x) \subseteq I_\beta(\sum_{n \in \omega} B_n) \). Thus \( \forall y \in \sum_{n \in \omega} B_n \forall x \leq y \in I_\beta(\sum_{n \in \omega} B_n) \), so by Proposition 5.13, \( \forall y \in \sum_{n \in \omega} B_n \forall x \leq y \in \sum_{n \in \omega} I_\beta(B_n) \). Now take any \( n \in \omega \). Define \( y \in \prod_{n \in \omega} B_n \) by

\[
y_m = \begin{cases} x_m & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}
\]

Then \( y \in \sum_{n \in \omega} B_n \) and \( y \leq x \), so \( y \in \sum_{n \in \omega} I_\beta(B_n) \]. Hence \( x_n = y_n \in I_\beta(B_n) \). Since \( n \) is arbitrary, \( x \in \ast I_\beta^B \).

Second, suppose that \( x \in \ast I_\beta^B \). Take any \( y \in h(x) \). Then \( y \in \sum_{n \in \omega} B_n \) and \( y \leq x \). Now \( \forall n \in \omega[x_n \in I_\beta(B_n)] \), so \( \forall n \in \omega[y_n \in I_\beta(B_n)] \). Then by Proposition 5.13, \( y \in I_\beta(\sum_{n \in \omega} B_n) \).

Let \( A \) be a special ABID. Let \( \langle B_n : n \in \omega \rangle \) be as in Propositions 5.11 or 5.12. Then \( A^* \overset{\text{def}}{=} \prod_{n \in \omega} B_n/\sum_{n \in \omega} B_n \). Now for each \( \beta \leq \sigma'(A) \) we define \( \Psi_\beta(A) = \{[a] \sum_{n \in \omega} B_n : a \in \ast I_\beta^B \} \).

Now if \( d \in A^* \), say \( d = [a] \sum_{n \in \omega} B_n \), with \( a \in \prod_{n \in \omega} B_n \). By Proposition 5.14, \( a \in \ast I_\beta^B \). Hence \( d \in \Psi_\beta'(A) \). For each \( d \in A^* \) let \( \rho(d) = \min\{\beta : d \in \Psi_\beta(A)\} \).

**Proposition 5.16.** Let \( A \) be a ABID, \( J \) an ideal in \( A \), and \( \beta \) an ordinal. Then

(i) \( J_\beta \overset{\text{def}}{=} \{[a] \mid a \in J \} \) is an ideal in \( A/I_\beta(A) \).

(ii) There is an isomorphism \( f \) of \( J/I_\beta(J) \) onto \( J_\beta(A) \) such that \( f([a]I_\beta(J)) = [a]I_\beta(A) \) for all \( a \in J \).

(iii) \( I_\beta(J) = J \cap I_\beta(A) \).

**Proof.** (i) \( J_\beta \) is clearly closed under \(+\). Suppose that \( a \in J \), \( b \in A \), and \( [b]I_\beta(A) \leq [a]I_\beta(A) \). Then \( b \cdot a \in J \) and \( b \Delta (b \cdot a) = b \setminus a \in I_\beta(A) \). So \( [b]I_\beta(A) = [a \cdot b]I_\beta(A) \in J_\beta \). So (i) holds.

Now we prove (ii) and (iii) by induction on \( \beta \). They are clear for \( \beta = 0 \). For (iii) for \( \beta + 1 \), first suppose that \( a \in I_{\beta + 1}(J) \). Then we can write \( [a]I_\beta(J) = \sum_{i < m}[y_i]I_\beta(J) \), where each \([y_i]I_\beta(J)\) is an atom of \( J/I_\beta(J) \). By (ii) for \( \beta \), \([a]I_\beta(A) = \sum_{i < m}[y_i]I_\beta(A) \). Also, (ii) implies that each \([y_i]I_\beta(A)\) is an atom of \( A/I_\beta(A) \). Hence \( a \in I_{\beta + 1}(A) \). The converse is similar.

45
For $\beta + 1$ for (ii), if $a, a' \in J$ then

$$[a]_{I_{\beta+1}(J)} = [a']_{I_{\beta+1}(J)} \quad \text{iff} \quad (a \triangle a') \in I_{\beta+1}(J)$$

iff $(a \triangle a') \in I_{\beta+1}(A)$ by (iii) for $\beta + 1$,

iff $[a]_{I_{\beta+1}(A)} = [a']_{I_{\beta+1}(A)}$.

Now (iii) follows.

The limit step is clear. \hfill \Box

**Proposition 5.17.** Let $A$ be a special $\text{ABID}$, let $B$ be as in Proposition 5.11 or 5.12, and let $x \in \prod_{n \in \omega} B_n$. Then $\rho \left( [x] \sum_{n \in \omega} B_n \right) = \sigma'(h(x))$.

**Proof.** Let $A' = \sum_{n \in \omega} B_n$. By Proposition 4.26 there is an $a \in h(x)$ such that $h(x) \cong (A' \uparrow a) \times \{ b \in A' : b \cdot a = 0 \}$, with $\sigma'(h(x)) = \sigma'(\{ b \in A' : b \cdot a = 0 \})$. Hence $I_{\sigma'(h(x))}(\{ b \in A' : b \cdot a = 0 \}) = \{ b \in A' : b \cdot a = 0 \}$. By Proposition 4.30, $\{ b \in A' : b \cdot a = 0 \} \subseteq I_{\sigma'(h(x))}(A')$. Now $a \in h(x) \subseteq \sum_{n \in \omega} B_n \subseteq \prod_{n \in \omega} B_n$. So we can form $-a \in \prod_{n \in \omega} B_n$.

(1) $\{ b \in A' : b \cdot a = 0 \} = h(-a)$.

In fact, $(b \in A' \text{ and } b \cdot a = 0)$ iff $(b \in A' \text{ and } b \leq -a)$ iff $b \in h(-a)$. So (1) holds.

Hence $h(x) \cong (A' \uparrow a) \times h(-a)$, so $(\prod_{n \in \omega} B_n \uparrow x) \cong (\prod_{n \in \omega} B_n \uparrow h^{-1}(a)) \times (\prod_{n \in \omega} B_n \uparrow (-a))$. Thus $h(-a) \subseteq I_{\sigma'(h(x))}(A')$. Hence by Proposition 5.15, $-a \in \ast I^{B}_{\sigma'(h(x))}$. So $[-a]_{A'} \in \Psi_{\sigma'(h(x))}(A)$. Now $a \in \sum_{n \in \omega} B_n$, so $[-a]_{A'} = [x]_{A'}$. It follows that $\rho([x]_{A'}) \leq \sigma'(h(x))$.

Suppose that $\gamma \overset{\text{def}}{=} \rho([x]_{A'}) < \sigma'(h(x))$. Then $[x]_{A'} \in \Psi_{\gamma}(A)$. Hence there is a $b \in \ast I^{B}_{\gamma}$ such that $[x]_{A'} = [b]_{A'}$. Now $h(b) \subseteq I_{\gamma}(\sum_{n \in \omega} B_n)$, so $\sigma'(h(b)) \leq \gamma$. Choose $c \in h(b)$ so that $\{ x \in h(b) : x \cdot c = 0 \}$ is special. Thus $\sigma'(\{ x \in h(b) : x \cdot c = 0 \}) = \sigma'(h(b))$. Hence $h(b \cdot -c) = \{ x \in h(b) : x \cdot c = 0 \} \subseteq I_{\sigma'(h(b))}(A')$. Hence by Proposition 5.15, $b \cdot -c \in \ast I^{B}_{\sigma'(h(b))}$. Then $x \triangle (b \cdot -c) = x \triangle b \triangle b \triangle (b \cdot -c) \in A'$. It follows that there is a finite $F \subseteq \omega$ such that $\forall n \in \omega \setminus F[x_n = b_n \cdot -c_n]$. Define $e \in \sum_{n \in \omega} B_n$ by

$$e_n = \begin{cases} 1 & \text{if } n \in F, \\ 0 & \text{if } n \notin F. \end{cases}$$

Also, define $e' \in \sum_{n \in \omega} B_n$ by

$$e'_n = \begin{cases} x_n & \text{if } n \in F, \\ 0 & \text{if } n \notin F. \end{cases}$$

Then $h(x) \cong (h(x) \uparrow e') \times \{ y \in h(x) : y \cdot e' = 0 \}$, so $\{ y \in h(x) : y \cdot e' = 0 \}$ is special with $\sigma'(h(x)) = \sigma'(\{ y \in h(x) : y \cdot e' = 0 \})$. Also, $\{ y \in h(x) : y \cdot e = 0 \} = \{ y \in h(x) : y \cdot e' = 0 \}$.

Now define $e'' \in \sum_{n \in \omega} B_n$ by

$$e''_n = \begin{cases} b_n \cdot -c_n & \text{if } n \in F, \\ 0 & \text{if } n \notin F. \end{cases}$$

Let $M = \{ y \in h(b \cdot -c) : y \cdot e'' = 0 \}$. Then $M$ is special with $\sigma'(M) = \sigma'(h(b \cdot -c))$. 46
Proposition 5.18. Let $A$ be a special ABID, and let $\langle B_n : n \in \omega \rangle$ be as above. Then $\rho$ maps $A^*$ into $\sigma'(A) + 1$, $\rho$ is additive, and $\rho(d) = 0$ implies that $d = 0$.

Proof. By Proposition 4.30, for any $x \in \prod_{n \in \omega} B_n$, $I_{\sigma'(A)}(h(x)) = h(x) \cap I_{\sigma'(A)}(A) = h(x) \cap I_{\sigma'(A)+1}(A) = I_{\sigma'(A)+1}(h(x))$ and so by Proposition 5.17, $\rho([x] \sum_{n \in \omega} B_n = \sigma'(h(x)) \leq \sigma'(A))$. Hence $\rho$ maps $A^*$ into $\sigma'(A) + 1$. Clearly $\rho(0) = 0$ and $\rho(d) = 0$ implies that $d = 0$. Now suppose given $d, e \in A^*$; say $d = [d'] \sum_{n \in \omega} B_n$ and $e = [e'] \sum_{n \in \omega} B_n$. Then

$$\rho(d + e) = \rho([d'] \sum_{n \in \omega} B_n + [e'] \sum_{n \in \omega} B_n = \rho([d' + e'] \sum_{n \in \omega} B_n) = \sigma'(h(d' + e')) = \sigma'(h(d')) + \sigma'(h(e')) \text{ by Proposition 5.10}$$

$$= \sigma'(h(d')) + \sigma'(h(e')) = \rho([d'] \sum_{n \in \omega} B_n) \rho([e'] \sum_{n \in \omega} B_n) = \rho(d) + \rho(e).$$

Proposition 5.19. For any special ABID $A$, any $d \in A^*$, and any $\beta \leq \rho(d)$ there is a $d_0 \in A$ such that $d_0 \leq d$, $\rho(d_0) = \beta$, and $\rho(d \setminus d_0) = \rho(d)$.

Proof. Suppose that $A$ is a ABID, $d \in A^*$, and $\beta \leq \rho(d)$. Say $\rho(d) = \gamma$. If $\beta = 0$, take $d_0 = 0$. Assume that $\beta \neq 0$. Let $\delta_0 \leq \delta_1 \leq \cdots$ be such that each $\delta_i$ is a successor ordinal and $\sup_{n \in \omega} \delta_n = \beta$. Choose $k \in \prod_{n \in \omega} B_n$ so that $d = [k] \sum_{n \in \omega} B_n$.

5.19(1) $\forall \varepsilon < \gamma \{n \in \omega : k(n) \notin I_{\varepsilon}(B_n)\}$ is infinite.

Otherwise, let $F = \{n \in \omega : k(n) \notin I_{\varepsilon}(B_n)\}$. Let $g \upharpoonright (\omega \setminus F) = k \upharpoonright (\omega \setminus F)$ with $\forall n \in \omega \upharpoonright g(n) \in I_{\varepsilon}(B_n)$. Then $d = [k] \sum_{n \in \omega} B_n = [g] \sum_{n \in \omega} B_n$, contradicting $\rho(d) = \gamma$.

Case 1. $\forall n \in \omega \delta_n < \beta$. Let $i \in \omega$. By 5.19(1), choose $n_i \in \omega$ such that $k(n_i) \notin I_{\delta_i}(B_{n_i})$. Say $\tau_n(B_{n_i} \upharpoonright k(n_i)) = (\xi, 0, m)$. Then $B_{n_i} \upharpoonright k(n_i) \simeq \text{Intalg}(\omega^\xi \cdot m)$ and so $\delta_i < \xi$. Say $g$ is an isomorphism from $B_{n_i} \upharpoonright k(n_i)$ onto $\text{Intalg}(\omega^\xi \cdot m)$. Let $a = [0, \omega^\delta_i]$, an element of $\text{Intalg}(\omega^\xi \cdot m)$. Then $\tau_n(\text{Intalg}(\omega^\xi \cdot m) \upharpoonright a) = (\delta_i, 0, 1)$. Let $d_i \overset{\text{def}}{=} g^{-1}(a) \in B_{n_i} \upharpoonright k(n_i)$. Then $\tau_n(B_{n_i} \upharpoonright d_i) = (\delta_i, 0, 1)$. Now we define

$$h'(n) = \begin{cases} 
2i & \text{if } n = n_{2i}, \\
0 & \text{if } n \notin \{n_0, n_2, \ldots\}.
\end{cases}$$

Let $d' = [h'] \sum_{n \in \omega} B_n$. Then by Proposition 5.17, $\rho(d') = \sigma'(h(h'))$. Now $h' \in ^*I_{\beta}(A)$, so by Proposition 5.15, $h(h') \subseteq I_{\beta}(A)$. Hence by Proposition 4.30, $h(h') \subseteq I_{\beta}(h(h'))$, so
Let $\sigma'(h'(h')) \leq \beta$. If $\varepsilon < \beta$, then clearly $h' \notin \star I_\varepsilon(A)$. So $\sigma'(h(h')) \not\leq \varepsilon$. Thus $\sigma'(h(h')) = \beta$. So $\rho(d') = \beta$. If $\beta < \gamma$, then $\rho(d \setminus d') = \gamma$ by Proposition 5.18. Suppose that $\beta = \gamma$. Then define

$$h''(n) = \begin{cases} d_{2i+1} & \text{if } n = n_{2i+1}, \\ 0 & \text{if } n \notin \{n_1, n_3, \ldots\}. \end{cases}$$

Let $d'' = [h'']\sum_{n \in \omega} B_n$. Then as above, $\rho(d'') = \beta$. Clearly $d'' \leq d \setminus d'$, so $\rho(d \setminus d') = \beta$.

Case 2. $\beta = \delta + 1$ for some $\delta$. By 5.9(1), choose distinct $n_i \in \omega$ for $i < \omega$ such that $k(n_i) \notin I_\delta(B_{n_i})$. Say $\tau_s(B_{n_i} \upharpoonright k(n_i)) = (\xi, 0, m)$. Then $B_{n(i)} \upharpoonright k(n_i) \cong \text{Intalg}(\omega^\xi \cdot m)$ and so $\delta < \xi$. Say $g$ is an isomorphism from $B_{n_i} \upharpoonright k(n_i)$ onto $\text{Intalg}(\omega^\xi \cdot m)$. Let $a = [0, \omega^\delta)$, an element of $\text{Intalg}(\omega^\xi \cdot m)$. Then $\tau_s(\text{Intalg}(\omega^\xi \cdot m) \upharpoonright a) = (\delta, 0, 1)$. Hence $d_i \overset{\text{def}}{=} g^{-1}(\text{Intalg}(\omega^\xi \cdot m) \upharpoonright a) \leq k(n_i)$ and $\tau_s(d_i) = (\delta, 0, 1)$. Now we define

$$h'(n) = \begin{cases} d_{2i} & \text{if } n = n_{2i}, \\ 0 & \text{if } n \notin \{n_0, n_2, \ldots\}. \end{cases}$$

Let $d' = [h']\sum_{n \in \omega} B_n$. By Proposition 5.17, $\rho(d') = \sigma'(h(h'))$. Now $h' \in \star I_\beta(A)$, so by Proposition 5.15, $h(h') \subseteq I_\beta(A)$. Hence by Proposition 4.30, $I_\beta(h(h')) = h(h') \cap I_\beta(A) = h(h')$. Hence $h(h') \subseteq I_\beta(K)$, so $\sigma'(h(h')) \leq \beta$.

5.19(2) $\sigma'(h(h')) = \beta$.

In fact, suppose that $\sigma'(h(h')) = \varepsilon < \beta$. Since $\rho(d') = \sigma'(h(h'))$, it follows that there is an $e \in \star I_\beta^2$ such that $d' = [e]\sum_{n \in \omega} B_n$. Thus $[h']\sum_{n \in \omega} B_n = [e]\sum_{n \in \omega} B_n$, so $h' \triangle e \in \sum_{n \in \omega} B_n$.

Let $F$ be finite such that $\forall n \in \omega \setminus F[h'(n) = e(n)]$. Take $i$ with $n_{2i} \notin F$. Then $h'(n_{2i}) = d_{2i} = e(n_{2i}) \in I_\varepsilon(B_n)$, contradiction.

So $\rho(d') = \beta$. If $\beta < \gamma$, then $\rho(d \setminus d') = \gamma$ by Proposition 5.18. Suppose that $\beta = \gamma$. Then define

$$h''(n) = \begin{cases} d_{2i+1} & \text{if } n = n_{2i+1}, \\ 0 & \text{if } n \notin \{n_1, n_3, \ldots\}. \end{cases}$$

Let $d'' = [h'']\sum_{n \in \omega} B_n$. Then as above, $\rho(d'') = \beta$. Clearly $d'' \leq d \setminus d'$, so $\rho(d \setminus d') = \beta$. \hfill \square

**Proposition 5.20.** Let $B$ be a special ABID, $\alpha = \sigma'(B)$, $A$ a countable BA, and $r : A \rightarrow \alpha + 1$ an additive function, with $r(1) = \alpha$. Then there is a homomorphism $f : A \rightarrow B^*$ such that $\forall d \in A[r(d) = \rho(f(d))]$.

**Proof.** Assume the hypotheses. Let $R = \{(a, b) \in A \times B^* : r(a) = \rho(b) \text{ and } r(-a) = \rho(-b)\}$. We verify the conditions (V5) and (V6) in the definition of a weak $V$-relation. Condition (V5) is clear. Now suppose that $(a, b) \in R$ and $c \in A$. Let $r(a) = \beta$, $r(-a) = \gamma$, $r(a \cdot c) = \delta$, $r(a \setminus c) = \eta$. By the additivity of $r$, $\max(\beta, \gamma) = \alpha$ and $\max(\delta, \eta) = \beta$.

Case 1. $\delta \leq \eta$. In Proposition 5.19 replace $\beta$ and $d$ by $\delta$ and $b$. So $\delta \leq \beta = r(a) = \rho(b)$ gives a $d_0 \leq b$ such that $\rho(d_0) = \delta = r(a \cdot c)$ and $\rho(b \setminus d_0) = \rho(b) = r(a) = \beta = \eta = r(a \setminus c)$. Also, $\rho(b \cdot d_0) = \rho(d_0) = r(a \cdot c)$.
Case 2. \( \eta < \delta \). So \( \eta \leq \beta = r(a) = \rho(b) \) gives a \( d_1 \leq b \) such that \( \rho(d_1) = \eta = r(a \setminus c) \) and \( \rho(b \setminus d_1) = \rho(b) = r(a) = \beta = \delta = r(a \cdot c) \). Also, \( \rho(b \cdot d_1) = \rho(d_1) = r(a \setminus c) \). Setting \( d_0 = -d_1 \) we get \( \rho(b \cdot d_0) = r(a \cdot c) \) and \( \rho(b \setminus d_0) = r(a \setminus c) \).

Now
\[
r(-a + c) = r(-a + a \setminus c) = \max(r(-a), r(a \setminus c)) = \max(\rho(-b), \rho(b \setminus d_0)) = \rho(-b + b \setminus d_0);
\]
\[
r(-a + c) = r(-a + a \setminus c) = \max(r(-a), r(a \setminus c)) = \max(\rho(-b), \rho(b \setminus d_0)) = \rho(-b + b \setminus d_0).
\]

This shows that \( (a \cdot c, d_0), (a \setminus c, b \setminus d_0) \in R \).

Now \(-a = c \cdot a + c \cdot a - a \), so \( r(-a) = \max(r(c \cdot a), r(-c \cdot a)) \).

Case 1. \( r(-a) = r(-c \cdot a) \). By Proposition 5.19 there is a \( d_1 \leq -b \) such that \( \rho(d_1) = r(c \setminus a) \) and \( \rho(-b \cdot d_1) = \rho(-b) \). Thus \( \rho(d_1 \setminus b) = \rho(d_1) = r(c \setminus a) \). Then \( r(a + c) = r(a + (c \setminus a)) = \max(r(a), r(c \setminus a)) = \max(\rho(b), \rho(d_1 \setminus b)) = \rho(d_1 + b) \). Also, \( r(-a - c) = r(-a) = \rho(-b) = \rho(-b \cdot d_1 - d_1) \). Also, \( r(-c + a) = r(-c - a + a) = \max(r(-c - a), r(a)) = \max(\rho(-b \cdot d_1), \rho(b)) = \rho(-b \cdot d_1 - b) = \rho(-d_1 + b) \). Hence \( (a + c, b + d_1), (c \setminus a, d_1 \setminus b) \in R \).

Case 2. \( r(-a) = r(c \cdot a) \). By Proposition 5.19 there is a \( d_1 \leq -b \) such that \( \rho(d_1) = r(-c \cdot a) \) and \( \rho(-b \cdot d_1) = \rho(-b) \). Thus \( \rho(d_1 \setminus b) = \rho(d_1) = r(-c \cdot a) \). Next,
\[
r(a + c) = r(a + (c \setminus a)) = \max(r(a), r(c \setminus a)) = \max(r(a), r(-a)) = r(1) = \alpha = \rho(1) = \max(\rho(b), \rho(-b))
\]
\[
\]
Now we claim that for any $\alpha$

5.21(1) $D$ is a subalgebra of the BA $\text{Fr}(\omega) \times \text{rng}(h)$.

5.21(2) $I'$ is an ideal in $D$.

5.21(3) $w$ is a homomorphism from $D$ into $\text{Fr}(\omega)$

5.21(4) $t$ is an isomorphic embedding of $C$ into $D$.

5.21(5) $l$ is well-defined, and is an isomorphism of $D/I'$ onto $\text{Fr}(\omega)$.

In fact, clearly $l$ is well-defined, and is a homomorphism of $D/I'$ onto $\text{Fr}(\omega)$. To see that it is one-one, suppose that $l([(u, h(x))]_{I'}) = 0$. Thus $u = 0$. Since $(u, h(x)) \in D$, we have $x \in \sum_{n \in \omega} B_n$. Say $x = f(c)$ with $c \in C$. Then $(u, h(x)) = (0, h(f(c))) \in I'$, as desired.

Now let $E, t', l'$ be such that $C \subseteq E$, $t'$ is an isomorphism of $E$ onto $D$, $t' \mid C = t$, and $\forall b \in E[l'((b)_C) = l([(t'(b))_{I'}])]$. So $l'$ is an isomorphism from $E/C$ onto $\text{Fr}(\omega)$. Now for any $d \in \text{Fr}(\omega)$ write $d = l'([c]_C)$. Note by Proposition 5.1 that $I_{\text{att}(E)}(E) \subseteq C$. Hence

5.21(6) $r_{E l'}(d) = r_{E l'}(l'([c]_C)) = r_{E l'}(l'([e]_{I_{\text{att}(E)}}(E))) = \alpha_*(E \uparrow e) = \alpha_*(D \uparrow t'(e)).$

Say $t'(e) = (u, h(x))$. Now

$$f'(d) = f'(l'([c]_C)) = f'(l([t'(e)]_C)) = f'(l([(u, h(x))]_C)) = f'(u) = [x] \sum_{n \in \omega} B_n.$$ 

Hence by Proposition 5.17 we get

5.21(7) $s(d) = \rho(f'(d)) = \alpha_*(h(x)).$

Now we claim that for any $\alpha$,

5.21(8) $I_\alpha(D \uparrow (u, h(x))) = \{0\} \times I_\alpha(h(x)).$

5.21(9) For any $(p, q) \in D \uparrow (u, h(x))$ let $f_{\alpha}([(p, q)]_{I_\alpha(D \uparrow (u, h(x))}) = (p, [q]_{I_\alpha(h(x))})$. Then $f_{\alpha}$ is well-defined, and is an isomorphism of $(D \uparrow (u, h(x))/I_\alpha(D \uparrow (u, h(x))))$ onto $\{(u', [v])_{I_\alpha(h(x))} : u' \leq u, v \leq h(x), f(u') = v\}$.

Note that 5.12(9) with $\alpha = \alpha_*(D \uparrow (u, h(x)))$ gives $\alpha_*(D \uparrow t'(e)) = \alpha_*(h(x))$. Together with 5.21(6) and 5.21(7) this finishes the proof.

5.21(8) and 5.21(9) are clear for $\alpha = 0$. Now assume them for $\alpha$. Suppose that $(s, t) \in I_{\alpha+1}(D \uparrow (u, h(x)))$. Then there are atoms $[(z_i, y_i)]_{I_\alpha(D \uparrow (u, h(x)))}$ of $(D \uparrow (u, h(x))/I_\alpha(D \uparrow (u, h(x))))$ for $i < m$ such that $[(s, t)]_{I_\alpha(B\uparrow (u, h(x)))} = \sum_{i < m} [(z_i, y_i)]_{I_\alpha(D \uparrow (u, h(x)))}$. Then $s, t \Delta \sum_{i < m} (z_i, y_i) \in I_\alpha(D \uparrow (u, h(x)))$. So by 5.21(8) for $\alpha$, $s \Delta \sum_{i < m} z_i = 0$ and $t \Delta \sum_{i < m} y_i \in I_\alpha(v)$. For any $i < m$, by 5.21(9) for $\alpha$, $[(z_i, y_i)]_{I_\alpha(h(x))}$ is an atom of $\{(u', [v'])_{I_\alpha(h(x))} : u' \leq u, v' \leq h(x), f(u') = v'\}$.

5.21(10) $z_i = 0$.

For, suppose that $z_i \neq 0$. Now $f(z_i) = y_i$. Let $w$ be such that $0 < w < z_i$, and set $u = f(w)$. Then $(0, 0) < [(w, u)]_{I_\alpha(h(x))} < (z_i, [y_i]_{I_\alpha(h(x))}$, contradiction. So 5.21(10) holds.
It follows that \([y_i]_{I_\alpha(h(x))}\) is an atom of \(h(x)/I_\alpha(h(x))\). Hence \(s = 0\) and \(t \in I_{\alpha+1}(h(x))\).

Conversely, suppose that \(t \in I_{\alpha+1}(h(x))\). Then there exist atoms \([y_i]_{I_\alpha(h(x))}\) of \(h(x)/I_\alpha(h(x))\) for \(i < m\) such that \([t]_{I_\alpha(h(x))} = \sum_{i<m}[y_i]_{I_\alpha(h(x))}\). Then \(t \triangle \sum_{i<m}y_i \in I_\alpha(h(x))\) hence by \(5.21(8)\) for \(\alpha\), \((0, t \triangle \sum_{i<m}y_i) \in I_\alpha(D \upharpoonright (u, h(x)))\). Hence \([0, t]_{I_\alpha(D\upharpoonright (u, h(x)))} = \sum_{i<m}[(0, y_i)]_{I_\alpha(D\upharpoonright (u, h(x)))}\). Now each \([(0, y_i)]_{I_\alpha(D\upharpoonright (u, h(x)))}\) is an atom of \(D \upharpoonright (u, h(x))\) by \(5.21(9)\) for \(\alpha\). Hence \((0, t) \in I_{\alpha+1}(D \upharpoonright (u, h(x)))\).

This proves \(5.21(8)\) for \(\alpha + 1\).

For \(5.21(9)\) for \(\alpha + 1\), suppose that \((p, q), (p', q') \in D \upharpoonright (u, h(x))\). Then

\[
\begin{align*}
([p, q]_{I_{\alpha+1}(D\upharpoonright (u, h(x)))} &= [(p', q')]_{I_{\alpha+1}(D\upharpoonright (u, h(x)))} \\
\text{iff} \quad ((p, q) \triangle (p', q')) \in I_{\alpha+1}(D \upharpoonright (u, h(x))) \\
\text{iff} \quad p = p' \text{ and } q \triangle q' \in I_{\alpha+1}(h(x)) \quad \text{by \(5.21(8)\)} \\
\text{iff} \quad p = p' \text{ and } [q]_{I_\alpha(h(x))} = [q']_{I_\alpha(h(x))}.
\end{align*}
\]

\(5.21(9)\) for \(\alpha + 1\) now easily follows.

The limit step is clear.

So \(5.21(8)\) and \(5.21(9)\) hold. \(\square\)

For any additive function \(s: Fr(\omega) \rightarrow \omega_1\) the BA of Proposition 5.21 is denoted by \(B_s\); an isomorphism \(g: B_s^* \rightarrow Fr(\omega)\) is denoted by \(g_s\). Thus \(r_{B_s}g_s = s\).

**Proposition 5.22.** If \(A\) is a countable non-superatomic BA, then there exist BAs \(B, C\) such that \(A \cong B \times C\) and:

(i) \(B\) is normalized.

(ii) Either \(|C| = 1\) or \(C\) is superatomic with atomic type of the form \((\alpha, 0, n)\) with \(\alpha \geq \sigma'(B)\).

**Proof.** Let \(\tau(A) = (\alpha, \beta, n)\).

Case 1. \(n = 0\). Then by Proposition 5.10, \(\alpha = \beta\). Hence \(I_{ar(A)}(A)\) is special. Since \(A/I_{ar(A)}(A)\) is atomless, \(A\) is normalized. We can then take \(B = A\) and \(|C| = 1\).

Case 2. \(n > 0\). Now by Proposition 4.22 there is an element \(d \in I_{ar(A)}(A)\) with \(\tau_s(I_{ar(A)}(A) \upharpoonright d) = (\alpha, 0, n)\) and \(\tau_s(d^{\perp}) = (\beta, \beta, 0)\). Now

\[
d^{\perp} = \{a \in I_{ar(A)}(A) : a \cdot d = 0\} = I_{ar(A)}(A) \cap (A \upharpoonright (-d)) \\
= I_{ar(A)}(A \upharpoonright (-d)) = I_{ar(A^{\perp})}(A \upharpoonright (-d)).
\]

Now

\[
\tau(A \upharpoonright d) = \tau_s(I_{ar(A^{\perp})}(A \upharpoonright d)) = \tau_s(I_{ar(A)}(A) \upharpoonright d) = (\alpha, 0, n)
\]

and

\[
\tau(A \upharpoonright (-d)) = \tau_s(I_{ar(A^{\perp})}(A \upharpoonright (-d))) = \tau_s(d^{\perp}) = (\beta, \beta, 0).
\]

\(\square\)

**Proposition 5.23.** Suppose that \(A\) is a countable nonsuperatomic BA, and \(A \cong B \times C \cong B' \times C',\) and
Now if $\alpha \geq \sigma'(B)$, $\beta \geq \sigma'(B')$, $\tau(C) = (\alpha, 0, n)$, and $\tau(C') = (\beta, 0, p)$. Then $B \cong B'$ and $C \cong C'$.

**Proof.** Assume the hypotheses. If $|C| = |C'| = 1$, then the desired conclusion is clear. Assume now that $\alpha, \beta$ are ordinals and $n, p$ are positive integers such that $\alpha \geq \sigma'(B)$, $\beta \geq \sigma'(B')$, $\tau(C) = (\alpha, 0, n)$, and $\tau(C') = (\beta, 0, p)$. Say $\beta \leq \alpha$. Let $f : A \to B \times C$ be an isomorphism. Then by Proposition 4.11, $f[I_{\alpha+1}(A)] = I_{\alpha+1}(B \times C) = I_{\alpha+1}(B) \times I_{\alpha+1}(C)$. Similarly, let $g : A \to B' \times C'$ be an isomorphism. Then by Proposition 4.11, $g[I_{\alpha+1}(A)] = I_{\alpha+1}(B' \times C') = I_{\alpha+1}(B') \times I_{\alpha+1}(C')$. Now $\tau_s(I_{\alpha+1}(B)) = (\sigma'(B), \sigma'(B), 0)$. By the above, $\tau_s(C) = (\alpha, 0, n)$. Also, $\tau_s(I_{\alpha+1}(B')) = (\sigma'(B'), \sigma'(B'), 0)$. By the above, $\tau_s(C') = (\beta, 0, p)$. Then by Proposition 4.14, $\tau_s(I_{\alpha+1}(B) \times I_{\alpha+1}(C)) = (\alpha, \sigma'(B), n)$. Similarly, $\tau_s(I_{\alpha+1}(B') \times I_{\alpha+1}(C')) = (\beta, \sigma'(B'), n)$. Now $I_{\alpha+1}(B) \times I_{\alpha+1}(C) \cong I_{\alpha+1}(B') \times I_{\alpha+1}(C')$, so $\alpha = \beta$, $\sigma'(B) = \sigma'(B')$, and $n = p$. Then $\tau_s(I_{\alpha+1}(B)) = \tau_s(I_{\alpha+1}(B'))$, so $I_{\alpha+1}(B) \cong I_{\alpha+1}(B')$ by Proposition 4.28. Similarly, $I_{\alpha+1}(C) \cong I_{\alpha+1}(C')$. Hence $C \cong C'$. Now $A/I_{\alpha+1}(A) \cong B/I_{\alpha+1}(A) \times C/I_{\alpha+1}(A) \cong B/I_{\alpha+1}(B)$, since $|C/I_{\alpha+1}(C)| = 1$. Similarly, $A/I_{\alpha+1}(A) \cong B'/I_{\alpha+1}(A)$. Let $h$ be an isomorphism from $B/I_{\alpha+1}(B)$ onto $\text{Fr}(\omega)$. Now since $I_{\alpha+1}(B) \cong I_{\alpha+1}(B')$, let $B''$ and $j$ be such that $j$ is an isomorphism from $B'$ onto $B''$ and $j[I_{\alpha+1}(B')] = I_{\alpha+1}(B)$. Let $k$ be an isomorphism from $B''/I_{\alpha+1}(B)$ onto $\text{Fr}(\omega)$. For each $e \in A$ let $t([e]_{I_{\alpha+1}(A)}) = 1^{st}([f(e)]_{I_{\alpha+1}(B)})$. Then $t$ is an isomorphism from $A/I_{\alpha+1}(A)$ onto $B/I_{\alpha+1}(B)$.

Let $l$ be an isomorphism of $A/I_{\alpha+1}(A)$ onto $\text{Fr}(\omega)$. Now if $x \in \text{Fr}(\omega)$ is given, choose $e \in A$ so that $l([e]_{I_{\alpha+1}(A)}) = x$. Then $l \circ t^{-1}$ is an isomorphism of $B/I_{\alpha+1}(B)$ onto $\text{Fr}(\omega)$.

Hence
\[
l(t^{-1}(t([e]_{I_{\alpha+1}(A)}))) = l([e]_{I_{\alpha+1}(A)}) = x.
\]

Hence
\[
\begin{align*}
r_{B, l \circ t^{-1}}(x) &= r_{B, l \circ t^{-1}}(l(t^{-1}(t(e \triangle I_{\alpha+1}(A)))) \\
&= r_{B, l \circ t^{-1}}(l(t^{-1}(1^{st}(f(e)) \triangle I_{\alpha+1}(B)) = \sigma'(1^{st}(f(e)) = \sigma'(e).
\end{align*}
\]

Let $g'$ be an isomorphism of $A$ onto $B'' \times C$. For each $e \in A$ let $s([e]_{I_{\alpha+1}(A)}) = 1^{st}([g'(e)]_{I_{\alpha+1}(B)})$. Then $s$ is an isomorphism from $A/I_{\alpha+1}(A)$ onto $B''/I_{\alpha+1}(B)$. As above, $r_{B''}(x) = \sigma'(e)$. Hence $B \cong B'' \cong B'$.

Now if $A$ is a BA which is nonsuperatomic and is not normalized, then $\text{INV}(A) = \{2, \beta, \alpha, n, \{r_B \circ f : f \in \text{aut}(\text{Fr}(\omega)) \}$, where there is a superatomic BA $C$ such that $A \cong B \times C$, with $B$ normalized, $\tau(B) = (\beta, \beta, 0)$, $\tau(C) = (\alpha, 0, n)$, and $\alpha \geq \beta$. Note by Proposition 5.23 that the entries in $\text{INV}(A)$ do not depend on the particular $B, C$ chosen.

**Theorem C.** For any BAs $A, A'$ the following are equivalent:

(i) $A \cong A'$.

(ii) $\text{INV}(A) = \text{INV}(A')$.

**Proof.** For $A, A'$ superatomic, or $A, A'$ normal, see Theorems A and B. Now suppose that $A, A'$ are not superatomic, and not normal.
(i)⇒(ii): Suppose that \( A \cong A' \). By Proposition 5.22 there are BAs \( B, C \) such that \( A \cong B \times C \), \( B \) is normalized, and \( C \) is superatomic with some type \((\alpha, 0, n)\), with \( \alpha \geq \sigma'(B) \). Then clearly \( \text{INV}(A) = \text{INV}(A') \).

(ii)⇒(i). Suppose that \( \text{INV}(A) = (2, \beta, \alpha, n, \{r_B \circ f : f \in \text{aut}(\text{Fr}(\omega))\}) \), where there is a superatomic BA \( C \) such that \( A \cong B \times C \), with \( B \) normalized, \( \tau(C) = (\alpha, 0, n) \), and \( \alpha \geq \beta \) and \( \text{INV}(A') = (2, \beta', \alpha', n', \{r_B' \circ f : f \in \text{aut}(\text{Fr}(\omega))\}) \), where there is a superatomic BA \( C' \) such that \( A' \cong B' \times C' \), with \( B' \) normalized, \( \tau(B') = (\beta, \beta, 0) \), \( \tau(C') = (\alpha', 0, n') \), and \( \alpha' \geq \beta' \). Then \( \alpha = \alpha' \), \( \beta = \beta' \), \( n = n' \), and there is an \( f \in \text{Aut}(\text{Fr}(\omega)) \) such that \( r_B = r_B' \circ f \). Now \( \tau_s(I_{\beta}(B)) = \tau_s(I_{\beta}(B')) \), so by Proposition 4.23, \( I_{\beta}(B) \cong I_{\beta}(B') \); say that \( h \) is an isomorphism from \( I_{\beta}(B) \) onto \( I_{\beta}(B') \). Then there exists an isomorphism \( h' \) of \( B \) onto a BA \( B'' \) such that \( I_{\beta}(B') \subseteq B'' \) and \( h' \upharpoonright I_{\beta}(B) = h \):

\[
\begin{array}{c}
B'' \subseteq I_{\beta}(B') \\
B \ni h' \ni h \ni I_{\beta}(B)
\end{array}
\]

Now since \( B \) is normalized, there is an isomorphism \( l \) from \( B/I_{\beta}(B) \) onto \( \text{Fr}(\omega) \). Similarly, there is an isomorphism \( k \) from \( B'/I_{\beta}(B') \) onto \( \text{Fr}(\omega) \). For any \( a \in B'' \) define \( g([a]_{I_{\beta}(B')}) = f(l([h'^{-1}(a)]I_{\beta}(B))) \). Then \( g \) is well-defined, since if \( a \in I_{\beta}(B') \) then \( h'^{-1}(a) = h^{-1}(a) \in I_{\beta}(B) \). Clearly then \( g \) is an isomorphism from \( B''/I_{\beta}(B') \) onto \( \text{Fr}(\omega) \). Hence if \( b \in B'' \) then

\[
\begin{align*}
\tau_{B''}g(g([b]_{I_{\beta}(B')})) &= \sigma'(b) = \sigma'(h'^{-1}(b)) \\
&= r_{B,I_{\beta}(h'^{-1})}(l([h'^{-1}(b)]I_{\beta}(B))) = r_{B',I_{\beta}(h'^{-1})}(l([h'^{-1}(b)]I_{\beta}(B))) \\
&= r_{B''}g([b]_{I_{\beta}(B')})
\end{align*}
\]

Thus \( r_{B''}g = r_{B''}g \), and so by Proposition 5.8, \( B'' \cong B' \). Hence \( B \cong B' \).

\( C \cong C' \) by Proposition 4.28. It follows that \( A \cong A' \). \( \square \)

Also, we define

\[
\text{INV} = \{(0, \alpha, n) : \alpha \text{ an ordinal, } n \in \omega \setminus \{0\}\} \\
\cup \{(1, \alpha, \{r \circ f : f \in \text{aut}(\text{Fr}(\omega))\}) : \alpha \text{ a nonzero ordinal, } \}
\]

\[
r : \text{Fr}(\omega) \to \alpha + 1 \text{ is additive, } r(1) = \alpha
\]

\[
\cup \{(2, \beta, \alpha, n, \{r \circ f : f \in \text{aut}(\text{Fr}(\omega))\}) : \alpha, \beta \text{ are ordinals, } n \in \omega \setminus \{0\}, \}
\]

\[
\alpha \geq \beta \neq 0, r : \text{Fr}(\omega) \to \beta + 1 \text{ is additive, } r(1) = \beta
\]

**Theorem 5.24.** If \( \Gamma \in \text{INV} \), then there is a BA \( A \) such that \( \text{INV}(A) = \Gamma \).

**Proof.** Suppose that \( \Gamma \in \text{INV} \). For \( 1^{st}(\Gamma) = 0 \) or \( 1 \), apply Propositions 4.19 and 5.21. Now suppose that \( 1^{st}(\Gamma) = 2 \). Say \( \Gamma = (2, \beta, \alpha, n, \{r \circ f : f \in \text{aut}(\text{Fr}(\omega))\}) \), where
\[ \alpha, \beta \text{ are ordinals, } n \in \omega \setminus \{0\}, \alpha \geq \beta \neq 0, r : \text{Fr}(\omega) \to \beta + 1 \text{ is additive, and } r(1) = \beta. \]

Let \( A_0 = \text{Intalg}_\omega(\omega^\beta) \). Thus \( A_0 \) is a special ABID such that \( \tau(A_0) = (\beta, \beta, 0) \). By Proposition 5.21 let \( B \) be a BA such that \( \tau(B) = (\beta, \beta, 0) \) and let \( g \) be an isomorphism of \( B^* \) onto \( \text{Fr}(\omega) \) such that \( r_B g = r \). Let \( A = B \times C \), where \( C = \text{Intalg}(\omega^\alpha \cdot n) \). Then by Proposition 4.11, \( (B \times C)/I_\beta(B \times C) \cong (B/I_\beta(B)) \times (C/I_\beta(C)) \). Thus \( B \) is normalized and \( \tau(C) = (\alpha, 0, n) \). So \( \text{Inv}(A) = \{2, \beta, \alpha, n, \{r \circ f : f \in \text{aut}(\text{Fr}(\omega))\}\} \). \( \square \)

**Proposition 5.25.** If \( A \) is a superatomic BA with \( \tau(A) = (\alpha, 0, n) \) and \( B \) is a normalized BA with \( \tau(B) = (\beta, \beta, 0) \), and \( \alpha < \beta \), then \( A \times B \cong B \).

**Proof.** We have \( (A \times B)/I_\beta(A \times B) \cong (A/I_\beta(A)) \times (B/I_\beta(B)) \cong (B/I_\beta(B)) \cong \text{Fr}(\omega) \).

Also, \( I_\beta(A \times B) = I_\beta(A) \times I_\beta(B) \). By Proposition 4.14, \( \tau_s(I_\beta(A \times B)) = (\beta, \beta, 0) \). Thus \( A \times B \) is normalized. By Proposition 4.27, \( I_\beta(A \times B) \cong I_\beta(B) \). Let \( f \) be an isomorphism of \( (A \times B)/I_\beta(A \times B) \) onto \( \text{Fr}(\omega) \). Let \( g \) be an isomorphism of \( A \times B \) onto a BA \( C \) such that \( g[I_\beta(A \times B)] = I_\beta(B) \). Let \( g'([a, b]_{I_\beta(A \times B)}) = [g([a, b])_{I_\beta(C)}] \) for all \( (a, b) \in A \times B \).

Thus \( g' \) is an isomorphism of \( (A \times B)/I_\beta(A \times B) \) onto \( C/I_\beta(C) \). Let \( h \) be an isomorphism of \( B/I_\beta(B) \) onto \( \text{Fr}(\omega) \). Now \( f \circ g' \) is an isomorphism of \( C/I_\beta(C) \) onto \( \text{Fr}(\omega) \). Let \( x \in \text{Fr}(\omega) \). Choose \( c \in C \) such that \( f(g'([c]_{I_\beta(C)})) = x \). Then

\[
    r_{C, f \circ g' \circ f}^{-1}(x) = r_{C, f \circ g' \circ f}^{-1}((f(g'([c]_{I_\beta(C)})))) = \sigma'(C \upharpoonright c) = \sigma'((A \times B) \upharpoonright g^{-1}(c)).
\]

Say \( g^{-1}(c) = (a, b) \). Now there is an automorphism \( l \) of \( \text{Fr}(\omega) \) such that for all \( (u, v) \in A \times B \), \( l(f([u, v]_{I_\beta(A \times B)})) = h([v]_{I_\beta(B)}) \).

Hence

\[
    r_B h(l(x)) = r_B h(l(f(g'([c]_{I_\beta(C)})))) = r_B h(l(f(g'([g^{-1}(c)]_{I_\beta(C)}))))
    = r_B h(l(f(g'([g([a, b])_{I_\beta(C)}])))) = r_B h(l(f([a, b]_{I_\beta(A \times B)})))
    = r_B h([b]_{I_\beta(B)}) = \sigma'(B \upharpoonright b) = \sigma'((A \times B) \upharpoonright g^{-1}(c)).
\]

It follows that \( B \cong C \cong A \times B \). \( \square \)

**Proposition 5.26.** Suppose that \( B \) and \( C \) are normalized, with \( \tau(B) = (\beta, \beta, 0) \) and \( \tau(C) = (\gamma, \gamma, 0) \). Then

(i) if \( \beta < \gamma \), then \( B \times C \cong C \);

(ii) if \( \gamma < \beta \), then \( B \times C \cong B \);

(iii) if \( \beta = \gamma \), then \( \tau(B \times C) = (\beta, \beta, 0) \), \( B \times C \) is normalized, and \( \forall x \in \text{Fr}(\omega) \exists y, z \in \text{Fr}(\omega) [x = y + z \text{ and } r_{A \times B, u}(x) = \max(r_{A, u}(y), r_{B, w}(z))] \) for some \( u, v, w \).

**Proof.** (i): Assume that \( \beta < \gamma \). By Proposition 4.14, \( \tau(B \times C) = (\gamma, \gamma, 0) \), and by Proposition 4.11, \( I_\gamma(B \times C) \cong I_\gamma(C) \). Then

\[
    (B \times C)/I_\gamma(B \times C) \cong (B/I_\gamma(B)) \times (C/I_\gamma(C)) \cong C/I_\gamma(C) \cong \text{Fr}(\omega).
\]

Hence \( B \times C \) is normalized. Let \( f \) be an isomorphism of \( B \times C \) onto a BA \( D \) such that \( f[I_\gamma(B \times C)] = I_\gamma(D) \). Let \( g \) be an isomorphism from \( C/I_\gamma(C) \) onto \( \text{Fr}(\omega) \), and let \( h \) be an
isomorphism from $D/I_\gamma(D)$ onto $\text{Fr}(\omega)$. Define $f'([(u,v)]_{I_\gamma(B \times C)}) = [f(u,v)]_{I_D(D)}$. Then $f'$ is an isomorphism of $(B \times C)/I_\gamma(B \times C)$ onto $D/I_\gamma(D)$. Define

$$k(h(f'([(u,v)]_{I_\gamma(B \times C)}))) = g([v]_{I_\gamma(C)}).$$

Then $k$ is an automorphism of $\text{Fr}(\omega)$. Now given $x \in \text{Fr}(\omega)$, choose $(b, c) \in B \times C$ such that $h(f'([(b,c)]_{I_\gamma(B \times C)})) = x$. Let $g([c]_{I_\gamma(C)}) = y$. Then $h \circ f'$ is an isomorphism from $(B \times C)/I_\gamma(B \times C)$ onto $\text{Fr}(\omega)$. Further, $k(x) = g([c]_{I_\gamma(C)})$, and

$$r_{D,h \circ f'}(x) = r_{D,h \circ f'}(h(f'((b,c)\Delta I_\gamma(B \times C)))) = \sigma'(f(b,c));$$
$$r_{C,g}(k(x)) = r_{C,g}(g([c]_{I_\gamma(C)})) = r_{C,g}(y) = \sigma'(c) = \sigma'(f(b,c)).$$

Hence $B \times C \cong C$.

(ii): this is symmetric to (i).

(iii): By Proposition 4.14, $\tau(B \times C) = (\beta, \beta, 0)$. Also,

$$(B \times C)/I_\beta(B \times C) \cong (B/I_\beta(B)) \times (C/I_\beta(C)) \cong \text{Fr}(\omega) \times \text{Fr}(\omega) \cong \text{Fr}(\omega).$$

So $B \times C$ is normalized.

Let $f$ be an isomorphism of $(B \times C)/I_\beta(B \times C)$ onto $\text{Fr}(\omega)$. Given $x \in \text{Fr}(\omega)$ choose $(b, c) \in B \times C$ such that $f([(b,c)]_{I_\beta(B \times C)}) = x$. Then

$$r_{B \times C,f}(x) = r_{B \times C,f}(f([(b,c)]_{I_\beta(B \times C)})) = \sigma'(b,c) = \max(\sigma'((0,b)), \sigma'((0,c))).$$

Now $[(b,0)]_{I_\beta(B \times C)} + [(0,c)]_{I_\beta(B \times C)} = [(b,c)]_{I_\beta(B \times C)}$. Let $f([(b,0)]_{I_\beta(B \times C)}) = y$ and $f([(0,c)]_{I_\beta(B \times C)}) = z$. Then $x = y + z$. Let $g$ be an isomorphism of $B/I_\beta(B)$ onto $((B \times C)/I_\beta(B \times C)) \uparrow (([1,0])_{I_\beta(B \times C)})$ and $h$ an isomorphism of $C/I_\beta(C)$ onto $((B \times C)/I_\beta(B \times C)) \uparrow (\{0,1\})_{I_\beta(B \times C)}$. Then $r_{B,f \circ g}(y) = r_{B,f \circ g}(g([b]_{I_\beta(B)})) = \sigma'((b,0))$ and similarly $r_{C,f \circ h}(z) = \sigma'((0,c))$. So $r_{B \times C,f}(x) = \max(r_{B,f \circ g}(y), r_{C,f \circ h}(z)).$ \hfill $\Box$

**Proposition 5.27.** Let $A, B$ be nontrivial countable BAs.

(i) $\text{INV}(A) = (0, \alpha, n), \text{INV}(B) = (0, \beta, m) \Rightarrow \text{INV}(A \times B) = (0, \max(\alpha, \beta, p), \text{where}

$$p = \begin{cases} 
    m & \text{if } \alpha < \beta, \\
    n & \text{if } \beta < \alpha, \\
    m+n & \text{if } \alpha = \beta.
\end{cases}$$

(ii) If $\text{INV}(A) = (0, \alpha, n)$ and $\text{INV}(B) = (1, \beta, \{r_B \circ f : f \in \text{aut}(\text{Fr}(\omega))\})$, where $B$ is normalized and $\tau(B) = (\beta, \beta, 0)$, then

(a) if $\alpha < \beta$, then $A \times B \cong B$ and so $\text{INV}(A \times B) = \text{INV}(B)$,

(b) if $\beta \leq \alpha$, then $\text{INV}(A \times B) = (2, \beta, \alpha, n, \{r_B \circ f : f \in \text{aut}(\text{Fr}(\omega))\})$.

(iii) If $\text{INV}(A) = (0, \alpha, n)$ and $\text{INV}(B) = (2, \beta, \gamma, m, \{r_B \circ f : f \in \text{aut}(\text{Fr}(\omega))\})$, then

(a) if $\alpha < \beta$, then $A \times B \cong B$ and so $\text{INV}(A \times B) = \text{INV}(B)$,

(b) if $\beta \leq \alpha$, then $\text{INV}(A \times B) = (2, \beta, \alpha, n, \{r_B \circ f : f \in \text{aut}(\text{Fr}(\omega))\})$.

(iv) If $\text{INV}(A) = (0, \alpha, m)$ and $\text{INV}(B) = (2, \beta, \gamma, n, \{r_B \circ f : f \in \text{aut}(\text{Fr}(\omega))\})$, then

(a) if $\alpha < \beta$, then $A \times B \cong B$, and so $\text{INV}(A \times B) = \text{INV}(B)$. 

55
(b) if $\beta \leq \alpha$, then $\text{INV}(A \times B) = (2, \beta, \delta, p, \{r_B \circ f : f \in \text{aut}(\text{Fr}(\omega))\})$, where $\delta = \max(\alpha, \gamma)$ and

$$p = \begin{cases} n & \text{if } \alpha < \gamma, \\ m & \text{if } \gamma < \alpha, \\ m + n & \text{if } \alpha = \gamma. \end{cases}$$

(v) If $\text{INV}(A) = (1, \beta, \{r_A \circ f : f \in \text{aut}(\text{Fr}(\omega))\})$, where $A$ is normalized and $\tau(A) = (\beta, \beta, 0)$ and $\text{INV}(B) = (1, \gamma, \{r_B \circ f : f \in \text{aut}(\text{Fr}(\omega))\})$, where $B$ is normalized and $\tau(B) = (\gamma, \gamma, 0)$, then $\text{INV}(A \times B)$ is given by Proposition 5.29.

(vi) If $\text{INV}(A) = (1, \alpha, \{r_A \circ f : f \in \text{aut}(\text{Fr}(\omega))\})$, where $A$ is normalized and $\tau(A) = (\alpha, \alpha, 0)$ and $\text{INV}(B) = (2, \beta, \gamma, n, \{r_B \circ f : f \in \text{aut}(\text{Fr}(\omega))\})$, then

(a) if $\alpha < \beta$, then $A \times B \cong B$ and $\text{INV}(A \times B) = \text{INV}(B)$;
(b) if $\gamma < \alpha$, then $A \times B \cong A$ and $\text{INV}(A \times B) = \text{INV}(A)$;
(c) if $\beta < \alpha \leq \gamma$, then $\text{INV}(A \times B) = (2, \alpha, \gamma, n, \{r_A \circ f : f \in \text{aut}(\text{Fr}(\omega))\})$;
(d) If $\alpha = \beta$, then $\text{INV}(A \times B) = (2, \alpha, \gamma, n, \{r_{A \times B} \circ f : f \in \text{aut}(\text{Fr}(\omega))\})$;

(vii) If $\text{INV}(A) = (2, \alpha, \gamma, n, \{r_A \circ f : f \in \text{aut}(\text{Fr}(\omega))\})$ and $\text{INV}(B) = (2, \beta, \delta, m, \{r_B \circ f : f \in \text{aut}(\text{Fr}(\omega))\})$, then

(a) if $\gamma < \beta$ then $A \times B \cong B$ and $\text{INV}(A \times B) = \text{INV}(B)$;
(b) if $\alpha < \beta \leq \gamma$, then $\text{INV}(A \times B) = (2, \beta, \varepsilon, p, \{r_B \circ f : f \in \text{aut}(\text{Fr}(\omega))\})$, where $\varepsilon$ and $p$ are determined by Proposition 5.14;
(c) if $\beta < \alpha$, then $\text{INV}(A \times B) = (2, \alpha, \varepsilon, p, \{r_A \circ f : f \in \text{aut}(\text{Fr}(\omega))\})$, where $\varepsilon$ and $p$ are determined by Proposition 5.14;
(d) If $\beta = \alpha$, then $\text{INV}(A \times B) = (2, \varphi, \varepsilon, p, \{r_{A \times B} \circ f : f \in \text{aut}(\text{Fr}(\omega))\})$, where $\varphi$, $\varepsilon$ and $p$ are determined by Proposition 5.14.

(ii)(a): by Proposition 5.25.
(ii)(b): clear.
(iii): Let $C$ be normalized and $D$ be superatomic such that $B \cong C \times D$, $\tau(C) = (\beta, \beta, 0)$, and $\tau(D) = (\gamma, 0, m)$.
(a): Then $A \times C \cong C$ by Proposition 5.25, and hence $A \times B \cong B$.
(b): obvious.
(iii) Let $C$ be normalized and $D$ be superatomic such that $B \cong C \times D$, $\tau(C) = (\beta, \beta, 0)$, and $\tau(D) = (\gamma, 0, m)$.
(a): clear.
(b): clear.
(iv)(a): by Proposition 5.25
(vi)(a): by Proposition 5.25.
(vi)(b): by Propositions 5.24, 5.25.
(vi)(c): by Proposition 5.25.
(vi)(d): by Proposition 5.25.
(vii)(a): by Propositions 5.24, 5.25.
(vii)(b): by Proposition 5.25.
(vii)(c),(d): by Propositions 5.24, 5.25. 
\[\square\]
6. More examples

Example 6.1. If $|A| = 1$, then $\tau(A) = \tau_s(A) = (0, 0, 0)$. □

Example 6.2. If $A$ is finite with $n > 0$ atoms, then $\tau(A) = \tau_s(A) = (0, 0, n)$. □

Example 6.3. If $A = \Intalg(\omega^\alpha \cdot n)$, with $\alpha$ a nonzero ordinal and $n$ a nonzero natural number, then $\tau(A) = \tau_s(A) = (\alpha, 0, n)$.

Proof. We claim that for all $\beta < \alpha$,

6.3(1) atoms($A/I_\beta(A)$) = \{[[\omega^\alpha \cdot i + \omega^\beta \cdot \xi, \omega^\alpha \cdot i + \omega^\beta \cdot (\xi + 1)]_{I_\beta(A)} : i < n, \xi < \omega^\alpha\}.

6.3(2) $A/I_\beta(A)$ is generated by the increasing sequence

\[
\begin{align*}
\{[[0, \omega^\beta \cdot \xi]]_{I_\beta(A)} : \xi < \omega^\alpha \} & \\
[[0, \omega^\alpha + \omega^\beta \cdot \xi]]_{I_\beta(A)} : \xi < \omega^\alpha \} & \\
\vdots \\
[[0, \omega^\alpha \cdot (n - 1) + \omega^\beta \cdot \xi]]_{I_\beta(A)} : \xi < \omega^\alpha \}.
\end{align*}
\]

We prove these statements by induction on $\beta$. They are clear for $\beta = 0$. Assume them for $\beta$. Then they are clear for $\beta + 1$. Now assume that $\gamma$ is limit $< \omega^\alpha$ and (1) and (2) hold for all $\beta < \gamma$. Then for any $i < n$ and $\xi < \omega^\alpha$, $[[\omega^\alpha \cdot i + \omega^\gamma \cdot \xi, \omega^\alpha \cdot i + \omega^\gamma \cdot (\xi + 1)]_{I_\gamma(A)}$ is an atom of $A/I_\gamma(A)$. In fact, first note that each member of $I_\gamma(A)$ is a finite union of intervals $[\varphi, \psi)$ with $\psi - \varphi < \gamma$; hence $[[\omega^\alpha \cdot i + \omega^\gamma \cdot \xi, \omega^\alpha \cdot i + \omega^\gamma \cdot (\xi + 1)]_{I_\gamma(A)} \neq 0$. Now suppose that $i < n$, $\xi < \omega^\alpha$, and $\delta < \omega^\gamma$. Choose $\beta < \gamma$ so that $\delta < \omega^\beta$. Say $\beta + \varepsilon = \gamma$.

\[
\begin{align*}
[[\omega^\alpha \cdot i + \omega^\gamma \cdot \xi, \omega^\alpha \cdot i + \omega^\gamma \cdot (\xi + \delta)]_{I_\gamma(A)} \\
= [[\omega^\alpha \cdot i + \omega^\beta \cdot \omega^\varepsilon \cdot \xi, \omega^\alpha \cdot i + \omega^\beta \cdot \omega^\varepsilon \cdot (\xi + \delta)]_{I_\gamma(A)} \\
\le [[\omega^\alpha \cdot i + \omega^\beta \cdot \omega^\varepsilon \cdot \xi, \omega^\alpha \cdot i + \omega^\beta \cdot (\omega^\varepsilon \cdot (\xi + 1))]_{I_\gamma(A)} \\
\le [[\omega^\alpha \cdot i + \omega^\beta \cdot (\omega^\varepsilon \cdot (\xi + 1))]_{I_\gamma(A)} = 0.
\end{align*}
\]

Now 6.3(1) and 6.3(2) follow for $\gamma$.

So 6.3(1) and 6.3(2) hold for all $\beta < \alpha$. It follows that $A/I_\alpha(A)$ is generated by

\[
\{[[0, \omega^\alpha]]_{I_\alpha(A)}, \ldots, [[\omega^\alpha \cdot (n - 1), \infty]]_{I_\alpha(A)}\}.
\]

So $A/I_\alpha(A)$ is finite with $n$ atoms. Hence $\tau(A) = \tau_s(A) = (\alpha, 0, n)$. □

Example 6.4. Let $L = \omega \times \mathbb{Q}$ ordered by second differences, and with a new zero $0'$ adjoined. Let $A = \Intalg_d(L)$. Then $\tau(A) = (1, 1, 0)$.

Proof.
atoms(A) = \{\{n\}, q : n \in \omega, q \in \mathbb{Q}\};
I_1(A) is generated by \{\{n\}, q : n \in \omega, q \in \mathbb{Q}\};
A/I_1(A) is atomless;
I_2(A) = I_1(A);
I_1(A)/I_0(A) is not principal.
\tau(A) = (1,1,0).
ar(A) = 1.
A is normal.
g : A/I_{ar(A)}(A) \rightarrow \text{Intalg}(\mathbb{Q}) is defined by g([0', (m, q)])_{I_1(A)} = [0, q]; with h an isomorphism of \text{Intalg}(\mathbb{Q}) onto \text{Fr}(\omega) we have INV(A) = (1,1,\{r_{A,h\circ g} \circ k : k \in \text{Aut}(\text{Fr}(\omega))\}). \qed

Example 6.5. Let \( L = \omega^2 \times \mathbb{Q} \) ordered by second differences, and with a new zero \( 0' \) adjoined. Let \( A = \text{Intalg}(L) \). Then \( \tau(A) = (2,2,0) \).

Proof.

atoms(A) = \{\{\alpha\}, q : \alpha < \omega_2, q \in \mathbb{Q}\};
I_1(A) is generated by \{\{\alpha\}, q : \alpha < \omega_2, q \in \mathbb{Q}\};
atoms(A/I_1(A)) = \{[[\omega \cdot n, \sigma \cdot (n + 1)], q]_{I_2(A)} : n \in \omega, q \in \mathbb{Q}\};
I_2(A) is generated by \( I_1(A) \cup \{[[\omega \cdot n, \sigma \cdot (n + 1)], q] : n \in \omega, q \in \mathbb{Q}\};
A/I_2(A) is atomless;
I_3(A) = I_2(A);
I_2(A)/I_1(A) is not principal;
\tau(A) = (2,2,0).
ar(A) = 2.
A is normal.
g : A/I_{ar(A)}(A) \rightarrow \text{Intalg}(\mathbb{Q}) is defined by g([0', (\alpha, q)])_{I_1(A)} = [0, q]; with h an isomorphism of \text{Intalg}(\mathbb{Q}) onto \text{Fr}(\omega) we have INV(A) = (2,2,\{r_{A,h\circ g} \circ k : k \in \text{Aut}(\text{Fr}(\omega))\}). \qed

Example 6.6. Let \( A = \text{Intalg}(\omega^2) \times \text{Intalg}(\mathbb{Q}) \). Then \( \tau(A) = (2,0,1) \).

Proof.

atoms(A) = \{\{\alpha\}, 0 : \alpha < \omega^2\};
I_1(A) is generated by \{\{\alpha\}, 0 : \alpha < \omega^2\};
atoms(A/I_1(A)) = \{[[\omega \cdot n, \omega \cdot (n + 1)], 0]_{I_2(A)} : n \in \omega\};
I_2(A) is generated by \( I_1(A) \cup \{[[\omega \cdot n, \omega \cdot (n + 1)], 0] : n \in \omega\};
atoms(A/I_2(A)) = \{[(1,0)]_{I_2(A)}\};
I_3(A) is generated by \( I_2(A) \cup \{(1,0)\};
A/I_3(A) is atomless;
I_4(A) = I_3(A)
I_3(A)/I_0(A) is generated by [(1,0)]_{I_0(A)};
A/I_2(A) has the one atom [(1,0)]_{I_2(A)}.
\tau(A) = (2,0,1). \qed

Example 6.7. Let \( A = \text{Intalg}(\omega) \times \text{Intalg}(\omega^2 \times \mathbb{Q}) \). Then \( \tau(A) = (2,2,0) \).
Proof.

atoms\(A\) = \(\{(\{m\}, 0) : m \in \omega\} \cup \{(0, (\{\alpha\}, q)) : \alpha < \omega^2, q \in \mathbb{Q}\}\);
\(I_1(A)\) is generated by \(\{(\{m\}, 0) : m \in \omega\} \cup \{(0, (\{\alpha\}, q)) : \alpha < \omega^2, q \in \mathbb{Q}\}\);
atoms\(A/I_1(A)\) = \(\{(1, 0) I_1(A)\} \cup \{(0, (\omega \cdot n, \omega \cdot (n + 1)), q) I_1(A) : n \in \omega, q \in \mathbb{Q}\}\);
\(I_2(A)\) is generated by \(I_1(A) \cup \{(1, 0) \} \cup \{(0, (\omega \cdot n, \omega \cdot (n + 1)), q) : n \in \omega, q \in \mathbb{Q}\}\);
\(A/I_2(A)\) is atomless;
\(I_3(A) = I_2(A)\);
\(I_2(A)/I_0(A)\) and \(I_2/I_1(A)\) are not principal;
\(\tau(A) = (2, 2, 0)\).
\(\square\)

Example 6.8. Let \(A = \text{Intalg}(\omega^2) \times \text{Intalg}(\omega^2 \times \mathbb{Q})\). Then \(\tau(A) = (2, 2, 1)\).

Proof.

atoms\(A\) = \(\{(\{\alpha\}, 0) : \alpha < \omega^2\} \cup \{(0, (\{\alpha\}, q)) : \alpha < \omega^2, q \in \mathbb{Q}\}\);
\(I_1(A)\) is generated by \(\{(\{\alpha\}, 0) : \alpha < \omega^2\} \cup \{(0, (\{\alpha\}, q)) : \alpha < \omega^2, q \in \mathbb{Q}\}\);
atoms\(A/I_1(A)\) = \(\{(\omega \cdot n, \omega \cdot (n + 1)), 0\} I_1(A) : n \in \omega\} \cup \{(0, (\omega \cdot n, \omega \cdot (n + 1)), q) I_1(A) : n \in \omega, q \in \mathbb{Q}\}\);
\(I_2(A)\) is generated by \(I_1(A) \cup \{(\omega \cdot n, \omega \cdot (n + 1)), 0\} : n \in \omega\} \cup \{(0, (\omega \cdot n, \omega \cdot (n + 1)), q) : n \in \omega, q \in \mathbb{Q}\}\);
\(A/I_2(A)\) is atomless;
\(I_3(A) = I_2(A)\);
\(I_2(A)/I_0(A)\) and \(I_2/I_1(A)\) are not principal;
\(\tau(A) = (2, 2, 1)\).
\(\square\)

Example 6.9. Let \(L = (\omega \times \mathbb{Q}) + (\omega^2 \times \mathbb{Q})\) with a new zero \(0'\) adjoined, \(A = \text{Intalg}(L)\).
\(\tau(A) = (2, 2, 0)\).

atoms\(A\) = \(\{(\{m\}, q) : q \in \mathbb{Q}\} \cup \{(\{\alpha\}, q) : \alpha < \omega^2, q \in \mathbb{Q}\}\).
\(I_1(A)\) is generated by \(\{(\{m\}, q) \cup \{(\{\alpha\}, q) : \alpha < \omega^2, q \in \mathbb{Q}\}\}\).
atoms\(A/I_1(A)\) = \(\{(\omega \cdot n, \omega \cdot (n + 1)), q\} I_1(A) : n \in \omega, q \in \mathbb{Q}\}\).
\(I_2(A)\) is generated by \(I_1(A) \cup \{(\omega \cdot n, \omega \cdot (n + 1)), q\} : n \in \omega, q \in \mathbb{Q}\}\).
\(A/I_2(A)\) is atomless.
\(I_3(A) = I_2(A)\).
\(\varphi(A) = (2, 2, 0)\).
\(\square\)

II. Ketonen’s theorem

7. Monoids and measures

An \(m\)-monoid is an algebra \((M, +, 0)\) such that + is a commutative and associative binary operation on \(M\), \(0 \in M\), \(\forall a \in M[a + 0 = a]\) and \(\forall a, b \in M[a + b = 0 \iff a = b = 0]\).

For any \(m\)-monoid \((M, +, 0)\) we set \(M^* = M \setminus \{0\}\). Note that \(M^*\) is closed under +, so that \((M^*, +)\) is a commutative semigroup.
A morphism of an \( m \)-monoid \((M, +, 0)\) to an \( m \)-monoid \((M', +', 0')\) is a function \( f : M \to M' \) such that \( \forall x, y \in M [f(x + y) = f(x) + f(y)] \), \( f(0) = 0' \), and \( f[M^*] \subseteq M'^* \).

**Proposition 7.1.** If \( L \) is a linear order with smallest element 0, then \((L, \max, 0)\) is an \( m \)-monoid, where \( \max(a, b) \) is the maximum of \( a \) and \( b \). □

\( \mathcal{W} \) is \( \omega_1 \) with a new smallest element \( o \) adjoined. So \((\mathcal{W}, \max, o)\) is an \( m \)-monoid.

If \( A \) is a BA and \( M \) is an \( m \)-monoid, then an \( M \)-measure on \( A \) is a mapping \( \sigma : A \to M \) such that

7.1(1) \( \sigma(x + y) = \sigma(x) + \sigma(y) \) for each pair \((x, y)\) of disjoint elements of \( A \).

7.1(2) \( \forall x \in A [\sigma(x) = 0 \iff x = 0] \).

The set of all \( M \)-measures on \( \text{Fr}(\omega) \) will be denoted by \( \mathcal{M}(M) \). \( \mathcal{M} \) is \( \mathcal{M}(\mathcal{W}) \).

**Proposition 7.2.** If \( M, N \) are \( m \)-monoids, \( \Phi : M \to N \) is a morphism, and \( \sigma \in \mathcal{M}(M) \), then \( \Phi \circ \sigma \in \mathcal{M}(N) \).

**Proof.** Clearly \( \Phi \circ \sigma : \text{Fr}(\omega) \to N \). If \( x, y \) are disjoint elements of \( \text{Fr}(\omega) \), then

\[
\Phi(\sigma(x + y)) = \Phi(\sigma(x) + \sigma(y)) = \Phi(\sigma(x)) + \Phi(\sigma(y)).
\]

Also, if \( x \in \text{Fr}(\omega) \), then

\[
\Phi(\sigma(x)) = 0 \iff \sigma(x) = 0 \iff x = 0. \quad \square
\]

**Proposition 7.3.** If \( k \) is an automorphism of \( \text{Fr}(\omega) \) and \( \sigma \in \mathcal{M}(M) \), then \( \sigma \circ k \in \mathcal{M}(M) \).

**Proof.** Assume that \( k \) is an automorphism of \( \text{Fr}(\omega) \) and \( \sigma \in \mathcal{M}(M) \). If \( x, y \) are disjoint elements of \( \text{Fr}(\omega) \), then \( k(x) \) and \( k(y) \) are disjoint, and

\[
\sigma(k(x + y)) = \sigma(k(x) + k(y)) = \sigma(k(x)) + \sigma(k(y)).
\]

For any \( x \in \text{Fr}(\omega) \),

\[
\sigma(k(x)) = 0 \iff k(x) = 0 \iff x = 0. \quad \square
\]

**8. Derived monoids**

For any set \( M \) we denote by \( \ll \omega M \) the set of all finite nonempty sequences of members of \( M \). If \((M, +, 0)\) is an \( m \)-monoid, then for any \( m \in \omega \setminus \{0\} \) the operation \( + \) on \( M \) is defined coordinatewise. We define the trace map \( T : \ll \omega M \to M \) by

\[
T(a) = \sum_{i < m} a_i \quad \text{where} \ a \in M^m \text{ with} \ m \in \omega \setminus \{0\}.
\]

If \( M \) is an \( m \)-monoid, \( m, n \in \omega \setminus \{0\} \), \( a \in M^m \), and \( b \in M^n \), then \( a \) is a refinement of \( b \), in symbols \( a \prec b \), iff there is a mapping \( \lambda : m \to n \) such that \( \forall j < n [b_j = \sum \{a_i : i < m, \lambda(i) = j\}] \). Note that if \( a \prec b \) with \( m, n, \lambda \) as indicated, then \( \sum_{j < n} b_j = \sum_{i < m} a_i \).
Proposition 8.1. $\propto$ is reflexive.

Proof. Suppose that $a \in {}^mM$. Let $\lambda(i) = i$ for all $i < m$. Then for all $j < m$, $a_j = \sum\{a_i : \lambda(i) = j\}$. □

Proposition 8.2. $\propto$ is transitive.

Proof. Suppose that $a \propto b < c$. Say $a \in {}^mM$, $b \in {}^nM$, and $c \in {}^pM$. Then there exist $\lambda : m \to n$ and $\mu : n \to p$ such that $\forall j < n[b_j = \sum\{a_i : i < m, \lambda(i) = j\}]$ and $\forall k < p[c_k = \sum\{b_j : j < n, \mu(j) = k\}]$. Then for all $k < p$,

$$c_k = \sum\{b_j : j < n, \mu(j) = k\} = \sum\bigl\{\sum\{a_i : i < m, \lambda(i) = j\} : j < n, \mu(j) = k\bigr\}$$

$= \sum\{a_i : i < m : \mu(\lambda(i)) = k\}$. □

Proposition 8.3. If $M$ is a $m$-monoid, $m \in \omega\setminus\{0\}$, $a \in {}^mM$, and $\lambda$ is a permutation of $m$, let $b = a \circ \lambda = \langle a_{\lambda(0)}, a_{\lambda(1)}, \ldots, a_{\lambda(m-1)} \rangle$. Then $a \propto b < a$.

Proof. $\forall j < n[b_j = a_{\lambda(j)}] = \sum\{a_i : i < n, \lambda^{-1}(i) = j\}]$ and $\forall j < n[a_j = b_{\lambda^{-1}(j)}] = \sum\{b_i : i < n, \lambda(i) = j\}]$. □

Proposition 8.4. If $M$ is a $m$-monoid, $m \in \omega\setminus\{0\}$, and $a \in {}^mM$, then $a \mathrel{\propto} \langle 0 \rangle = \langle a_0, \ldots, a_{m-1}, 0 \rangle$ and $a < a \mathrel{\propto} \langle 0 \rangle < a$.

Proof. Clearly $a \mathrel{\propto} \langle 0 \rangle = \langle a_0, \ldots, a_{m-1}, 0 \rangle$. Now let $b = a \mathrel{\propto} \langle 0 \rangle$. Let $\lambda : m \to m+1$ be the inclusion map. Then $\forall j < m+1[b_j = \sum\{a_i : i < n, \lambda(i) = j\}]$. So $a < b$. Now let $\mu : m+1 \to m$ be such that $\mu \upharpoonright m$ is the identity and $\mu(m) = m-1$. Then $\forall j < m[a_j = \sum\{b_i : i < m+1, \mu(i) = j\}]$, so $b < a$. □

Proposition 8.5. If $M$ is a $m$-monoid, $m, n \in \omega\setminus\{0\}$, $a \in {}^mM$, and $b \in {}^nM$, and $a < b$, then $T(a) = T(b)$.

Let $f : \omega \times \omega \to \omega$ be a bijection. Let $(M, +, 0)$ be an $m$-monoid. The derived monoid $\Delta M$ of $M$ is the set of all $\alpha \in [[[<\omega M]]^\omega$ satisfying the following conditions:

(i) (Collection property, (C.P.)): For all $a, b \in [[[<\omega M$, if $a \in \alpha$ and $a < b$, then $b \in \alpha$.

(ii) (Refinement property, (R.P.)): For all $m, n \in \omega\setminus\{0\}$ and all $a \in {}^mM$ and $b \in {}^nM$, if $a, b \in \alpha$ then there is a $c \in {}^{m \cdot n}M$ such that $c \in \alpha$, $\forall i < m[a_i = \sum\{c_{f(i,j)} : j < n\}]$ and $\forall j < n[b_j = \sum\{c_{f(i,j)} : i < m\}].$

(iii) (Splitting property, (S.P.)) $\forall m \in \omega\setminus\{0\}$ and all $a \in {}^mM$, if $a \in \alpha$ and $a_0 \neq o$, then there exist $b, c \in M\setminus\{0\}$ such that $a_0 = b + c$ and $\langle b, c, a_1, \ldots, a_{m-1} \rangle \in \alpha$.

For any $\alpha, \beta \in \Delta M$ let

$$\alpha + \beta = \{a + b : a \in \alpha, b \in \beta, \text{dmm}(a) = \text{dmm}(b)\}.$$

Also, let $O = \{\langle 0 \rangle, \langle 0, 0 \rangle, \ldots \}$.

Proposition 8.6. If $\alpha \in \Delta M$, then for each $m \in \omega\setminus\{0\}$, $\alpha$ has a member of length $m$. 

Proof. \(\alpha\) is infinite, so fix \(a \in \alpha\); say \(a\) has length \(m\). Let \(\lambda : m \to 1\) be the obvious function. Let \(b = \{(0, \sum\{a_i : i < m\})\}\). Then \(b_0 = \sum\{a_i : i < m, \lambda(i) = 0\}\), so \(a \prec b\). Hence by the C.P., \(b \in \alpha\). Thus \(\alpha\) has an element of length 1. By Proposition 8.4 and the C.P. it has elements of each positive length. \(\square\)

**Proposition 8.7.** If \(\alpha, \beta \in \Delta M\), then \(\alpha + \beta \in \Delta M\). Also \(O \in \Delta M\). Moreover, \((\Delta M, +, O)\) is an \(\omega\)-monoid.

**Proof.** By Proposition 8.6, \(\alpha + \beta \in [\omega \omega M]^{\omega}\). Now to check C.P. for \(\alpha + \beta\), suppose that \(a, b \in \omega \omega M, a \in \alpha + \beta\), and \(a \prec b\). Say \(a = c + d\) and \(m \in \omega \setminus \{0\}\) with \(c, d \in m M\), \(a = c + d\) and \(m \in \omega \setminus \{0\}\), \(\lambda : m \to n\), and \(\forall j < n[b_j = \sum\{a_i : i < m, \lambda(i) = j\}]\). Define \(c'\) and \(d'\) with domain \(n\) by setting, for \(j < n\), \(c'_j = \sum\{c_i : i < m, \lambda(i) = j\}\) and \(d'_j = \sum\{d_i : i < m, \lambda(i) = j\}\). Then \(c \prec c'\) and \(d \prec d'\), so by C.P. for \(\alpha\) and \(\beta\), \(c' \in \alpha\) and \(d' \in \beta\). Clearly \(b = c' + d'\), so \(b \in \alpha + \beta\). This proves C.P. for \(\alpha + \beta\).

For R.P., suppose that \(m, n \in \omega \setminus \{0\}, a \in m M, b \in n M\), and \(a, b \in \alpha + \beta\). Say \(c, d \in m M, e, f \in n M\), \(c, e \in \alpha\), \(d, f \in \beta\), \(a = c + d\), and \(b = e + f\). Choose \(g, h \in m \cdot n M\) such that \(g < a\), \(h < b\), \(\forall i < m[c_i = \sum\{g_{f(i,j)}(j : j < n)\}]\), \(\forall j < n[e_j = \sum\{g_{f(i,j)}(j : i < m)\}]\), \(\forall i < m[d_i = \sum\{h_{f(i,j)}(i : j < n)\}]\), and \(\forall j < n[f_j = \sum\{h_{f(i,j)}(i : j < m)\}]\). Let \(k = g + h\). Then \(k \in m \cdot n M\), \(k < \alpha + \beta\),

\[
\forall i < m[a_i = c_i + d_i = \sum\{k_{f(i,j)}(j : j < n)\}\] and 
\[
\forall j < n[b_j = e_j + f_j = \sum\{k_{f(i,j)}(i : j < m)\}\.
\]

This proves R.P. for \(\alpha + \beta\).

For S.P., suppose that \(m \in \omega \setminus \{0\}, a \in m M, a \in \alpha + \beta\), and \(a_0 \neq 0\). Say \(a = b + c\) with \(b \in \alpha\) and \(c \in \beta\).

Case 1. \(b_0 \neq 0 \neq c_0\). Choose \(d, e \in M \setminus \{0\}\) such that \(c_0 = d + e\) and

\[
\langle d, e, c_1, c_2, \ldots c_{m-1} \rangle \in \beta.
\]

Also, let \(f = \langle 0, 0, b_1, b_2, \ldots b_{m-1} \rangle\). Define \(\lambda : m \to m + 1\) by \(\lambda(i) = i + 1\) for all \(i < m\). Then \(f_0 = 0 = \sum\{b_i : \lambda(i) = 0\}\), \(f_1 = 0 = \sum\{b_i : i < n, \lambda(i) = 1\} = b_0\), and for \(j \geq 2\), \(f_j = b_{j-1} = \sum\{b_i : i < n, \lambda(i) = j\}\). So \(b \prec f\), hence \(f \in \alpha\). Clearly \(a_0 = d + e\) and

\[
\langle d, e, a_1, a_2, \ldots a_{m-1} \rangle = \langle d, e, c_1, c_2, \ldots c_{m-1} \rangle + f;
\]

hence \(\langle d, e, a_1, a_2, \ldots a_{m-1} \rangle \in \alpha + \beta\).

Case 2. \(b_0 \neq 0 = c_0\). This is symmetric to Case 1.

Case 3. \(b_0 \neq 0 \neq c_0\). Choose \(d, e \in M \setminus \{0\}\) such that \(c_0 = d + e\) and

\[
\langle d, e, c_1, c_2, \ldots c_{m-1} \rangle \in \beta.
\]

and choose \(d', e' \in M \setminus \{0\}\) such that \(b_0 = d' + e'\) and \(\langle d', e', b_1, b_2, \ldots b_{m-1} \rangle \in \alpha\). Then \(a_0 = b_0 + c_0 = d + e + d' + e'\) and

\[
\langle d + d', e + e', a_1, a_2, \ldots a_{m-1} \rangle \in \alpha + \beta.
\]
This proves S.P. for $\alpha + \beta$.

Hence $\alpha + \beta \in \triangle M$. Clearly $O \in \triangle M$. Clearly then $(\triangle M, +, O)$ is an $m$-monoid.

**Proposition 8.8.** If $a \in \alpha \in \triangle M$, with $a \in {}^m M$, and if $\lambda$ is a permutation of $m$, then $a \circ \lambda \in \alpha$.

**Proof.** By Proposition 8.3 and C.P.

**Proposition 8.9.** $\forall m \in \omega \setminus \{0\}$, all $a \in {}^m M$, and all $i < m$, if $a \in \alpha$ and $a_i \neq o$, then there exist $b, c \in M \setminus \{0\}$ such that $a_i = b + c$ and $(a_0, \ldots, a_{i-1}, b, c, a_{i+1}, \ldots, a_{m-1}) \in \alpha$.

**Proof.** Let $\lambda : m \rightarrow m$ be the permutation $(i, 0)$. Then

$$\langle a_i, a_1, \ldots, a_{i-1}, a_0, a_{i+1}, \ldots, a_{m-1} \rangle \in \alpha$$

by Proposition 8.8. By S.P. let $b, c \in M \setminus \{0\}$ be such that $a_i = b + c$ and

$$\langle b, c, a_1, \ldots, a_{i-1}, a_0, a_{i+1}, \ldots, a_{m-1} \rangle \in \alpha.$$ 

Let $d = \langle b, c, a_1, \ldots, a_{i-1}, a_0, a_{i+1}, \ldots, a_{m-1} \rangle$, and let $\lambda$ be the permutation of $m + 1$ such that

$$\lambda(0) = i + 1, \lambda(1) = 2, \ldots, \lambda(i - 1) = i,$$

$$\lambda(i) = 0, \lambda(i + 1) = 1, \lambda(i + 2) = i + 2, \ldots, \lambda(m) = m.$$

Then

$$d \circ \lambda = \langle d_{i+1}, d_2, \ldots, d_{i-1}, b, c, d_{i+1}, \ldots, d_m \rangle = \langle a_0, a_1, \ldots, a_{i-1}, b, c, a_{i+1}, \ldots, a_{m-1} \rangle.$$ 

Since $d \in \alpha$, also $d \circ \lambda \in \alpha$ by Proposition 8.8.

**Proposition 8.10.** If $\alpha$ is a countable subset of $\ll \omega M$ satistying R.P. and $a, b \in \alpha$, then $T(a) = T(b)$.

**Proof.** Say $a \in {}^m M$ and $b \in {}^n M$. Let $c \in \alpha$ be obtained by R.P.; thus $\forall i < m[a_i = \sum \{c_{f(i,j)} : j < n\}]$ and $\forall j < n[b_j = \sum \{c_{f(i,j)} : i < m\}]$. We claim that $a \prec c$. Let $\lambda(f(i,j)) = i$ for all $i < m, j < n$. Then

$$\forall i < m[a_i = \sum \{c_{f(i,j)} : j < n\} = \sum \{c_k : k < m \cdot n, \lambda(k) = i\}].$$

So $a \prec c$. By symmetry, $b \prec c$. Hence by Proposition 8.5, $T(a) = T(c) = T(b)$.

Now for any $\alpha \in \triangle M$ we let $T(\alpha) = T(a)$ for any $a \in \alpha$; this is justified by Proposition 8.10.

**Proposition 8.11.** $T : \triangle M \rightarrow M$ is a morphism of $m$-monoids.
**Proposition 9.3.** Suppose that \( n \in T \). Recall that \( W \in C.P. \), if \( \xi \in \{0, \sum \{a_i : i < m\}\} \). Then \( b_0 = \sum \{a_i : i < m, \lambda(i) = 0\} \), so \( a < b \). Hence by the C.P., \( b \in \alpha \).

**9. The derived monoid of \( \mathcal{W} \)**

Recall that \( \mathcal{W} \) is \( \omega_1 \) with a new smallest element \( o \) adjoined; it is an \( m \)-monoid.

For each \( \alpha \in (\Delta \mathcal{W})^* \) define \( \theta_\alpha : \omega_1 \to \omega + 1 \) by setting, for each \( \zeta < \omega_1 \),

\[
\theta_\alpha(\zeta) = \text{lub}\{\{|i \in \text{dmn}(a) : a_i = \zeta\} : a \in \alpha \}.
\]

**Proposition 9.1.** If \( \alpha \in (\Delta \mathcal{W})^* \), and \( a \in \alpha \) with \( a \in m \mathcal{W} \), then \( T(\alpha) = \max\{a_i : i < m, a_i \neq o, a \in \alpha\} \), a countable ordinal. \( \square \)

**Proposition 9.2.** Suppose that \( \alpha \in (\Delta \mathcal{W})^* \) and \( T(\alpha) = \eta \). Then

(i) \( \theta_\alpha(\zeta) = 0 \) for all \( \zeta > \eta \).

(ii) \( 0 < \theta_\alpha(\eta) \leq \omega \).

(iii) If \( \xi = \min\{\zeta < \omega_1 : \theta_\alpha(\zeta) \neq 0\} \), then

\( a \) \( \xi \leq \eta \).

(b) For each \( n \in \omega \) there is an \( a \in \alpha \) such that \( a \) has at least \( n \) entries equal to \( \xi \).

(c) \( \theta_\alpha(\xi) = \omega \).

(d) \( \forall \zeta < \xi[\theta_\alpha(\zeta) = 0] \).

**Proof.** (i), (ii), and (iii)(a) are clear. We prove (iii)(b) by induction on \( n \). It is clear for \( n = 0, 1 \). Now assume it for \( n \). Say \( a \in \alpha \) and \( a \) has at least \( n \) entries equal to \( \xi \). Say \( a \in m \mathcal{W} \), \( i < m \), and \( a_i = \xi \). By Proposition 8.10 there are \( b, c \in \mathcal{W}^* \) such that \( \xi = a_i = b + c \) and \( \langle a_0, \ldots, a_{i-1}, b, c, a_{i+1}, \ldots, a_m \rangle \in \alpha \). Since \( \max(b, c) = \xi \) by the minimality of \( \xi \) both \( \xi \leq b \) and \( \xi \leq c \), we have \( b = c = \xi \). Thus \( \langle a_0, \ldots, a_{i-1}, b, c, a_{i+1}, \ldots, a_m \rangle \) has one more \( \xi \) than \( a \), and hence it has at least \( n + 1 \) entries equal to \( \xi \). This proves (iii)(b). (iii)(c) follows from (iii)(b). (iii)(d) is obvious. \( \square \)

Let \( \mathcal{N}^* \) be the set of all mappings \( \theta : \omega_1 \to \omega + 1 \) such that there exist \( \xi \leq \eta \) ordinals in \( \omega_1 \) such that \( \theta(\zeta) = 0 \) for \( \zeta < \xi \) and \( \eta < \zeta < \omega_1 \), with \( \theta(\xi) = \omega \) and \( 1 \leq \theta(\eta) \leq \omega \).

Let \( \mathcal{N} = \mathcal{N}^* \) together with a new element \( o \). We define \( + \) on \( \mathcal{N} \) by setting, for \( \theta, \psi \in \mathcal{N}^* \),

\( o + o = o; \theta + o = \theta = \theta + o; (\theta + \psi)(\zeta) = \theta(\zeta) + \psi(\zeta) \) for each \( \zeta < \omega_1 \).

**Proposition 9.3.** Suppose that \( \alpha \in \Delta M \), \( m, n \in \omega \backslash \{0\} \), \( a \in m M \), \( b \in n M \), \( a, b \in \alpha \), \( \lambda \) and \( \mu \) are permutations of \( m, n \) respectively, \( c \in m-n M \), \( c \in \alpha \), \( \forall i < m[a_{\lambda(i)} = \sum\{c_{f(i,j)} : \}

64
By symmetry, \( b_i < n \), \( \forall j < n \), \( b_{\mu(j)} = \sum \{ c_f(i,j) : i < m \} \), \( d \in m \cdot n \), and \( \forall i < m \forall j < n \), \( d(f(i,j) = c_f(\lambda^{-1}(i), \mu^{-1}(j))) \).

Then \( d \in \alpha \), \( \forall i < m \not\exists_j \not\exists_k < n \), \( a_i = \sum \{ d_f(i,j) : j < n \} \), and \( \forall j < n \), \( b_j = \sum \{ d_f(i,j) : i < m \} \).

**Proof.** Let \( \rho \) be the permutation of \( m \cdot n \) such that \( \rho(f(i,j)) = f(\lambda^{-1}(i), \mu^{-1}(j)) \) for all \( i < m \) and \( j < n \). Then \( d = c \circ \rho \). Hence \( d \in \alpha \) by Proposition 8.8.

Now take any \( i < m \). Then

\[
a_i = a_{\lambda^{-1}(i)} = \sum \{ c_{f(\lambda^{-1}(i),j)} : j < n \} = \sum \{ c_{f(\lambda^{-1}(i), \mu^{-1}(j))} : j < n \} = \sum \{ d_f(i,j) : j < n \}.
\]

By symmetry, \( b_j = \sum \{ d_f(i,j) : i < m \} \) for all \( j < n \).

**Proposition 9.4.** Suppose that \( \alpha \in \Delta M \), \( m, n \in \omega \setminus \{0\} \), \( a \in m \cdot n \), \( b \in n \cdot M \), \( m < n \), \( a' \in n \cdot M \), \( a \subseteq a' \), \( a, a', b \in \alpha \), \( \forall i \in n \cdot m \), \( a'_i = o \), \( c \in n \cdot n \cdot M \), \( c \in \alpha \), \( \forall i < n \), \( a'_i = \sum \{ c_{f_{mn}(i,j)} : j < n \} \), \( d \in n \cdot m \cdot n \), and \( \forall i < m \forall j < n \), \( d_{f_{mn}(i,j)} = c_{f_{mn}(i,j)} \).

Then \( d \in \alpha \), \( \forall i < m \), \( a_i = \sum \{ d_{f_{mn}(i,j)} : j < n \} \) and \( \forall j < n \), \( b_j = \sum \{ d_{f_{mn}(i,j)} : i < m \} \).

**Proof.** Define \( \lambda : n \cdot n \rightarrow m \cdot n \) by setting, for any \( i < n \) and \( j < n \),

\[
\lambda(f_{mn}(i,j)) = \begin{cases} f_{mn}(i,j) & \text{if } i < m, \\ f_{mn}(m-1,j) & \text{if } m \leq i < n. \end{cases}
\]

Now \( \forall i \in n \cdot m \), \( o = a'_i = \sum \{ c_{f_{mn}(i,j)} : j < n \} \) and hence

\[
\forall i \in n \cdot m \forall j < n \), \( c_{f_{mn}(i,j)} = o \).
\]

Now \( \forall i < m \forall j < n \forall k < n \forall l < n \lambda(f_{mn}(k,l)) = f_{mn}(i,j) \) iff \( (k = i \text{ and } l = j) \) or \( (m \leq k < n \text{ and } i = m-1 \text{ and } l = j) \). So if \( i < m-1 \) and \( j < n \) then

\[
\sum \{ c_{f_{mn}(k,l)} : \lambda(f_{mn}(k,l)) = f_{mn}(i,j) \} = c_{f_{mn}(i,j)} = d_{f_{mn}(i,j)},
\]

while if \( i = m-1 \) and \( j < n \) then

\[
\sum \{ c_{f_{mn}(k,l)} : \lambda(f_{mn}(k,l)) = f_{mn}(i,j) \} = c_{f_{mn}(i,j)} + \sum \{ c_{f_{mn}(k,l)} : k \in n \cdot m, j < n \} = c_{f_{mn}(i,j)} = d_{f_{mn}(i,j)}.
\]

Hence \( c < d \), so \( d \in \alpha \) by C.P. Next, for any \( i < m \), \( a_i = a'_i = \sum \{ c_{f_{mn}(i,j)} : j < n \} = \sum \{ d_{f_{mn}(i,j)} : j < n \} \). By (*), for all \( j < n \), \( b_j = \sum \{ c_{f_{mn}(i,j)} : j < n \} = \sum \{ c_{f_{mn}(i,j)} : i < m \} = \sum \{ d_{f_{mn}(i,j)} : i < m \} \).

Note by symmetry that the modification of Proposition 9.4 by replacing \( m < n \) by \( n < m \) also holds.

**Proposition 9.5.** The following statement implies R.P.:
For all $\alpha \in \triangle \mathcal{W}$, all $m \in \omega \setminus \{0\}$ and for all $a, b \in m \mathcal{W}$, if $a, b \in \alpha$, $a_0 \geq a_1 \geq \cdots \geq a_{m-1}$ and $b_0 \geq b_1 \geq \cdots \geq b_{n-1}$, then there is a $c \in m \mathcal{mW}$ such that $c \in \alpha$ and 

\[ \forall i < m \{ a_i = \sum \{ c_{f_{mm}(i, j)} : j < m \} \} \] and 

\[ \forall j < m \{ b_j = \sum \{ c_{f_{mn}(i, j)} : i < j \} \} . \]

**Proof.** Assume the indicated statement, and suppose that $m, n \in \omega \setminus \{0\}$, $a \in m \mathcal{W}$, $b \in n \mathcal{W}$, and $a, b \in \alpha$. Let $\lambda$ be a permutation of $m$ and $\mu$ a permutation of $n$, such that 

\[ a\lambda(0) \geq a\lambda(1) \geq \cdots \geq a\lambda(m-1) \] and 

\[ b\mu(0) \geq b\mu(1) \geq \cdots \geq b\mu(n-1) . \] Let $a' = a\circ \lambda$ and $b' = b\circ \mu$. Wlog $m < n$. Let $a'' \in n \mathcal{W}$ be such that $a' \subseteq a''$ and $\forall i \in n \{ a''_i = a'_i \}$. Then by the assumed statement, there is a $c \in n \mathcal{mW}$ such that $c \in \alpha$ and 

\[ \forall i < n \{ a''_i = \sum \{ c_{f_{mn}(i, j)} : j < n \} \} \] and 

\[ \forall j < n \{ b'_j = \sum \{ c_{f_{mn}(i, j)} : i < j \} \} . \] Define $d \in m \mathcal{mW}$ by setting $d_{f_{mn}(i, j)} = c_{f_{mn}(i, j)}$ for all $i < m$ and $j < n$. Then by Proposition 9.4, $d \in \alpha$, $\forall i < m \{ a'_i = \sum \{ d_{f_{mn}(i, j)} : j < n \} \}$ and 

\[ \forall j < n \{ b'_j = \sum \{ d_{f_{mn}(i, j)} : i < j \} \} . \] Now define $e \in m \mathcal{mW}$ by setting, for $i < m$ and $j < n$, 

\[ e_{f_{mn}(i, j)} = d_{f_{mn}(\lambda^{-1}(i), \mu^{-1}(j))} . \] Now 

\[ \forall i < m \{ a_{\lambda(i)} = a'_i = \sum \{ d_{f_{mn}(i, j)} : j < n \} \} \] and 

\[ \forall j < n \{ b_{\mu(j)} = b'_j = \sum \{ d_{f_{mn}(i, j)} : i < j \} \} . \] Hence by Proposition 9.3, $e \in \alpha$, $\forall i \in m \{ a_i = \sum \{ f_{f_{mn}(i, j)} : j < n \} \}$, and 

\[ \forall j < n \{ b_j = \sum \{ f_{f_{mn}(i, j)} : i < m \} \} . \]

**Proposition 9.6.** Let $f(\theta) = \alpha$, and for any $\alpha \in (\triangle \mathcal{W})^*$ let $f(\alpha) = \theta_\alpha$. Then $f$ is an isomorphism of $(\triangle \mathcal{W}, +, O)$ onto $(\alpha, +, O)$.

**Proof.** By Proposition 9.2, $f$ maps $\triangle \mathcal{W}$ into $\mathcal{W}$. Now we check that $f$ preserves $+$. Suppose that $x, y \in \triangle \mathcal{W}$.

**Case 1.** $x = y = O$. Then $f(x + y) = f(O + O) = f(O) = 0 + 0 = f(x) + f(y)$.

**Case 2.** $x = \alpha \in (\triangle \mathcal{W})^*$ and $y = 0$. Then $f(x + y) = f(\alpha + 0) = f(\alpha) + f(0) = \theta_\alpha + 0 = f(x) + f(y)$.

**Case 3.** $x = 0$ and $y = \beta \in (\triangle \mathcal{W})^*$. Symmetric to Case 2.

**Case 4.** $x = \alpha \in (\triangle \mathcal{W})^* \text{ and } y = \beta \in (\triangle \mathcal{W})^*$. Then $f(x + y) = f(\alpha + \beta) = \theta_{\alpha + \beta}$, $f(x) = f(\alpha) = \theta_\alpha$, and $f(y) = f(\beta) = \theta_\beta$. So we want to show that $\theta_{\alpha + \beta} = \theta_\alpha + \theta_\beta$. Take any $\zeta < \omega_1$. Then if $a \in \alpha, b \in \beta$, and $\text{dmm}(a) = \text{dmm}(b)$, then 

\[ \{ i \in \text{dmm}(a) : a_i + b_i = \zeta \} \subseteq \{ i \in \text{dmm}(a) : a_i = \zeta \} \cup \{ i \in \text{dmm}(b) : b_i = \zeta \} , \]

and so $\theta_{\alpha + \beta}(\zeta) \leq \theta_\alpha(\zeta) + \theta_\beta(\zeta)$. On the other hand, if $a \in \alpha$ and $b \in \beta$, then $a \circ c \in \alpha$ and $d \circ b \in \beta$, where $c$ is the sequence of o’s with domain $\text{dmm}(b)$ and $d$ is the sequence of o’s with domain $\text{dmm}(a)$. Then $(a \circ c) + (d \circ b) \in \alpha + \beta$, and

\[ \{ i \in \text{dmm}(a) + \text{dmm}(b) : ((a \circ c) + (d \circ b))_i = \zeta \} \]

\[ = \{ i \in \text{dmm}(a) : a_i = \zeta \} \cup \{ i \in \text{dmm}(b) : b_i = \zeta \} . \]

It follows that $\theta_\alpha(\zeta) + \theta_\beta(\zeta) \leq \theta_{\alpha + \beta}(\zeta)$.

So $f$ preserves $+$.

Now we define a function $g$ with domain $\mathcal{W}$ which will turn out to be the inverse of $f$. Let $g(o) = O$. Now suppose that $\theta \in \mathcal{W}^*$. Say $\xi \leq \eta$ are ordinals in $\omega_1$ such that $\theta(\xi) = 0$ for $\zeta < \xi$ and $\eta < \zeta < \omega_1$, with $\theta(\xi) = \omega$ and $1 \leq \theta(\eta) \leq \omega$. Let

\[ g(\theta) = \alpha = \{ a \in << \omega \mathcal{W} : T(a) = \eta \} \text{ and } \forall \zeta < \omega_1 \{ \{ i \in \text{dmm}(a) : a_i = \zeta \} \leq \theta(\zeta) \} . \]

Now $\langle \eta \rangle, \langle \eta, o \rangle, \langle \eta, o, o \rangle, \ldots \in \alpha$, so $\alpha$ is infinite. Each member of $\alpha$ is a finite sequence of members of $\{ o \} \cup (\eta + 1)$, and so $\alpha$ is countable. Now we check that $\alpha \in \triangle \mathcal{W}$.
We prove the statement by induction on $\alpha$, $\beta < \omega \mathcal{W}$, and $\alpha < \beta$. Say $\text{dmm}(a) = m$, $\text{dmm}(b) = n$, $\lambda : m \to n$, and $\forall j < n[b_j = \sum \{a_i : i < m, \lambda(i) = j\}]$. Then $T(b) = T(a) = \eta$. For any $\zeta < \eta$ we have $|\{j < n : b_j = \zeta\}| \leq |\{i < m : a_i = \zeta\}| = \theta(\zeta)$. So C.P. holds.

S.P.: Suppose that $m \in \omega \setminus \{0\}$, $a \in m \mathcal{W}$, $\alpha \in \alpha$ and $a_0 \neq o$. Now $\xi \leq a_0$ and $a_0 = a_0 + \xi$. Clearly

$$\langle a_0, \xi, a_1, a_2, \ldots, a_{m-1} \rangle \in \alpha.$$  

So S.P. holds.

R.P.: By Proposition 9.5 it suffices to prove the following statement:

\((*)\) For all $m \in \omega \setminus \{0\}$ and for all $a, b \in m \mathcal{W}$, if $a, b \in \alpha$, $a_0 \geq a_1 \geq \ldots \geq a_{m-1}$ and $b_0 \geq b_1 \geq \cdots \geq b_{m-1}$, then there is a $c \in m \cup m \mathcal{W}$ such that $c \in \alpha$, $\forall i < m[a_i = \sum \{c_{f_{mn}(i,j)} : j < m\}]$, and $\forall j < m[b_j = \sum \{c_{f_{mn}(i,j)} : i < m\}]$.

We prove the statement by induction on $m$. First assume that $m = 1$ and $a, b \in \alpha$. Since $\text{dmm}(a) = \text{dmm}(b) = 1$ and $T(a) = T(b)$, it follows that $a_0 = \eta = b_0$, hence $a = b$. Then we can take $c = a$.

Now assume the result for $m$, and suppose that $a, b \in \alpha$, $a \in \alpha$, $a_0 \geq a_1 \geq \cdots \geq a_m$ and $b_0 \geq b_1 \geq \cdots \geq b_m$. Let $a' = a \upharpoonright m$ and $b' = b \upharpoonright m$. Let $\lambda : m + 1 \to m$ be such that $\lambda \in m$ is the identity, and $\lambda(m) = m - 1$. Then $\forall j < m[a'_j = a_j = \sum \{a_i : i < m + 1, \lambda(i) = j\}]$, so $a < a'$, hence $a' \in \alpha$ by C.P. Similarly, $b' \in \alpha$.

Now suppose that $c \in m \mathcal{W}$. Then $\forall i < m[a'_i = \sum \{c_{f_{i,m}(j)} : j < m\}]$, and $\forall j < m[b'_j = \sum \{c_{f_{i,j}} : i < m\}]$. Note that $a_m = \xi$ or $a_m = o$; similarly, $b_m = \xi$ or $b_m = o$. Let $d \in (m + 1) \mathcal{W}$. Extend $c$, with

$$d_{f_{m,0}} = a_m;$$
$$d_{f_{m,i}} = o\text{ if }0 < i \leq m;$$
$$d_{f_{0,m}} = b_m;$$
$$d_{f_{i,m}} = o\text{ if }0 < i < m.$$  

We have $a_m = \sum \{d_{f_{m,i}} : i \leq m\}$ and $b_m = \sum \{d_{f_{i,m}} : i \leq m\}$. Also, if $i < m$, then $\sum_{j \leq m} d_{ij} = \sum_{j < m} c_{ij} + d_{im} = a_i + o = a_i$, and similarly $\sum_{i \leq m} d_{ij} = b_j$. Also, $d \in \alpha$ by the definition of $g(\theta)$, proving $(*)$ for $m + 1$.

So R.P. holds.

Hence $g$ maps into $\triangle \mathcal{W}$.

Now we claim that $f(g(\theta)) = \theta$ for all $\theta \in \mathcal{N}$. For, $f(g(o)) = f(O) = o$. Now let $\theta \in \mathcal{N}^\ast$. Let $\alpha = g(\theta)$. Choose $\xi \leq \eta$ ordinals in $\omega_1$ such that $\forall \zeta \in \xi[\theta(\zeta) = 0]$, $\forall \zeta > \eta[\theta(\zeta) = 0]$, $\theta(\xi) = \omega$ and $1 \leq \theta(\eta) \leq \omega$. Now suppose that $\zeta < \omega_1$.

Case 1. $\zeta < \xi$. Then by the definition of $g(\theta)$, for any $a \in \alpha$, $|\{i \in \text{dmm}(a) : a_i = \zeta\}| \leq \theta(\zeta) = 0$, so $f(g(\theta))(\zeta) = 0 = \theta(\zeta)$.

Case 2. $\zeta = \xi$. Clearly there are members of $\alpha$ with an arbitrarily large number of $\zeta$'s, so $\theta_\alpha(\xi) = \omega = \theta(\xi)$.

Case 3. $\xi < \zeta \leq \eta$. Then by the definition of $g(\theta)$, $\forall a \in \alpha[|\{i \in \text{dmm}(a) : a_i = \zeta\}| \leq \theta(\zeta)]$. Hence $\theta_\alpha(\zeta) \leq \theta(\zeta)$. Now we consider subcases.

Subcase 3.1. $\zeta < \eta$ and $\theta(\zeta) < \omega$. Let $\text{dmm}(a) = \theta(\zeta) + 1$ with $a_0 = \eta$ and $a_i = \zeta$ for $0 < i \leq \theta(\zeta)$. Then $a \in \alpha$, so $\theta_\alpha(\zeta) = \theta(\zeta)$.
Subcase 3.2. \( \zeta = \eta \) and \( \theta(\zeta) < \omega \). Let \( \text{dmn}(a) = \theta(\zeta) \) with \( a_i = \zeta \) for \( i < \theta(\zeta) \). Then \( a \in \alpha \), so \( \theta_\alpha(\zeta) = \theta(\zeta) \).

Subcase 3.3. \( \theta(\zeta) = \omega \) and \( \zeta = \eta \). For each \( m \in \omega \setminus \{0\} \) let \( a \) have domain \( m \) with constant value \( \eta \). Then \( \theta_\alpha(\eta) = \omega = \theta(\eta) \).

Subcase 3.4. \( \theta(\zeta) = \omega \) and \( \zeta \neq \eta \). For each \( m \in \omega \setminus \{0\} \) let \( a \) have domain \( m + 1 \) with \( a_0 = \eta \) and \( a_i = \zeta \) for all \( i = 1, \ldots, m \). This shows that \( \theta_\alpha(\zeta) = \omega = \theta(\zeta) \).

Case 4. \( \eta < \zeta \). Since \( T(\alpha) = \eta \), for any \( a \in \alpha \) we have \( \{ i \in \text{dmn}(a) : a_i = \zeta \} = \emptyset \). Hence \( \theta_\alpha(\zeta) = 0 = \theta(\zeta) \).

This proves the claim that \( f(g(\theta)) = \theta \) for all \( \theta \in \mathcal{N} \).

Now to prove that \( g(f(\alpha)) = \alpha \) for all \( \alpha \in \triangle \mathbb{W} \), first note that \( g(f(\omega)) = g(\omega) = \omega \).

Now suppose that \( \alpha \in (\triangle \mathbb{W})^* \); we want to show that \( \alpha_{\theta_\alpha} = \alpha \). Let \( \eta = T(\alpha) \). If \( a \in \alpha \), then \( T(a) = \eta \) and \( \forall \zeta < \omega \Rightarrow \{ i \in \text{dmn}(a) : a_i = \zeta \} \leq \theta_\alpha(\zeta) \), hence \( a \in \alpha_{\theta_\alpha} \). Thus \( \alpha \leq \alpha_{\theta_\alpha} \).

Now since \( \alpha \in (\triangle \mathbb{W})^* \), \( \theta_\alpha \) is given by

\[
\theta_\alpha(\zeta) = \text{lub}\{\{ i \in \text{dmn}(a) : a_i = \zeta \} : a \in \alpha \}.
\]

Then we have \( \xi \) and \( \eta \) given by Proposition 9.2, and

\[
\alpha_{\theta_\alpha} = \{ a \in \triangle \mathbb{W}^* : T(a) = \eta \text{ and } \forall \zeta < \eta \Rightarrow \{ i \in \text{dmn}(a) : a_i = \zeta \} \leq \theta_\alpha(\zeta) \}.
\]

Now suppose that \( a \in \alpha_{\theta_\alpha} \). Say \( \zeta_0 > \cdots > \zeta_n \) are the distinct values of \( a \), with multiplicities \( k_0, \ldots, k_n \geq 1 \). Note that when R.P. is applied to \( a' \) and \( b' \), obtaining \( c \), if \( \zeta < \omega_1 \) occurs \( l \) times in \( a' \), then it occurs at least \( l \) times in \( c \). Now for each \( i \leq n \) we have \( k_i \leq \theta_\alpha(\zeta_i) \), so there is a \( b_i \in \alpha \) such that \( \{ j \in \text{dmn}(b_i) : b_{ij} = \zeta_i \} \geq k_i \). Applying R.P. to all the \( b_i \) in succession, we end up with \( c \in \alpha \) such that \( \forall i \leq n \Rightarrow \{ j \in \text{dmn}(c) : c_j = \zeta_i \} \geq k_i \). Note that \( c \) may have values in addition to \( \zeta_0, \ldots, \zeta_n \). Say \( m = \text{dmn}(c) \). By Proposition 8.8 we may assume that the entries of \( c \) are in decreasing order; say

\[
\begin{align*}
c_i &= \zeta_0 \text{ for } i = 0, \ldots, k_0 - 1; k_0 \leq l_1; \\
c_i &= \zeta_1 \text{ for } i = l_1, \ldots, l_1 + k_1 - 1; l_1 + k_1 \leq l_2; \\
&\vdots \\
c_i &= \zeta_n \text{ for } i = l_n, \ldots, l_n + k_n - 1; l_n + k_n \leq m.
\end{align*}
\]

Let \( p = \text{dmn}(a) \), and define \( \lambda : m \to p \) as follows:

\[
\begin{align*}
\lambda(j) &= j \text{ for } 0 \leq j < k_0; \\
\lambda(j) &= k_0 - 1 \text{ for } k_0 \leq j < l_1; \\
\lambda(j) &= j \text{ for } l_1 \leq j < l_1 + k_1; \\
\lambda(j) &= l_1 + k_1 - 1 \text{ for } l_1 + k_1 \leq j < l_2; \\
&\vdots \\
\lambda(j) &= j \text{ for } l_n \leq j < l_n + k_n; \\
\lambda(j) &= l_n + k_n - 1 \text{ for } l_n + k_n \leq j < m.
\end{align*}
\]
Now let \( d_j = \sum \{ c_i : i < m, \lambda(i) = j \} \) for all \( j < n \). Thus \( c \prec d \), so \( d \in \alpha \) by C.P. Now \( d \) has entries, in order, \( \zeta_0, \ldots, \zeta_n \) with multiplicities \( k_0, \ldots, k_n \). \( a \) is a permutation of \( d \), so \( a \in \alpha \) by Proposition 8.8.

\[ \square \]

10. Derived measures

For the notion of an \( m \)-monoid, \( M \)-measure, and \( \mathcal{M}(M) \) see section 7. We apply these notions with \( A = \text{Fr}(\omega) \). For any \( \sigma \in \mathcal{M}(M) \) we define \( \Delta \sigma \) with domain \( \text{Fr}(\omega) \) by

\[
(\Delta \sigma)(x) = \{ (\sigma(y_0), \ldots, \sigma(y_{n-1})) : n \in \omega, x = y_0 + y_1 + \cdots + y_{n-1} \}.
\]

**Proposition 10.1.** If \( \sigma \in \mathcal{M}(M) \), then \( \Delta \sigma \in \mathcal{M}(\Delta M) \).

**Proof.** Recall from Proposition 8.7 that \( \Delta M \) is an \( m \)-monoid. Clearly \( (\Delta \sigma)(0) = O \), the zero of \( \Delta M \). Conversely, if \( (\Delta \sigma)(x) = O \), then \( x = 0 \). Now suppose that \( x \neq 0 \). Then clearly \( (\Delta \sigma)(x) \) is a countable subset of \( <\omega M \). To show that \( (\Delta \sigma)(x) \in \Delta M \) we need to check C.P., R.P. and S.P.

C.P.: Suppose that \( a, b \in <\omega M \), \( a \in (\Delta \sigma)(x) \), and \( a \prec b \). Say \( a = (\sigma(y_0), \ldots, \sigma(y_{m-1})) \) with \( m \in \omega \) and \( x = y_0 + y_1 + \cdots + y_{m-1} \), \( b \) has length \( n \), \( \lambda : m \to n \), and \( \forall j < n [b_j = \sum \{ a_i : i < m, \lambda(i) = j \}] \). For each \( j < n \) let \( z_j = \sum \{ y_i : i < m, \lambda(i) = j \} \). Then \( x = z_0 + z_1 + \cdots + z_{n-1} \) and for all \( j < n \),

\[
b_j = \sum \{ \sigma(y_i) : i < n, \lambda(i) = j \} = \sigma \left( \sum \{ y_i : i < n, \lambda(i) = j \} \right) = \sigma(z_j).
\]

Thus \( b \in (\Delta \sigma)(x) \). This proves C.P.

R.P.: Suppose that \( m, n \in \omega \setminus \{ 0 \}, a \in ^m M, b \in ^n M \), and \( a, b \in (\Delta \sigma)(x) \). Say \( x = y_0 + y_1 + \cdots + y_{m-1} \), \( \forall i < m[a_i = \sigma(y_i)] \), \( x = z_0 + z_1 + \cdots + z_{n-1} \), and \( \forall i < n [b_i = \sigma(z_i)] \). For \( i < m \) and \( j < n \) let \( w_{f(i,j)} = y_i \cdot z_j \). Then \( x = \sum w_{f(i,j)} \) (disjoint sum). Let \( c_{f(i,j)} = \sigma(y_i \cdot z_j) \) for all \( i < m \) and \( j < n \). Then \( \forall i < m[a_i = \sum \{ c_{f(i,j)} : j < n \}] \) and \( \forall j < n [b_j = \sum \{ c_{f(i,j)} : i < m \}] \). So R.P. holds.

S.P.: Suppose that \( m \in \omega \setminus \{ 0 \}, a \in ^m M, a_0 \neq o \), and \( a \in (\Delta \sigma)(x) \). Say \( x = y_0 + y_1 + \cdots + y_{m-1} \) and \( a = (\sigma(y_0), \sigma(y_1), \ldots, \sigma(y_{m-1})) \). Thus \( \sigma(y_0) \neq o \). So \( y_0 \neq 0 \). Write \( y_0 = u+v \) with \( u, v \neq 0 \). Then \( x = u+v+y_1+\cdots+y_{m-1} \). Hence

\[
(\sigma(u), \sigma(v), \sigma(y_1), \ldots, \sigma(y_{m-1}) \in (\Delta \sigma)(x).
\]

This proves S.P.

Thus \( (\Delta \sigma)(x) \in \Delta M \).

To prove that \( (\Delta \sigma) \) preserves +, suppose that \( x, y \in \text{Fr}(\omega) \) are disjoint. Suppose that \( u \in (\Delta \sigma)(x) + (\Delta \sigma)(y) \). Say \( a \in (\Delta \sigma)(x) \), \( b \in (\Delta \sigma)(y) \), with \( \text{dmm}(a) = \text{dmm}(b) \), \( x = z_0 + z_1 + \cdots + z_{m-1}, a = (\sigma(z_0), \sigma(z_1), \ldots, \sigma(z_{m-1})) \), \( y = w_0 + w_1 + \cdots + w_{m-1} \),

\[
b = (\sigma(w_0), \sigma(w_1), \ldots, \sigma(w_{n-1})),
\]

and \( a + b = u \). Now

\[
x + y = (z_0 + w_0) + (z_1 + w_1) + \cdots + (z_{m-1} + w_{m-1})
\]
Thus \( v = a + b \in (\Delta \sigma)(x + y) \). If follows that \((\Delta \sigma)(x) + (\Delta \sigma)(y) \subseteq (\Delta \sigma)(x + y)\).

Conversely, suppose that \( v \in (\Delta \sigma)(x + y) \). Say \( x + y = z_0 + z_1 + \cdots + z_{m-1} \) and \( v = \langle \sigma(z_0), \sigma(z_1), \ldots, \sigma(z_{m-1}) \rangle \). Then \( x = (x \cdot z_0) + (x \cdot z_1) + \cdots + (x \cdot z_{m-1}) \), and hence \( a \triangleq \langle \sigma(x \cdot z_0), \sigma(x \cdot z_1), \ldots, \sigma(x \cdot z_{m-1}) \rangle \in (\Delta \sigma)(x) \). Similarly, \( b \triangleq \langle \sigma(y \cdot z_0), \sigma(y \cdot z_1), \ldots, \sigma(y \cdot z_{m-1}) \rangle \in (\Delta \sigma)(y) \). So \( a + b \in (\Delta \sigma)(x) + (\Delta \sigma)(y) \). Now

\[
a + b = \langle \sigma(x \cdot z_0) + \sigma(y \cdot z_0), \sigma(x \cdot z_1) + \sigma(y \cdot z_1), \ldots, \sigma(x \cdot z_{m-1}) + \sigma(y \cdot z_{m-1}) \rangle = \langle \sigma(z_0), \sigma(z_1), \ldots, \sigma(z_{m-1}) \rangle = v.
\]

Thus \( v \in (\Delta \sigma)(x) + (\Delta \sigma)(y) \).

\[\square\]

**Proposition 10.2.** \( \forall x \in \text{Fr}(\omega)[T((\Delta \sigma)(x)) = \sigma(x)] \).

**Proof.** We have \((\Delta \sigma)(x) \in \Delta M\) by Proposition 10.1. Hence by the definition before Proposition 8.11, \( T((\Delta \sigma)(x)) = T(a) \) for any element \( a \) of \((\Delta \sigma)(x)\). So suppose that \( x = y_0 + y_1 + \cdots + y_{m-1} \) and \( a = \langle \sigma(y_0), \sigma(y_1), \ldots, \sigma(y_{m-1}) \rangle \). Hence \( T((\Delta \sigma)(x)) = T(a) = \sum_{i < m} \sigma(y_i) = \sigma(\sum_{i < m} y_i) = \sigma(x) \).

The definition of \( \Delta \sigma \) can also be made with respect to any BA isomorphic to \( \text{Fr}(\omega) \), and 10.1,10.2 hold.

**Proposition 10.3.** If \( z \in \text{Fr}(\omega) \), \( k : \text{Fr}(\omega) \to \text{Fr}(\omega) \restriction z \) is an isomorphism, and \( \sigma : \text{Fr}(\omega) \restriction z \to M \) is an \( M \)-measure on \( \text{Fr}(\omega) \restriction z \), then \( \Delta(\sigma \circ k) = (\Delta' \sigma) \circ k \). Here \( \Delta' \) refers to the above construction with \( A = \text{Fr}(\omega) \restriction z \).

**Proof.** Let \( x \in \text{Fr}(\omega) \). First suppose that \( a \in (\Delta(\sigma \circ k))(x) \). Say \( x = y_0 + y_1 + \cdots + y_{m-1} \) and \( a = \langle \sigma(k(y_0)), \sigma(k(y_1)), \ldots, \sigma(k(y_{m-1})) \rangle \). Then

\[
k(x) = k(y_0) + k(y_1) + \cdots + k(y_{m-1}),
\]

and so \( a \in (\Delta' \sigma)(k(x)) \).

Conversely, suppose that \( a \in (\Delta' \sigma)(k(x)) \). Say \( k(x) = z_0 + z_1 + \cdots + z_{m-1} \) and \( a = \langle \sigma(z_0), \sigma(z_1), \ldots, \sigma(z_{m-1}) \rangle \). Let \( y_i = k^{-1}(z_i) \) for all \( i < m \). Then \( x = y_0 + y_1 + \cdots + y_{m-1} \) and \( a = \langle \sigma(k(y_0)), \sigma(k(y_1)), \ldots, \sigma(l(y_{m-1})) \rangle \). Hence \( a \in (\Delta(\sigma \circ k))(x) \).

\[\square\]

**Proposition 10.4.** Suppose that \( \sigma, \tau \) are \( M \) measures with respect to \( \text{Fr}(\omega) \) and \( k \) is an automorphism of \( \text{Fr}(\omega) \). Then (i) implies (ii):

(i) \( \sigma = \tau \circ k \).

(ii) \( (\Delta \sigma) = (\Delta \tau) \circ k \).

**Proof.** Assume that \( k \) is an automorphism \( k \) of \( \text{Fr}(\omega) \) such that \( \sigma = \tau \circ k \). Then by Proposition 10.3, \( (\Delta \tau) \circ k = \Delta(\tau \circ k) = \Delta \sigma \).

The measure \( \Delta \sigma \) is the first derivative of \( \sigma \).
11. Existence of measures

**Proposition 11.1.** If \( \sigma, \tau \in \mathcal{M}(M) \), \( k \) is an automorphism of \( \text{Fr}(\omega) \), and \( \sigma = \tau \circ k \), then \( (\triangle \sigma)(1) = (\triangle \tau)(1) \).

**Proof.** Let \( a \in (\triangle \sigma)(1) \). Say

\[
1 = y_0 + y_1 + \cdots + y_{m-1}
\]

and \( a = \langle \sigma(y_0), \sigma(y_1), \ldots, \sigma(y_{m-1}) \rangle \).

Thus

\[
a = \langle \tau(k(y_0)), \tau(k(y_1)), \ldots, \tau(k(y_{m-1})) \rangle \text{ and } 1 = k(y_0) + k(y_1) + \cdots + k(y_{m-1}),
\]

so \( a \in (\triangle \tau)(1) \). The converse is symmetric. \( \square \)

**Proposition 11.2.** Suppose that \( M \) is an \( m \)-monoid, \( \alpha \subseteq \omega \setminus \{0\} \), \( a \in mM \), \( b \in nM \), \( a, b \in \alpha \), and \( p \geq n \).

Then there is a \( c \in m_p M \) such that \( c \in \alpha \), \( \forall i < m \sum_{j<p} c_{f_{mp}(i,j)} = a_i \), and \( c < b \).

**Proof.** By R.P. let \( d \in m^n M \) be such that \( d \in \alpha \), \( \forall i < m \sum_{j<n} [a_i = \sum_{j<n} d_{f_{mn}(i,j)} : j < n] \)

and \( \forall j < n [b_j = \sum_{j<n} d_{f_{mn}(i,j)} : i < m] \). Define \( c \in m_p M \) so that \( c_{f_{mp}(i,j)} = d_{f_{mn}(i,j)} \) for \( i < m \) and \( j < n \), and \( c_{f_{mp}(i,j)} = 0 \) for \( i < m \) and \( n \leq j < p \). Define \( \lambda : m \cdot n \rightarrow m \cdot p \)

by \( \lambda(f_{mn}(i,j)) = f_{mp}(i,j) \) for all \( i < m \) and \( j < n \). Now if \( i < m \), \( j < p \), \( k < m \), \( l < n \),

then \( \lambda(f_{mn}(k,l)) = f_{mp}(k,l) \), and so \( \lambda(f_{mn}(k,l)) = f_{mp}(i,j) \) iff \( j < n \), \( k = i \), and \( l = j \).

so \( \forall i < m \forall j < p [c_{f_{mp}(i,j)} = \sum_{j<n} d_{f_{mn}(i,j)} : k < m, l < n, \lambda(f_{mn}(k,l)) = f_{mp}(i,j)] \). Hence \( d < c \) and so \( c \in \alpha \). We have

\[
\forall i < m \left[ \sum_{j<p} c_{f_{mp}(i,j)} = \sum_{j<n} c_{f_{mp}(i,j)} = \sum_{j<n} d_{f_{mn}(i,j)} = a_i \right].
\]

Now define \( \mu : m \cdot p \rightarrow n \) by setting, for \( i < m \) and \( j < p \),

\[
\mu(i, j) = \begin{cases} 
  j & \text{if } j < n, \\
  n-1 & \text{if } n \leq j < p.
\end{cases}
\]

Now take any \( j < n \), and any \( k < m, l < p \).

*Case 1. \( j < n - 1 \). Then \( \mu(k,l) = j \) iff \( l = j \), so

\[
\sum \{ c_{f_{mp}(k,l)} : \mu(k,l) = j \} = \sum \{ c_{f_{mp}(i,j)} : i < m \} = \sum \{ d_{f_{mn}(i,j)} : i < m \} = b_j.
\]

*Case 2. \( j = n - 1 \). Then \( \mu(k,l) = j \) iff \( n - 1 \leq j < p \) Hence

\[
\sum \{ c_{f_{mp}(k,l)} : \mu(k,l) = j \} = \sum \{ c_{f_{mp}(i,n-1)} : i < m \} + \sum \{ c_{f_{mp}(k,l)} : k < m, n \leq l < p \}
\]

\[
= \sum \{ c_{f_{mp}(i,n-1)} : i < m \} = b_{n-1}.
\]

\( \square \)
Proposition 11.3. Suppose that $M$ is an m-monom, $\alpha \subseteq \omega \setminus \{0\}$, $a \in mM, b \in nM, a, b \in \alpha$, and $p \geq n$. Also suppose that $\forall i < m[a_i \neq 0]$

Then there is a $c \in m \cdot pM$ such that $c \in \alpha$, $\forall i < m[\sum_{j < p} c_{f_{mp}(i,j)} = a_i]$, $c < b$, and $\forall i < m \forall j < p[c_{f_{mp}(i,j)} \neq 0]$

Proof. Assume all the hypotheses. By R.P. let $d \in m \cdot nM$ be such that $d \in \alpha$, $\forall i < m[a_i = \sum j < n\{d_{f_{mn}(i,j)} : j < n\}]$ and $\forall j < n[b_j = \sum i < m\{d_{f_{mn}(i,j)} : i < m\}]$. Then for every $i < m$ there is a $j(i) < n$ such that $d_{f_{mn}(i,j(i))} \neq 0$. For each $i < m$ let $r(i) = \{|j < n : d_{f_{mn}(i,j)} \neq 0|\}$. Thus $1 \leq r(i) \leq n$. Let $g$ be a bijection from $\prod_{i < m}(n + p - r(i))$ onto some integer $q$. Applying Proposition 8.9 many times, for each $i < m$ we get elements $e_0^i, \ldots, e_{p-r(i)}^i$ such that each $e_s^i \neq 0$ and $d_{f_{mn}(i,j(i))} = e_0^i + \ldots + e_{p-r(i)}^i$ and $d' \in \alpha$, where for each $i < m$ and $k < n + p - r(i)$ we have

$$d'_{g(i,k)} = \begin{cases} d_{f_{mn}(i,k)} & \text{if } k < j(i), \\ e_i^j & \text{if } k = j(i) + l \text{ with } l \leq p - r(i), \\ d_{f_{mn}(i,k-p+r(i))} & \text{if } j(i) + p - r(i) < k < n + p - r(i). \end{cases}$$

Note that for each $i < m$ there are $n - r(i)$ 0’s among $\{d_{f_{mn}(i,j)} : j < n\}$. So if we delete all 0’s in $d'$ we obtain a sequence $c \in \alpha$ of length $\sum_{i < m}(r(i) + p - r(i)) = \sum_{i < m}p = m \cdot p$. We claim that $c$ is as desired.

For each $i < m$,

$$a_i = \sum j < n\{d_{f_{mn}(i,j)} : j < n\} = \sum j < n\{d'_{g(i,j)} : j < n + p - r(i)\} = \sum j < n c_{f_{mp}(i,j)}.$$ 

Now define $\lambda : q \rightarrow n$ as follows, with $i < m$:

$$\lambda(g(i,v)) = \begin{cases} v & \text{if } v < j(i); \\ j(i) & \text{if } v = j(i) + l \text{ with } l \leq p - r(i); \\ v - p + r(i) & \text{if } j(i) + p - r(i) < v < n + p - r(i). \end{cases}$$

Then for all $k < n$,

$$\sum \{d'_{g(i,v)} : i < m, j < n + p - r(i), \lambda(g(i,v)) = k\} = \sum \{d'_{g(i,k)} : i < m, v = k < j(i)\}$$

$$+ \sum \{d'_{g(i,v)} : i < m, k = j(i), v = j(i) + l, l \leq p - r(i)\}$$

$$+ \sum \{d'_{g(i,v)} : k = v - p + r(i), j(i) + p - r(i) < v < n + p - r(i)\}$$

$$= \sum \{d'_{g(i,k)} : i < m, v = k < j(i)\}$$

$$+ \sum \{d'_{g(i,v)} : i < m, k = j(i), v = j(i) + l, l \leq p - r(i)\}$$

$$+ \sum \{d'_{g(i,v)} : i < m, j(i) < k < n, j(i) + p - r(i) < v < n + p - r(i)\}$$

$$= 72$$
\[= \sum \{ d_{mn(i,k)} : i < m, k < j(i) \} + \sum \{ e_i^j : i < m, k = j(i), l \leq p - r(i) \} + \sum \{ d_{mn(i,k)} : i < m, j(i) < k \} = \sum \{ d_{mn(i,k)} : i < m \} = b_k. \]

Thus \( d' < b \). By Propositions 8.3 and 8.4, \( c < d' \). So \( c < b \). Clearly \( \forall i < m \forall j < p[c_{fmp(i,j)} \neq 0]. \)

We let \( \mathcal{D} = \langle \omega_2 \rangle \), and for each \( n \in \omega, \mathcal{D}_n = \langle n \rangle \). If \( a : \mathcal{D}_n \rightarrow M \), then we let \( a(\mathcal{D}_n) = \langle a(\text{lex}_{\mathcal{D}_n}(i)) : i < |\mathcal{D}_n| \rangle \), where \( \text{lex}_{\mathcal{D}_n} \) enumerates \( \mathcal{D}_n \) in lexicographic order. Thus \( a(\mathcal{D}_n) \in 2^n M \). Let \( a_n = a(\mathcal{D}_n) \). So \( a_n \in 2^n M \); \( a_n = \langle a(\text{lex}_{\mathcal{D}_n}(i)) : i < 2^n \rangle \).

Let \( \alpha \in [\langle \omega \rangle]^\omega \). An \( \alpha \)-tree is a mapping \( a : \mathcal{D} \rightarrow M \) such that:

(t1) \( \forall n \in \omega [a_n \in \alpha] \).

(t2) If \( f \in \mathcal{D}_n \) and \( n \leq m \), then \( a(f) = \sum \{ a(g) : f \subseteq g \in \langle m \rangle 2 \} \).

An \( \alpha \)-tree is dense iff:

(t3) \( \forall b \in \alpha \exists n \in \omega [a_n < b] \).

An \( \alpha \)-tree is homogeneous iff \( \forall f \in \mathcal{D} [a(f) \neq 0] \).

Proposition 11.4. If \( \alpha \in [\langle \omega \rangle]^\omega \) and \( \forall a, b \in \alpha [T(a) = T(b) \defeq T(\alpha)] \), and if \( a \) is an \( \alpha \)-tree, then \( a_0 = (T(\alpha)) \).

Proof. By (t1) we have \( a_0 \in \alpha \), so \( T(a_0) = T(\alpha) \). Now \( a_0 = a(\mathcal{D}_0) = a(\{\emptyset\}) = \langle a(\emptyset) \rangle \), so \( T(\alpha) = T(a_0) = a(\emptyset) \), hence \( a_0 = (T(\alpha)) \).

Proposition 11.5. If \( \alpha \in [\langle \omega \rangle]^\omega \), \( a \) is an \( \alpha \)-tree, and \( n \leq m \), then \( a_m < a_n \).

Proof. Define \( \lambda : 2^m \rightarrow 2^n \) by setting, for any \( i < 2^m \), \( \lambda(i) = \text{lex}_{\mathcal{D}_n}^{-1}(\text{lex}_{\mathcal{D}_m}(i) \upharpoonright n) \). Then for any \( j < 2^n \),

\[
\sum \{ a_m(i) : i < 2^m \text{ and } \lambda(i) = j \} = \sum \{ a_m(i) : i < 2^m \text{ and } \text{lex}_{\mathcal{D}_m}(\text{lex}_{\mathcal{D}_m}(i) \upharpoonright n) = j \} = \sum \{ a_m(i) : i < 2^m \text{ and } \text{lex}_{\mathcal{D}_m}(i) \upharpoonright n = \text{lex}_{\mathcal{D}_m}(j) \} = \sum \{ a(g) : g \in \langle m \rangle 2 \text{ and } g \upharpoonright n = \text{lex}_{\mathcal{D}_m}(j) \} = a(\text{lex}_{\mathcal{D}_m}(j)) = a_n(j). \]

Proposition 11.6. Let \( M \) be an \( m \)-monoid, and \( \alpha \) a countable subset of \( \langle \omega \rangle^\omega \) which satisfies C.P. and R.P. Then there is a dense \( \alpha \)-tree \( a \).

Proof. Note that \( (T(\alpha)) \in \alpha \) by the proof of Proposition 8.6. Let \( \alpha = \{b_0, b_1, \ldots\} \) with \( b_0 = (T(\alpha)) \). We define \( m_n \) and \( a : \bigcup_{k \leq m_n} \mathcal{D}_k \rightarrow M \) by induction on \( n \) so that the following conditions hold:

73
11.6(1) \( \forall k \leq m_n [a_k \in \alpha] \).

11.6(2) \( \forall k \leq m_n \forall f \in \mathcal{D}_k [a(f) = \sum \{ a(g) : g \in \mathcal{D}_{m_n}, f \subseteq g \}] \).

11.6(3) \( a_{m_n} \) refines each \( b_j \) with \( j \leq n \).

11.6(4) \( n \leq m_n \).

Note that \( \mathcal{D}_0 = \{0\} \). We let \( m_0 = 0 \) and \( a(\emptyset) = \langle T(\alpha) \rangle \). Thus \( a_0 = a(\mathcal{D}_0) = a(\emptyset) = \langle T(\alpha) \rangle \). So 11.6(1)–11.6(4) hold for \( n = 0 \). Now suppose that \( n \) and \( a : \bigcup_{k \leq m_n} \mathcal{D}_k \to M \) have been defined so that 11.6(1)–11.6(4) hold. Note that \( a_{m_n} \) and \( b_{n+1} \) are in \( \alpha \). Say \( b_{n+1} \in pM \). Take any \( 2^s \geq p + 1 \). Thus \( s \geq 1 \) since \( p \geq 1 \). Now we apply Proposition 11.2 with \( m, n, a, b, \alpha, p \) replaced by \( 2^m, n, a_{m_n}, b_{n+1}, m, 2^s \). This gives a \( c \in 2^{m_n} \cdot 2^s M \) such that

11.6(5) \( c \in \alpha \).

11.6(6) \( \forall i < 2^{m_n} \sum_{j < 2^s} c_{f2^{m_n}2^s(i,j)} = a_{m_n}(i) = a(\text{lex}_{D_{m_n}}(i)) \),

and

11.6(7) \( c < b_{n+1} \).

Let \( m_{n+1} = m_n + s \). Since \( s > 0 \), we have \( n + 1 \leq m_{n+1} \). So 11.6(4) holds for \( n + 1 \). For each \( g \in \mathcal{D}_{m_{n+1}} \) let \( a(g) = c_{f2^{m_n}2^s(i,j)} \) with \( i = \text{lex}^{-1}_{\mathcal{D}_{m_n}}(g \upharpoonright m_n) \), and \( j = \text{lex}^{-1}_s(\langle g_{m_n + k} : k < s \rangle) \). For \( m_n < k < m_{n+1} \) and \( g \in \mathcal{D}_k \) let \( a(g) = \sum \{ a(h) : h \in \mathcal{D}_{m_{n+1}}, g \subseteq h \} \). Now suppose that \( g \in \mathcal{D}_{m_n} \). Let \( \text{lex}_{\mathcal{D}_{m_n}}(i) = g \). Then

11.6(8) \( a(g) = a(\text{lex}_{\mathcal{D}_{m_n}}(i)) = \sum_{j < 2^s} c_{f2^{m_n}2^s(i,j)} = \sum \{ a(h) : h \in \mathcal{D}_{m_{n+1}}, g \subseteq h \} \).

Now for \( k \leq m_n \) and \( g \in \mathcal{D}_k \) we have

\[
a(g) = \sum \{ a(h) : h \in \mathcal{D}_{m_n}, g \subseteq h \} \quad \text{by 11.6(2) for } n
\]

\[
= \sum \{ a(h) : h \in \mathcal{D}_{m_{n+1}}, g \subseteq h \} \quad \text{by 11.6(8)}
\]

For \( m_n < k < m_{n+1} \) and \( g \in \mathcal{D}_k \) we have

\[
a(g) = \sum \{ a(h) : h \in \mathcal{D}_{m_{n+1}}, g \subseteq h \}.
\]

Hence 11.6(2) holds for \( n + 1 \).

Now suppose that \( k < 2^{m_{n+1}} \). Let \( g = \text{lex}_{\mathcal{D}_{m_{n+1}}}(k) \). Then by the above, \( a(g) = c_{f2^{m_n}2^s(i,j)} \) with \( i = \text{lex}^{-1}_{\mathcal{D}_{m_n}}(g \upharpoonright m_n) \), and \( j = \text{lex}^{-1}_s(\langle g_{m_n + k} : k < s \rangle) \). Now for \( i < 2^m \) and \( j < 2^s \) let \( \mu(f2^{m_n}2^s(i,j)) = \text{lex}_{\mathcal{D}_{2^m}}(i) \cap \text{lex}_{\mathcal{D}_{2^s}}(j) \). Then

\[
\forall k < 2^{m_{n+1}} [a(\text{lex}_{\mathcal{D}_{m_{n+1}}}(k)) = \sum \{ c_{f2^{m_n}2^s(i,j)} : i < 2^m, j < 2^s, \mu(f2^{m_n}2^s(i,j)) = k \}].
\]
It follows that \( c < a_{m_{n+1}} \), and so \( a_{m_{n+1}} \in \alpha \). Next, suppose that \( m_n \leq k < m_{n+1} \). For any \( h \in \mathcal{D}_{m_{n+1}} \) let \( \lambda(\text{lex}
olimits^{-1}_{\mathcal{D}_{m_{n+1}}}(h)) = \text{lex}
olimits^{-1}_{\mathcal{D}_k}(h \upharpoonright k) \). Thus \( \lambda : 2^{m_{n+1}} \to 2^k \). For any \( i < 2^k \), with \( g = \text{lex}
olimits_{\mathcal{D}_k}(i) \) we have 
\[
a_k(i) = a(\text{lex}
olimits_{\mathcal{D}_k}(i)) = a(g) \\
= \sum \{ a(h) : h \in \mathcal{D}_{m_{n+1}}, g \subseteq h \} \\
= \sum \{ a(\text{lex}
olimits_{\mathcal{D}_{m_{n+1}}}(j)) : g \subseteq \text{lex}
olimits_{\mathcal{D}_{m_{n+1}}}(j) \} \\
= \sum \{ a(\text{lex}
olimits_{\mathcal{D}_{m_{n+1}}}(j)) : (\text{lex}
olimits_{\mathcal{D}_{m_{n+1}}}(j)) \upharpoonright k = g \} \\
= \sum \{ a(\text{lex}
olimits_{\mathcal{D}_{m_{n+1}}}(j)) : \text{lex}
olimits_{\mathcal{D}_{m_{n+1}}}(j) \upharpoonright k = \text{lex}
olimits_{\mathcal{D}_k}(i) \} \\
= \sum \{ a(\text{lex}
olimits_{\mathcal{D}_{m_{n+1}}}(j)) : \lambda(j) = i \}
\]
The last equation holds since 
\[
\lambda(j) = \lambda(\text{lex}
olimits^{-1}_{\mathcal{D}_{m_{n+1}}} \langle \text{lex}
olimits_{\mathcal{D}_{m_{n+1}}}(j) \rangle) = \text{lex}
olimits^{-1}_{\mathcal{D}_k}(\text{lex}
olimits_{\mathcal{D}_{m_{n+1}}}(j) \upharpoonright k),
\]
hence \( \lambda(j) = i \) iff \( \text{lex}
olimits_{\mathcal{D}_{m_{n+1}}}(j) \upharpoonright k = \text{lex}
olimits_{\mathcal{D}_k}(i) \).

Thus \( \langle a(\text{lex}
olimits_{\mathcal{D}_{m_{n+1}}}(j)) : j < 2^{m_{n+1}} \rangle \prec \langle a(\text{lex}
olimits_{\mathcal{D}_k}(j)) : j < 2^k \rangle \), so \( \langle a(\text{lex}
olimits_{\mathcal{D}_k}(j)) : j < 2^k \rangle \in \alpha \). Hence 11.6(1) holds for \( n + 1 \).

Now if \( s \leq n \), then \( \langle a(\text{lex}
olimits_{\mathcal{D}_s}(i)) : i < 2^{m_s} \rangle \prec b_s \) by 11.6(3) for \( n \). By the above with \( k = 2^{m_n} \) we have 
\[
\langle a(\text{lex}(j)) : j < 2^{m_{n+1}} \rangle \prec \langle a(\text{lex}(j)) : j < 2^{m_n} \rangle \prec b_s.
\]

Thus 11.6(3) holds for \( n + 1 \). \( \square \)

**Proposition 11.7.** Let \( M \) be an \( m \)-monoid, and \( \alpha \neq O \) a countable subset of \( << \omega M \) which satisfies C.P., R.P., and S.P. Then there is a dense homogeneous \( \alpha \)-tree \( \mathcal{D} \).

**Proof.** We add to the proof of Proposition 11.6. We apply Proposition 11.3 instead of Proposition 11.2, and obtain in addition to the properties of \( c \) given in the proof of Proposition 11.6 the condition \( \forall i < 2^{m_n} \forall j < 2^s [c_{fj_{2^{m_n}2^s}(i,j)} \neq 0] \). Clearly this gives the homogeneous property. \( \square \)

**Proposition 11.8.** (1.16.4) If \( M \) is an \( m \)-monoid and \( O \neq \alpha \in \Delta M \), then there is a \( \sigma \in \mathcal{M}(M) \) such that \( (\Delta \sigma)(1) = \alpha \).

**Proof.** For each \( n \in \omega \) and each \( f \in \mathcal{D}_n \) let 

\[
11.8(1) \ x(f) = \{ g \in \omega 2 : f \subseteq g \}.
\]

Then \( \{ x(f) : f \in \mathcal{D} \} \) is a base for \( \text{clop}(\omega 2) \), which is a denumerable atomless BA.

11.8(2) If \( k \leq i \in \omega \) and \( f \in \mathcal{D}_k \), then 
\[
11.8(2) \ x(f) = \bigcup \{ x(g) : g \in \mathcal{D}_i, f \subseteq g \},
\]
and this union is pairwise disjoint.
11.8(3) If $1 = y_0 + y_1 + \cdots + y_{n-1}$ with each $y_i$ clopen in $\omega^2$, then there is a $k \in \omega$ and a partition $\mathcal{D}_k = G_0 \cup G_1 \cup \ldots \cup G_{n-1}$ such that \( \forall j < n[y_j = \sum\{x(f) : f \in G_j\}] \).

In fact, since \( \{x(f) : f \in \mathcal{D}\} \) is a base for clop($\omega^2$) and each $y_i$ is clopen and hence compact, there exist finite subsets $\mathcal{D}^i$ of $\mathcal{D}$ for $i < n$ such that \( \forall i < n[y_i = \bigcup\{x(f) : f \in \mathcal{D}^i\}] \). Let $k$ be greater than the domain of $f$, for all $f \in \bigcup_{i < n} \mathcal{D}^i$. By 11.8(2) we may assume that each $\mathcal{D}^i$ is a subset of $\mathcal{D}_k$. Clearly the sets $\mathcal{D}^i$ are pairwise disjoint. For any $f \in k^2$ the set $x(f)$ is a nonempty set, and hence $x(f) \cap y_i \neq \emptyset$ for some $i < n$. So we must actually have $f \in \mathcal{D}^i$. This proves 11.8(3).

By Proposition 11.7 let $a$ be a dense homogeneous $\alpha$-tree. Now let $z \in \text{clop}(\omega^2)$. Then by compactness there is a finite $X \subseteq \mathcal{D}$ such that $z = \bigcup\{x(f) : f \in X\}$. Hence there is a $k \in \omega$ and a $G \subseteq \mathcal{D}_k$ such that $z = \sum\{x(f) : f \in G\}$. (disjoint sum) We then define

\[
\sigma(z) = \sum\{a(f) : f \in G\}.
\]

This definition does not depend on the particular choice of $k$ and $G$. For, suppose that also $l \in \omega$, $H \subseteq \mathcal{D}_l$, and $z = \sum\{x(f) : f \in H\}$; we claim that $\sum\{a(f) : f \in G\} = \sum\{a(f) : f \in H\}$. For, suppose that $k \leq l$, and let $K = \{f \in \mathcal{D}_l : \exists g \in G[g \subseteq f]\}$. Then $z = \sum\{x(f) : f \in K\}$ by 11.8(2). Since $x(f) \cap x(g) = \emptyset$ for distinct $f, g \in \mathcal{D}_l$, it follows that $K = H$. Then by 11.8(2) we get

\[
\sum\{a(f) : f \in G\} = \sum\{a(g) : g \in K\} = \sum\{a(f) : f \in H\}.
\]

Now to show that $\sigma$ is additive, suppose that $z, z' \in \text{clop}(\omega^2)$ and $z \cdot z' = 0$. Choose $k, k', G, G'$ such that $G \subseteq \mathcal{D}_k, G' \subseteq \mathcal{D}_{k'}$, $z = \sum\{x(f) : f \in G\}$, and $z' = \sum\{x(f) : f \in G'\}$; thus $\sigma(z) = \sum\{a(f) : f \in G\}$ and $\sigma(z') = \sum\{a(f) : f \in G'\}$. Wlog $k \leq k'$. Let $H = \{f \in \mathcal{D}_{k'} : \exists g \in G[g \subseteq f]\}$. Then by 11.8(2), $z = \sum\{x(f) : f \in H\}$ and $\sigma(z) = \sum\{a(f) : f \in H\}$. Since $z \cdot z' = 0$ we have $H \cap G' = \emptyset$. Now $z + z' = \sum\{x(f) : f \in H \cup G'\}$, so

\[
\sigma(z + z') = \sum\{a(f) : f \in H \cup G'\} = \sum\{a(f) : f \in H\} + \sum\{a(f) : f \in G'\} = \sigma(z) + \sigma(z').
\]

Since $a$ is homogeneous, clearly $\sigma(z) = 0$ iff $z = 0$. So $\sigma \in \mathcal{M}(M)$.

It remains only to show that $(\triangle \sigma)(1) = \alpha$. Suppose that $b \in (\triangle \sigma)(1)$. Say $n \in \omega$, $1 = z_0 + z_1 + \cdots + z_{n-1}$ and $b = (\sigma(z_0), \sigma(z_1), \ldots, \sigma(z_{n-1}))$. Now by 11.8(3) let $k \in \omega$ and $G_0, \ldots, G_{n-1}$ be such that $\mathcal{D}_k = G_0 \cup G_1 \cup \cdots \cup G_{n-1}$ and $\forall j < n[z_j = \sum\{x(f) : f \in G_j\}]$. So $\forall j < n[\sigma(z_j) = \sum\{a(f) : f \in G_j\}]$. Define $\lambda : 2^n \to n$ by $\lambda(m) = j$ iff $\text{lex}(m) \in G_j$. Then $\forall j < n[\sigma(z_j) = \sum\{a(\text{lex}(m)) : m < 2^n, \lambda(m) = j\}]$. Hence $a_k < b$, hence $b \in \alpha$ by (t1) and C.P. This shows that $(\triangle \sigma)(1) \subseteq \alpha$.

For the other inclusion, first note that

11.8(4) $\forall f \in \mathcal{D}[(\triangle \sigma)(a(f)) = a(f)]$

In fact, let $f \in \mathcal{D}$. Let $z = x(f)$ and $G = \{f\}$. Then by definition $\sigma(x(f)) = a(f)$.

11.8(5) $\forall k \in \omega[a_k \in (\triangle \sigma)(1)]$. 

76
For, let $k \in \omega$. Then

$$a_k = a(\mathcal{D}_k) = \langle a(\text{lex}(i)) : i < 2^k \rangle = \langle \sigma(x(\text{lex}(i))) : i < 2^k \rangle \in (\triangle \sigma)(1)$$

since $1 = \sum\{x(\text{lex}(i)) : i < 2^k\}$, a disjoint sum. So 11.8(5) holds.

Now take any $b \in \alpha$. Since $a$ is dense there is a $k \in \omega$ such that $a_k < b$. Now $(\triangle \sigma)(1) \in \Delta M$ by Proposition 10.1, so $b \in (\triangle \sigma)(1)$ by (t1) and C.P.

So $(\triangle \sigma)(1) = \alpha$.

12. Stable measures

Proposition 12.1. If $x, y \in \text{Fr}(\omega)$, $\sigma$ is a $M$-measure, and $(\triangle \sigma)(x) = (\triangle \sigma)(y)$, then $\sigma(x) = \sigma(y)$.

Proof. By Proposition 10.2.

An $M$-measure $\sigma$ is stable iff $\forall x, y \in \text{Fr}(\omega)$, $\sigma(x) = \sigma(y)$ implies that $(\triangle \sigma)(x) = (\triangle \sigma)(y)$.

Proposition 12.2. Let $M$ be an $m$-monoid, and $\sigma, \tau$ stable $M$-measures. Then the following are equivalent:

(i) There exists an automorphism $k \in \text{Fr}(\omega)$ such that $\sigma = \tau \circ k$.
(ii) $(\triangle \sigma)(1) = (\triangle \tau)(1)$.

Proof. (i)$\Rightarrow$(ii): by Proposition 10.4.

(ii)$\Rightarrow$(i): Assume (ii). Define $xRy$ iff $\sigma(x) = \tau(y)$.

12.2(1) $R$ is a $V$-relation.

Recall from Section 3 the notion of a $V$-relation.

1R1: By Proposition 10.2, $\sigma(1) = T((\triangle \sigma)(1)) = T(\triangle \tau(1)) = \tau(1)$.

0R0: By definition, $\sigma(0) = 0 = \tau(0)$.

For (V9), we want to show that $\forall x \in \text{Fr}(\omega)[xR0 \rightarrow x = 0]$. Suppose that $\sigma(x) = \tau(0)$. Since $\tau(0) = 0$, we have $\sigma(x) = 0$, and hence $x = 0$.

(V10) is symmetric to the above.

To complete the proof of 12.2(1) by symmetry it suffices to prove (V11). So suppose that $\sigma(x) = \tau(y+z)$. Now $1 = x+yz + (-y \cdot -z)$, so by definition $\langle \tau(y), \tau(z), \tau(-y \cdot -z) \rangle \in \triangle \tau(1) = (\triangle \sigma)(1)$. Hence there exist $u_0, u_1, u_2$ such that $1 = u_0 + u_1 + u_2$, $\tau(y) = \sigma(u_0)$, $\tau(z) = \sigma(u_1)$, and $\tau(-y \cdot -z) = \sigma(u_2)$. Let $v = u_0 + u_1$. Then $\sigma(v) = \sigma(u_0) + \sigma(u_1) = \tau(y)+\tau(z) = \tau(y+z) = \sigma(x)$. Then since $\sigma$ is stable, $(\triangle \sigma)(v) = (\triangle \sigma)(x)$. Now $v = u_0 + u_1$, so $\langle \sigma(u_0), \sigma(u_1) \rangle \in (\triangle \sigma)(v) = (\triangle \sigma)(x)$. Hence there exist $w_0, w_1$ such that $x = w_0 + w_1$, $\tau(y) = \sigma(u_0) = \sigma(w_0)$, and $\tau(z) = \sigma(u_1) = \sigma(w_1)$. Hence $w_0 Ry$ and $w_1 Rz$. This proves (V11). So we have proved 12.2(1).
Now by 12.2(1) and Theorem 3.3, there is an automorphism $k$ of $\text{Fr}(\omega)$ such that for all $x \in \text{Fr}(\omega)$ there exist $n \in \omega$ and a disjoint $y \in n\text{Fr}(\omega)$ such that $x = \sum_{i < n} y_i$ and $\forall i < n[y_i R k(y_i)]$. So $\forall i < n[\sigma(y_i) = \tau(k(y_i))]$. Hence

$$\sigma(x) = \sigma\left(\sum_{i < n} y_i\right) = \sum_{i < n} \sigma(y_i) = \sum_{i < n} \tau(k(y_i)) = \tau\left(k\left(\sum_{i < n} y_i\right)\right) = \tau(k(x)).$$

Thus $\sigma = \tau \circ k$. 

\[ \square \]

13. Fragments

Let $M$ be an $m$-monoid. For $\alpha \subseteq \ll \omega M$ we define

$$\Phi(\alpha) = \{ b \in \ll \omega M : \exists c \in \ll \omega M[b \sim c \in \alpha]\}.$$ 

If also $a \in M$, then

$$\Phi_a(\alpha) = \{ b \in \Phi(\alpha) : T(b) = a\}.$$ 

Elements of $\Phi(\alpha)$ are called fragments of $\alpha$; elements of $\Phi_a(\alpha)$ are called $a$-fragments of $\alpha$.

**Proposition 13.1.** If $\alpha$ is a countable subset of $M$, then $\Phi(\alpha)$ and $\Phi_a(\alpha)$ are countable. 

**Proposition 13.2.** (i) $\alpha \subseteq \Phi(\alpha)$.

(ii) $\Phi(\alpha \cup \beta) = \Phi(\alpha) \cup \Phi(\beta)$.

(iii) $\Phi(\emptyset) = \emptyset$.

(iv) $\Phi(\Phi(\alpha)) = \Phi(\alpha)$.

(v) If $\alpha \subseteq \beta$, then $\Phi(\alpha) \subseteq \Phi(\beta)$.

**Proposition 13.3.** If $\sigma$ is an $M$-measure, then $\Phi((\Delta \sigma)(1)) = \bigcup_{x \in \text{Fr}(\omega)} (\Delta \sigma)(x)$.

**Proof.** Suppose that $b \in \Phi((\Delta \sigma)(1))$. Choose $c \in \ll \omega M$ such that $b \sim c \in (\Delta \sigma)(1)$. Say $b$ has length $m$ and $c$ has length $n$. Then we can write

$$1 = x_0 \dot{+} \cdots \dot{+} x_{m-1} \dot{+} x_m \dot{+} \cdots \dot{+} x_{m+n-1},$$

and

$$\langle \sigma(x_0), \ldots, \sigma(x_{m-1}), \sigma(x_m), \ldots, \sigma(x_{m+n-1}) \rangle = b \sim c.$$

Let $y = x_0 \dot{+} \cdots \dot{+} x_{m-1}$. Then $b = \langle \sigma(x_0), \ldots, \sigma(x_{m-1}) \rangle \in (\Delta \sigma)(y)$. This proves $\subseteq$ in the proposition.

Conversely, suppose that $x \in \text{Fr}(\omega)$ and $b \in (\Delta \sigma)(x)$. Say $x = y_0 \dot{+} \cdots \dot{+} y_{m-1}$ and $b = \langle \sigma(y_0), \ldots, \sigma(y_{m-1}) \rangle$. Then $1 = y_0 \dot{+} \cdots \dot{+} y_{m-1} \dot{+} (-x)$, and so

$$\langle \sigma(y_0), \ldots, \sigma(y_{m-1}), \sigma(-x) \rangle \in (\Delta \sigma)(1),$$

and hence $b \in \Phi((\Delta \sigma)(1))$. 

\[ \square \]
Proposition 13.4. If \( \sigma \) is an \( M \)-measure and \( a \in M \), then

\[
\Phi_a((\Delta \sigma)(1)) = \bigcup_{z \in \text{Fr}(\omega) \atop \sigma(z) = a} (\Delta \sigma)(x).
\]

**Proof.** Suppose that \( b \in \Phi_a((\Delta \sigma)(1)) \). Then \( T(b) = a \). Choose \( c \in \langle \omega \rangle M \) such that \( b \prec c \in (\Delta \sigma)(1) \). Say \( b \) has length \( m \) and \( c \) has length \( n \). Then we can write

\[
1 = x_0 \dot{+} \cdots \dot{+} x_{m-1} \dot{+} x_m \dot{+} \cdots \dot{+} x_{m+n-1}, \quad \text{and}
\]

\[
\langle \sigma(x_0), \ldots, \sigma(x_{m-1}), \sigma(x_m), \ldots, \sigma_{m+n-1} \rangle = b \prec c.
\]

Let \( y = x_0 \dot{+} \cdots \dot{+} x_{m-1} \). Then \( b = \langle \sigma(x_0), \ldots, \sigma(x_{m-1}) \rangle \in (\Delta \sigma)(y) \). Also, \( \sigma(y) = \sigma(x_0) + \cdots + \sigma(x_{m-1}) = T(b) = a \). This proves \( \subseteq \) in the proposition.

Conversely, suppose that \( x \in \text{Fr}(\omega) \), \( \sigma(x) = a \), and \( b \in (\Delta \sigma)(x) \). Say \( x = y_0 \dot{+} \cdots \dot{+} y_{m-1} \) and \( b = \langle \sigma(y_0), \ldots, \sigma(y_{m-1}) \rangle \). Then \( 1 = y_0 \dot{+} \cdots \dot{+} y_{m-1} \dot{+} (-x) \), and so

\[
\langle \sigma(y_0), \ldots, \sigma(y_{m-1}), \sigma(-x) \rangle \in (\Delta \sigma)(1),
\]

and hence \( b \in \Phi((\Delta \sigma)(1)) \). Also,

\[
T(b) = \sigma(y_0) + \cdots + \sigma(y_{m-1}) = \sigma(y_0 + \cdots + y_{m-1}) = \sigma(x) = a.
\]

\( \square \)

Proposition 13.5. If \( \alpha \subseteq \langle \omega \rangle M \) has C.P., then \( \Phi(\alpha) \) and \( \Phi_a(\alpha) \) have C.P.

**Proof.** Assume that \( \alpha \subseteq \langle \omega \rangle M \) has C.P. Suppose that \( x \in \Phi(\alpha) \) and \( x \prec y \). Choose \( z \in \langle \omega \rangle M \) so that \( x \sim z \in \alpha \). Say \( x \in {}^m M \), \( y \in {}^n M \), and \( z \in {}^p M \). Since \( x \prec y \), there is a \( \lambda : m \to n \) such that \( \forall j < n[y_j = \sum \{x_i : i < m, \lambda(i) = j\}] \). Let \( \mu : m + p \to n + p \) extend \( \lambda \) by setting \( \mu(m + k) = n + k \) for all \( k < p \). Let \( w = y \sim z \) and \( v = x \sim z \). Then

\[
\forall j < n + p[w_j = \sum \{v_i : i < m + p, \lambda(i) = j\}].
\]

Hence \( (x \sim z) \prec (y \sim z) \) so, since \( \alpha \) has C.P., \( y \sim z \in \alpha \). It follows that \( y \in \Phi(\alpha) \).

For \( \Phi_a(\alpha) \), we assume in the above argument that in addition \( T(x) = a \). By Proposition 8.5, \( T(y) = a \). \( \square \)

Proposition 13.6. If \( \alpha \subseteq \langle \omega \rangle M \) has S.P., then \( \Phi(\alpha) \) and \( \Phi_a(\alpha) \) have S.P.

**Proof.** Assume that \( \alpha \subseteq \langle \omega \rangle M \) has S.P. Suppose that \( m \in \omega \setminus \{0\} \), \( a \in {}^m M \), \( a \in \Phi(\alpha) \), and \( a_0 \neq a \). Choose \( c \in \langle \omega \rangle M \) so that \( a \sim c \in \alpha \). Choose \( u, v \in M \setminus \{0\} \) such that \( a_0 = u + v \) and \( \langle u, v, a_1, \ldots, a_{m-1}, 0, c_1 \ldots \rangle \in \alpha \). Then \( \langle u, v, a_1, \ldots, a_{m-1} \rangle \in \Phi(\alpha) \).

For \( \Phi_d(\alpha) \), assuming that \( T(a) = d \), we have

\[
T(\langle u, v, a_1, \ldots, a_{m-1} \rangle) = u + v + a_1 + a_2 + \cdots + a_{m-1} = a_0 + a_1 + \cdots + a_{m-1} = T(a) = d.
\]

\( \square \)

Lemma 13.7. Let \( \sigma \in \mathcal{M}(M) \) be stable, and let \( \alpha = (\Delta \sigma)(1) \). Suppose that \( a \in M \), \( c \in \langle \omega \rangle M \), and \( \langle a \rangle \sim c \in \alpha \). Then
(i) $\Phi_a(\alpha) \in \triangle M$.
(ii) $\forall b \in \Phi_a(\alpha)[b \prec c \in \alpha]$.

**Proof.** Assume the hypotheses. To prove (i), by Proposition 10.1 it suffices to find $x \in \text{Fr}(\omega)$ such that $\Phi_a(\alpha) = (\triangle \sigma)(x)$. Now by Proposition 13.4,

$$\Phi_a(\alpha) = \Phi_a((\triangle \sigma)(1)) = \bigcup_{x \in \text{Fr}(\omega)} (\triangle \sigma)(x)$$

Note that if $x, y \in \text{Fr}(\omega)$ and $\sigma(x) = a = \sigma(y)$, then $(\triangle \sigma)(x) = (\triangle \sigma)(y)$ since $\sigma$ is stable. Since $(a) \prec c \in \alpha = (\triangle \sigma)(1)$, there is an $x \in \text{Fr}(\omega)$ such that $\sigma(x) = a$. So (i) follows.

Now for (ii), suppose that $b \in \Phi_a(\alpha)$ with dmn$(b) = m$. Let dmn$(c) = n$. Since $(a) \prec c \in \alpha = (\triangle \sigma)(1)$, there are pairwise disjoint $x, y_0, \ldots, y_{n-1}$ such that

$$1 = x + y_0 + y_1 + \cdots + y_{n-1}$$

and $a = \sigma(x)$ and $c_i = \sigma(y_i)$ for all $i < n$. By the argument in the preceding paragraph, $b \in (\triangle \sigma)(x)$. Say $x = z_0 + \cdots + z_{m-1}$ and $b = \langle \sigma(z_0), \ldots, \sigma(z_{m-1}) \rangle$. Then

$$1 = z_0 + \cdots + z_{m-1} + y_0 + \cdots + y_{n-1} \quad \text{and} \quad b \prec c = \langle \sigma(z_0), \ldots, \sigma(z_{m-1}), \sigma(y_0), \ldots, \sigma(y_{m-1}) \rangle.$$

If $M$ is an $m$-monoid, a countable subset $\alpha$ of $\prec \omega M$ has the local refinement property, L.P., iff the following conditions hold:

(L1) $\forall a \in M[\Phi_a(\alpha) = \emptyset$ or $\Phi_a(\alpha)$ has R.P.$].$

(L2) $\forall a \in M \forall c \in \prec \omega M[\langle a \rangle \prec c \in \alpha \rightarrow \forall b \in \Phi_a(\alpha)[b \prec c \in \alpha]]$.

**Proposition 13.8.** If $\sigma$ is a stable $M$ measure, then $(\triangle \sigma)(1)$ satisfies L.P.

**Proof.** By Proposition 13.7. □

**Proposition 13.9.** Suppose that $\alpha$ is a countable subset of $\prec \omega M$ which satisfies C.P., R.P., and L.P. Also suppose that $a \in \alpha$ has domain $n$, $a \in \alpha$, $q \leq n$, $c = \sum_{i<q} a_i$, $b$ has domain $m$, $b \in \Phi_c(\alpha)$, and $p \geq m$.

Then there is a $d$ with domain $n \cdot p$ such that $d \in \alpha$, $\forall i < n[a_i = \sum_{j<p} d_{i,j}]$, and $\langle d_{i,j} : i < q, j < p \rangle < b$.

**Proof.** Let $e = \sum_{q \leq i < n} a_i$.

13.9(1) $\langle c, e \rangle \in \alpha$.

For, let $u = \langle c, e \rangle$. Define $n \to 2$ by setting, for $i < n$,

$$\lambda(i) = \begin{cases} 
0 & \text{if } i < q, \\
1 & \text{if } q \leq i < n.
\end{cases}$$

80
Then \( c = u_0 = \sum \{a_j : \lambda(j) = 0\} \) \( e = u_1 = \sum \{a_j : \lambda(j) = 1\} \). So \( a < u \), and 13.9(1) follows by C.P.

Now \( \Phi_e(\alpha) \neq \emptyset \) since \( b \in \Phi_e(\alpha) \). So \( \Phi_e(\alpha) \) has R.P. since \( \alpha \) has L.P. Now \( a = \langle a_i : i < q \rangle \setminus \langle a_i : q \leq i < n \rangle \), and \( a \in \alpha \), so by Proposition 8.3, \( \langle a_i : q \leq i < n \rangle \setminus \langle a_i : i < q \rangle \in \alpha \), and so \( \langle a_i : q \leq i < n \rangle \setminus \langle a_i : i < q \rangle \in \Phi_e(\alpha) \). Hence \( \Phi_e(\alpha) \neq \emptyset \), so \( \Phi_e(\alpha) \) has R.P. since \( \alpha \) has L.P. By Proposition 13.5, \( \Phi_e(\alpha) \) also have C.P. We now apply Proposition 11.2, with \( \alpha, m, n, a, b, p \) replaced by \( \Phi_e(\alpha), q, m, a \uparrow q, b, p \) to get \( c' \in q \cdot p \cdot \Phi \) such that \( c' \in \Phi_e(\alpha), \forall i < q | \sum_{j < p} c'_{q, i, j} = a_i \rangle \) and \( c' < b \). Now let \( g_i = a_{q+i} \) for all \( i < n - q \). We now apply Proposition 11.2 with \( \alpha, m, n, a, b, p \) replaced by \( \Phi_e(\alpha), n - q, m, g, g, p \). This gives \( c'' \in (n - q) \cdot p \cdot \Phi \) such that \( c'' \in \Phi_e(\alpha) \forall i < n - q | \sum_{j < p} c''_{q(n - q) \cdot p, i, j} = g_i \rangle \) and \( c'' < g \).

Now define \( d \) with domain \( n \cdot p \) by setting, for \( i < n \) and \( j < p \),

\[
d_{f_{n, p}(i, j)} = \begin{cases} c'_{q, i, j} & \text{if } i < q, \\ c''_{q(n - q) \cdot p, i, j} & \text{if } q \leq i < n. \end{cases}
\]

Then for all \( i < n \) we have

\[
\sum_{j < p} d_{f_{n, p}(i, j)} = \begin{cases} \sum_{j < p} c'_{q, i, j} = a_i & \text{if } i < q, \\ \sum_{j < p} c''_{q(n - q) \cdot p, i, j} = g_{i - q} = a_i & \text{if } q \leq i < n. \end{cases}
\]

We have \( c' \in \Phi_e(\alpha) \), so by 13.9(1) and (L2), \( c' \setminus \langle e \rangle \in \alpha \). Hence by Proposition 8.3, \( \langle e \rangle \setminus c' \in \alpha \). Since \( c'' \in \Phi_e(\alpha) \), it follows by 13.9(1) that \( c'' \setminus c' \in \alpha \). Then by Proposition 8.3 again we get \( d \in \alpha \). Finally, if \( i < q \) and \( j < p \), then \( d_{f_{n, p}(i, j)} = c'_{q, i, j} \), so \( \langle d_{f_{n, p}(i, j)} : i < q, j < p \rangle = c' \). \( \square \)

**Proposition 13.10.** Assume the hypotheses of Proposition 13.9, and assume also that \( \alpha \) satisfies S.P., and \( \forall i < n | a_i \neq 0 \). Then in the conclusion we may assert that also \( \forall i < n \forall j < p | d_{f_{n, p}(i, j)} \neq 0 \).

**Proof.** Assume the hypotheses of Proposition 13.9, and assume also that \( \alpha \) satisfies S.P., and \( \forall i < n | a_i \neq 0 \). By Proposition 13.6, \( \Phi_e(\alpha) \) and \( \Phi_e(\alpha) \) have S.P. Then by Proposition 11.3, in the proof of Proposition 13.9 we may assume that \( \forall i < q \forall j < p | c'_{q, f_{n, p}(i, j)} \neq 0 \) and \( \forall i < n - q \forall j < p | c''_{q(n - q) \cdot p, f_{n, p}(i, j)} \neq 0 \). Hence \( \forall i < n \forall j < p | d_{f_{n, p}(i, j)} \neq 0 \). \( \square \)

Let \( \alpha \) be a countable subset of \( \langle \omega M \rangle \). An \( \alpha \)-tree \( a \) is uniformly dense iff

\[
\forall n \in \omega \forall G \in \mathcal{D}_n \forall b \in \Phi_{T(a(G))}(\alpha) \exists m \geq n | a(G \uparrow m) \prec b |.
\]

**Proposition 13.11.** If \( \alpha \) is a countable subset of \( \langle \omega M \rangle \) and \( a \) is a uniformly dense \( \alpha \)-tree, then \( a \) is dense.

**Proof.** Assume that \( \alpha \) is a countable subset of \( \langle \omega M \rangle \) and \( a \) is a uniformly dense \( \alpha \)-tree. Let \( b \in \alpha \). Then \( T(b) = T(\alpha) = T(a_0) \), so \( b \in \Phi_{T(a(D_0))} \), hence there is an \( m \geq 0 \) such that \( a(D_0 \uparrow m) \prec b \). Since \( D_0 \uparrow m = D_m \), this means that \( a_m \prec b \). Thus \( a \) is dense. \( \square \)
Lemma 13.12. If $\alpha$ is a countable subset of $\langle\omega M, \mathcal{V}_{S}^{M}\rangle$ that satisfies C.P., R.P., and L.P., then there is a uniformly dense $\alpha$-tree $a$.

Proof. Recall from Proposition 8.12 that $\langle T(\alpha) \rangle \in \alpha$. Let $\{b_0, b_1, \ldots \}$ be an enumeration of $\Phi(\alpha)$ with $b_0 = \langle T(\alpha) \rangle$. We now define $m_n \in \omega$ and $a \upharpoonright \bigcup_{r \leq m_n} \mathcal{D}_r$ by induction on $n$ so that the following conditions hold:

13.12(1) $n < m_n$.
13.12(2) $\forall r \leq m_n[a_r \in \alpha]$.
13.12(3) $\forall r, s \leq m_n \forall f \in \mathcal{D}_r[r \leq s \rightarrow a(f) = \sum_{f \leq g \in \mathcal{D}_s} a(g)]$.
13.12(4) $\forall k \leq n \forall G \subseteq \mathcal{D}_k[b_k \in \Phi_{T(\alpha)}(\alpha) \rightarrow a(G \upharpoonright m_n) < b_k]$.

Let $m_0 = 1$, $a(\emptyset) = T(\alpha)$, $a(\langle\rangle) = 0$, $a(\langle 1 \rangle) = T(\alpha)$. Now $\langle a(\text{lex}(i)) : i < 1 \rangle = \langle T(\alpha) \rangle \in \alpha$ by Proposition 8.12, and $\langle a(\text{lex}(i)) : i < 2 \rangle = \langle 0, T(\alpha) \rangle \in \alpha$ by Proposition 8.4 and C.P. Hence 13.12(1)-13.12(4) hold for $n = 0$. Now suppose that 13.12(1)-13.12(4) hold for $n$. List the set $\{(k, G) : k \leq n + 1, G \subseteq \mathcal{D}_{n+1}, b_k \in \Phi_{T(\alpha)}(\alpha)\}$ as $\{(k_0, G_0), (k_1, G_1), \ldots, (k_{i-1}, G_{i-1})\}$. Then

13.12(5) There is a $m'_0 > m_n$ and an extension of $a$ to $\bigcup_{r \leq m'_0} \mathcal{D}_r$ such that

(a) $\forall r \leq m'_0 \langle a(\text{lex}(i)) : i < 2^r \rangle \in \alpha$.
(b) $\forall r, s \leq m'_0 \forall f \in \mathcal{D}_r[r \leq s \rightarrow a(f) = \sum_{f \leq g \in \mathcal{D}_s} a(g)]$.
(c) $a(G_0 \upharpoonright m'_0) < b_{k_0}$.

To prove 13.12(5), first note that $b_{k_0} \in \Phi_{T(\alpha)(G_0)}(\alpha)$. Now by 13.12(3) for $m_n$, since $n < m_n$, $a_{n+1} \in \alpha$. Note that $T(b_{k_0}) = T(a(G_0)) = T(\langle a(\text{lex}(i)) : i < |G_0| \rangle)$. Let $l$ be such that $n + 1 \leq l$ and $\text{dmm}(b_{k_0}) \leq 2^l$. By adjoining 0s to $a_{m_n}$ we can obtain a sequence $e \in \alpha$ of length $2^l$. By a permutation of $e$ we obtain $e' \in \alpha$ such that $\langle a(\text{lex}(i)) : i < |G_0| \rangle$ is an initial segment of $e'$. Now we apply Proposition 13.9 with $a, n, \alpha, q, p, m, b$ replaced by $e', 2^l, \alpha, |G_0|, 2^l, \text{dmm}(b_{k_0}), b_{k_0}$ to obtain $d \in \alpha$ of length $2^l \cdot 2^l$ such that $\forall i < 2^l \langle e'_i = \sum_{j < 2^l} d_{f_{2^l,2^l}(i,j)} \rangle$ and $\langle d_{f_{2^l,2^l}(i,j)} : i < |G_0|, j < 2^l \rangle \in b_{k_0}$. Let $m'_0 = 2l$. Let $g : 2^{2l} \rightarrow 2^l \times 2^l$ be a bijection. For each $i < 2^{2l}$ let $a(\text{lex}(i)) = d_{g_{f_{2^l,2^l}(i,j)}}(g(i)_0, (g(i))_1)$. Now for $s$ such that $m_n < s \leq m'_0$ and $f \in \mathcal{D}_s$ let $a(f) = \sum \{a(k) : f \subseteq k \in \mathcal{D}_{2^l}\}$.

Clearly (b) holds, and (a) holds for $r = m'_0$. Now suppose that $m_n < r < m'_0$. For $i < 2^{2l}$ let $\lambda(i) = \text{lex}^{-1}(\text{lex}(i) \upharpoonright r)$. Then for any $j < 2^r$, $a(\text{lex}(j)) = \sum_{\text{lex}(k) \subseteq h \subseteq 2^l} a(\text{lex}(k)) = \sum \{a(\text{lex}(i)) : i < 2^{2l}, \lambda(i) = j\}$. Hence $\langle a(\text{lex}(i)) : i < 2^{2l} \rangle \prec \langle a(\text{lex}(j)) : j < 2^r \rangle$ and so $\langle a(\text{lex}(j)) : j < 2^r \rangle$. Thus (a) holds. Now we claim that $a(G_0 \upharpoonright m'_0) < \langle d_{f_{2^{2l},2^{2l}}(i,j)} : i < |G_0|, j < 2^l \rangle$, hence $a(G_0 \upharpoonright m'_0) < b_{k_0}$, giving (c). For, if $i < |G_0|$ and $j < 2^l$, then

$$d_{f_{2^l,2^l}(i,j)} = \sum \{a(\text{lex}(k)) : \text{lex}^{-1}(\text{lex}(k) \upharpoonright (n + 1)) = i\}.$$  

Now we repeat this argument for $(k_1, G_1), \ldots, (k_{i-1}, G_{i-1})$, ending up with $m_{n+1}$ satisfying 13.12(1)-13.12(4) for $n + 1$.

Finally we get $a : \mathcal{D} \rightarrow M$, and this is clearly the desired uniformly dense $\alpha$-tree.

Lemma 13.13. If $\alpha \neq \emptyset$ is a countable subset of $\langle\omega M, \mathcal{V}_{S}^{M}\rangle$ that satisfies C.P., R.P., L.P. and S.P., then there is a uniformly dense homogeneous $\alpha$-tree $a$. 

82
Proof. We modify the proof of Lemma 13.12 by adding the condition that \( a(f) \neq 0 \) for all \( f \). Instead of adding 0s in the proof, we split the last element. We apply Proposition 13.10 instead of 13.9.

### Proposition 13.14

Let \( \alpha \in \triangle M \). Then the following are equivalent:

(i) There is a stable \( M \)-measure \( \sigma \) on \( \operatorname{Fr}(\omega) \) such that \( ((\triangle \sigma))(1) = \alpha \).

(ii) \( \alpha \) has L.P.

**Proof.** (i) \( \Rightarrow \) (ii): Assume (i). Then (ii) holds by Proposition 13.8.

(ii) \( \Rightarrow \) (i): Assume (ii). Thus \( \alpha \) satisfies C.P., R.P., S.P., and L.P. Hence by Proposition 13.13 there is a uniformly dense homogeneous \( \alpha \)-tree \( a \).

For each \( n \in \omega \) and each \( f \in D_n \) let

\[
13.14(1) \quad x(f) = \{ g \in \omega^2 : f \subseteq g \}.
\]

Then \( \{ x(f) : f \in D \} \) is a base for \( \operatorname{clop}(\omega^2) \), which is a denumerable atomless BA.

13.14(2) If \( k \leq i \in \omega \) and \( f \in D_k \), then \( x(f) = \bigcup \{ x(g) : g \in D_i, f \subseteq g \} \), and this union is pairwise disjoint.

13.14(3) If \( 1 = y_0 \uplus y_1 \uplus \cdots \uplus y_{n-1} \) with each \( y_i \) clopen, then there is a \( k \in \omega \) and a partition

\[
D_k = G_0 \cup G_1 \cup \cdots \cup G_{n-1}
\]

such that \( \forall j < n \{ x_j = \sum \{ x(f) : f \in G_j \} \} \).

In fact, since \( \{ x(f) : f \in D \} \) is a base for \( \operatorname{clop}(\omega^2) \) and each \( y_i \) is clopen and hence compact, there exist finite subsets \( D^i \) of \( D \) for \( i < n \) such that \( \forall i < n \{ y_i = \bigcup \{ x(f) : f \in D^i \} \} \). Let \( k \) be greater than the domain of \( f \), for all \( f \in \bigcup_{i < n} D^i \). By 13.14(2) we may assume that each \( D^i \) is a subset of \( D_k \). Clearly the sets \( D^i \) are pairwise disjoint. For any \( f \in k^2 \) the set \( x(f) \) is a nonempty set, and hence \( x(f) \cap y_i \neq \emptyset \) for some \( i < n \). So we must actually have \( f \in D^i \). This proves 13.14(3).

Now let \( z \in \operatorname{clop}(\omega^2) \). Then by compactness there is a finite \( X \subseteq D \) such that \( z = \bigcup \{ x(f) : f \in X \} \). Hence by 13.14(2) there is a \( k \in \omega \) and a \( G \subseteq D_k \) such that \( z = \sum \{ x(f) : f \in G \} \) (disjoint sum) We then define

\[
\sigma(z) = \sum \{ a(f) : f \in G \}.
\]

This definition does not depend on the particular choice of \( k \) and \( G \). For, suppose that also \( l \in \omega \), \( H \subseteq D_l \), and \( z = \sum \{ x(f) : f \in H \} \); we claim that \( \sum \{ a(f) : f \in G \} = \sum \{ a(f) : f \in H \} \). For, suppose that \( k \leq l \), and let \( K = \{ f \in D_l : \exists g \in G \} \subseteq f \}. \) Then \( z = \sum \{ x(f) : f \in K \} \) by (88). Since \( x(f) \cap x(g) = \emptyset \) for distinct \( f, g \in D_l \), it follows that \( K = H \). Then by 13.14(2) we get

\[
\sum \{ a(f) : f \in G \} = \sum \{ a(g) : g \in K \} = \sum \{ a(f) : f \in H \}.
\]

Now to show that \( \sigma \) is additive, suppose that \( z, z' \in \operatorname{clop}(\omega^2) \) and \( z \cdot z' = 0 \). Choose \( k, k', G, G' \) such that \( G \subseteq D_k, G' \subseteq D_{k'}, z = \sum \{ x(f) : f \in G \} \), and \( z' = \sum \{ x(f) : f \in G' \} \); thus \( \sigma(z) = \sum \{ a(f) : f \in G \} \) and \( \sigma(z') = \sum \{ a(f) : f \in G' \} \). Wlog \( k \leq k' \). Let \( H = \{ f \in D_{k'} : \exists g \in G \} \subseteq f \}. \) Then by 13.14(2), \( z = \sum \{ x(f) : f \in H \} \) and \( \sigma(z) = \sum \{ a(f) : f \in H \} \) and \( \sigma(z) = \sum \{ a(f) : f \in H \} \).
\[ \sum \{a(f) : f \in H\}. \] Since \( z \cdot z' = 0 \) we have \( H \cap G' = \emptyset \). Now \( z + z' = \sum \{x(f) : f \in H \cup G'\} \), so
\[ \sigma(z + z') = \sum \{a(f) : f \in H \cup G'\} = \sum \{a(f) : f \in H\} + \sum \{a(f) : f \in G'\} = \sigma(z) + \sigma(z'). \]

Since \( a \) is homogeneous, clearly \( \sigma(z) = 0 \) iff \( z = 0 \). So \( \sigma \in \mathcal{M}(M) \).

Next we show that \( (\Delta \sigma)(1) = \alpha \). Suppose that \( b \in (\Delta \sigma)(1) \). Say \( n \in \omega \), \( 1 = z_0 + z_1 + \cdots + z_{n-1} \) and \( b = (\sigma(z_0), \sigma(z_1), \ldots, \sigma(z_{n-1})) \). \((89)\) gives \( k \in \omega \) and \( G_0, \ldots, G_{n-1} \) such that \( \mathcal{D}_k = G_0 \cup G_1 \cup \cdots \cup G_{n-1} \) and \( \forall j < n \) \( \sigma(z_j) = \sum \{a(f) : f \in G_j\} \). So \( \forall j < n \) \( \sigma(z_j) = \sum \{a(f) : f \in G_j\} \). Define \( \lambda : 2^k \to n \) by \( \lambda(m) = j \) iff \( \text{lex}(m) \in G_j \). Then \( \forall j < n \) \( \sigma(z_j) = \sum \{a(\text{lex}(m)) : m < 2^k, \lambda(m) = j\} \). Hence \( a_k < b \), hence \( b \in \alpha \) by \((63)\) and C.P. This shows that \( (\Delta \sigma)(1) \subseteq \alpha \).

For the other inclusion, first note that
\[ 13.14(4) \forall f \in \mathcal{D}[\sigma(x(f)) = a(f)] \]
In fact, let \( f \in \mathcal{D} \). Let \( z = x(f) \) and \( G = \{f\} \). Then by definition \( \sigma(x(f)) = a(f) \).
\[ 13.14(5) \forall k \in \omega \left[ a_k \in (\Delta \sigma)(1) \right]. \]
For, let \( k \in \omega \). Then
\[ a_k = a(\mathcal{D}_k) = \langle a(\text{lex}(i)) : i < 2^k \rangle = \langle \sigma(\text{lex}(i)) : i < 2^k \rangle \in (\Delta \sigma)(1) \]
since \( 1 = \sum \{x(\text{lex}(i)) : i < 2^k\} \), a disjoint sum. So \( 13.14(5) \) holds.

Now take any \( a \in \alpha \). Since \( a \) is dense there is a \( k \in \omega \) such that \( a_k < b \). Now \( (\Delta \sigma)(1) \subseteq \Delta M \) by Proposition 10.1, so \( b \in (\Delta \sigma)(1) \) by \( 13.14(5) \) and C.P.

So \( (\Delta \sigma)(1) = \alpha \).

It remains only to show that \( \sigma \) is stable. Suppose that \( y, z \in \text{Fr}(\omega) \) and \( \sigma(y) = \sigma(z) \). Choose \( n \in \omega \) and \( G \subseteq \mathcal{D}_n \) so that \( y = \sum_{f \in G} x(f) \). Then \( \sigma(y) = \sum_{f \in G} \sigma(x(f)) = \sum_{f \in G} a(f) \) by \( 13.14(4) \). Thus \( \sigma(y) = T(a(G)) \). Now suppose that \( b \in ((\Delta \sigma))(z) \). Then \( T(b) = T((\Delta \sigma)(z)) = \sigma(z) \) (by Proposition 10.2) = \( \sigma(y) = T(a(G)) \). Since \( a \) is uniformly dense, there is an \( m \geq n \) such that \( a(G \upharpoonright m) < b \). Now \( \sum_{f \in G \upharpoonright m} x(f) = \sum_{f \in G} x(f) = y \), so \( a(G \upharpoonright m) = \langle \sigma(x(\text{lex}(i))) : i < |G \upharpoonright m| \rangle \in ((\Delta \sigma))(y) \). Hence \( b \in ((\Delta \sigma))(y) \) by C.P.

This proves that \( ((\Delta \sigma))(z) \subseteq ((\Delta \sigma))(y) \). The other inclusion follows by symmetry, so \( ((\Delta \sigma))(z) = ((\Delta \sigma))(y) \). Thus \( \sigma \) is stable. \( \square \)

### 14. Iterated derivatives

**Proposition 14.1.** Let \( M \) be an \( m \)-monoid. Then there are functions \( \langle \Delta^\zeta M : \zeta < \omega_1 \rangle \) and \( \langle T^\eta_\zeta : \eta \leq \zeta < \omega_1 \rangle \) with the following properties, for any \( \zeta < \omega_1 \) and \( \eta \leq \zeta \): 

14.1(1) If \( \zeta = 0 \), then \( \Delta^\zeta M = M \).

14.1(2) \( T^\zeta_\zeta \) is the identity on \( \Delta^\zeta M \).

14.1(3) If \( \zeta = \eta + 1 \), then \( \Delta^\zeta M = \Delta(\Delta^\eta M) \).

14.1(4) If \( \zeta = \eta + 1 \) and \( \xi \leq \eta \), then \( T^\zeta_\xi = T^\eta_\xi \circ T \), with \( T : \Delta(\Delta^\xi M) \to \Delta^\xi M \).
14.1(5) If $\xi \leq \eta \leq \zeta$, then $T^\eta_\xi \circ T^\zeta_\eta = T^\zeta_\xi$

14.1(6) $\forall \eta \leq \zeta[\triangle M \text{ is an m-mono-}

14.1(7) If $\zeta$ is a limit ordinal, then

$$\triangle M = \{a : \text{dom}(a) = \zeta, \forall \eta < \zeta[a_\eta \in \triangle M], \forall \xi, \eta < \zeta[\xi < \eta \rightarrow T^n_\eta(a_\eta) = a_\xi]\}.$$ 

14.1(8) If $\zeta$ is a limit ordinal, then $\forall \eta < \zeta \forall a \in \triangle M[T^\zeta_\eta(a) = a_\eta]$. 

14.1(9) If $\eta \leq \zeta < \omega_1$ then $T^\zeta_\eta : \triangle M \rightarrow \triangle M$. 

14.1(10) If $\zeta$ is a limit ordinal, $x, y \in \triangle M$, and $T^\zeta_\eta(x) = T^\zeta_\eta(y)$ for all $\eta < \zeta$, then $x = y$. 

**Proof.** We construct $\triangle M$ and $T^\zeta_\eta$ by recursion on $\zeta$. For $\zeta = 0$ we take 14.1(1) and 14.1(2) as definitions. Clearly the other conditions hold. For the successor case we take 14.1(1)–14.1(4) as definitions. For 14.1(5), with $\zeta = \eta + 1$, assume that $\xi \leq \rho \leq \zeta$. If $\rho = \zeta$ the conclusion is clear. If $\rho < \zeta$, then $T^\rho_\xi \circ T^\zeta_\rho = T^\rho_\rho \circ T = T^\rho_\xi \circ T = T^\zeta_\xi$. 14.1(6) and 14.1(7) are clear.

Now for $\zeta$ limit, we use 14.1(2), 14.1(7), and 14.1(8) as definitions. Then 14.1(9) is clear. To check 14.1(5), suppose that $\xi \leq \eta < \zeta$. Then $\forall a \in \triangle M[T^\eta_\xi(a) = T^\eta_\xi(a_\eta) = a_\xi = T^\zeta_\eta(a)]$. To check 14.1(6), suppose $a, b \in \triangle M$. Then $\forall \eta < \zeta[\langle (a+b)_\eta = a_\eta + b_\eta \in \triangle M\rangle].$ Also, if $\xi < \eta < \zeta$, then $T^\eta_\xi((a+b)_\eta) = T^\eta_\xi(a_\eta + b_\eta) = T^\eta_\xi(a_\eta) + T^\eta_\xi(b_\eta) = a_\xi + b_\xi = (a + b)_\xi$. So this shows that $a + b \in \triangle M$. Now the monoid conditions are clear.

It remains to check 14.1(10). So assume that $\zeta$ is a limit ordinal, $x, y \in \triangle M$, and $T^\zeta_\eta(x) = T^\zeta_\eta(y)$ for all $\eta < \zeta$. Then for all $\eta < \zeta$, $x_\eta = T^\zeta_\eta(x) = T^\zeta_\eta(y) = y_\eta$. So $x = y$. 

**Proposition 14.2.** Let $M$ be an m-mono- and $\sigma \in M(M)$. Then for each $\zeta < \omega_1$ there is a unique $\triangle \zeta \sigma \in M(\triangle M)$ such that the following conditions hold:

14.2(1) For $\zeta = 0$, $\triangle \zeta \sigma = \sigma$. 

14.2(2) For $\zeta = \xi + 1$, $\triangle \zeta \sigma = \triangle(\triangle \xi \sigma)$. 

14.2(3) $\forall \eta \leq \zeta < \omega_1[T^\zeta_\eta \circ \triangle \zeta \sigma = \triangle^\eta \sigma]$. 

14.2(4) For $\zeta$ limit, $\forall x \in \text{Fr}(\omega)[((\triangle \zeta \sigma)(x) = (\langle \triangle \eta \sigma\rangle(x) : \eta < \zeta).$ 

**Proof.** We define $\triangle \zeta \sigma$ by recursion on $\zeta$. For $\zeta = 0$ we take 14.2(1) as a definition. For $\zeta = \xi + 1$ we take 14.2(2) as a definition. Then clearly 14.2(2) holds for $\eta = \zeta$, while for $\eta < \zeta, T^\eta_\xi \circ \triangle \zeta \sigma = T^\zeta_\eta \circ T \circ \triangle (\triangle \xi \sigma) = T^\zeta_\eta \circ \triangle \zeta \sigma$ (by Proposition 10.2) $= \triangle^\eta \sigma$. For $\zeta$ limit we take 14.2(3) as a definition. We check that $\triangle \zeta \sigma \in M(\triangle M)$. Take any $x \in \text{Fr}(\omega)$, and let $a = \langle (\triangle \eta \sigma)(x) : \eta < \zeta$. Then $a$ has domain $\zeta$, and for any $\eta < \zeta$, $a_\eta = \triangle^\eta \sigma(x) \in \triangle M$. If $\xi < \eta < \zeta$, then $T^\eta_\xi(a_\eta) = T^\eta_\xi(\langle (\triangle \eta \sigma)(x)\rangle) = (T^\eta_\xi \circ \triangle \eta \sigma)(x) = (\triangle \xi \sigma)(x) = a_\xi$. This checks that $(\triangle \zeta \sigma)(x) \in \triangle M$. Hence clearly $\triangle \zeta \sigma \in M(\triangle M)$. Finally we check 14.2(2). Suppose that $\eta < \zeta$. Then for any $x \in \text{Fr}(\omega)$, 

$$(T^\zeta_\eta \circ \triangle \zeta \sigma)(x) = T^\zeta_\eta(\langle (\triangle \xi \sigma)(x)\rangle) = T^\zeta_\eta(\langle (\triangle \rho \sigma)(x) : \rho < \zeta\rangle) = (\triangle \eta \sigma)(x)$$

**Proposition 14.3.** If $k$ is an automorphism of $\text{Fr}(\omega)$, $\sigma = \tau \circ k$, and $\xi$ is any ordinal, then $\triangle \xi \sigma = (\triangle \xi \tau) \circ k$. 

85
Proposition 14.2(2), \( \Delta^{\xi+1} = \Delta(\Delta^\xi) = \Delta(\Delta^\xi) \circ k \) (by Proposition 10.4) = \( \Delta^{\xi+1} \circ k \).

Now suppose that \( \xi \) is a limit ordinal and it holds for all \( \eta < \xi \). Then for any \( x \in \text{Fr}(\omega) \),

\[
(\Delta^\xi)(x) = (\{(\Delta^\eta)(x) : \eta < \xi\}) = (\{(\Delta^\eta)(k(x)) : \eta < \xi\}) = (\Delta^\xi)(k(x)).
\]

\[\square\]

15. The depth of measures

Recall the definition of stable measure from section 12.

Proposition 15.1. If \( M \) is an \( m \)-monoid and \( \sigma \in \mathcal{M}(M) \), then there is a smallest ordinal \( d(\sigma) < \omega_1 \) such that \( \forall \zeta \geq d(\sigma) [\Delta^\zeta \sigma \text{ is stable}] \).

Proof. For each \( \zeta < \omega_1 \) let \( E_\zeta = \{(x, y) \in \text{Fr}(\omega) \times \text{Fr}(\omega) : \Delta^\zeta \sigma(x) = \Delta^\zeta(y)\} \). Clearly \( E_\zeta \) is an equivalence relation on \( \text{Fr}(\omega) \).

15.1(1) If \( \eta \leq \zeta < \omega_1 \), then \( E_\zeta \subseteq E_\eta \).

In fact, suppose that \( \eta \leq \zeta < \omega_1 \) and \( (x, y) \in E_\zeta \). By Proposition 14.2(3), \( T_\eta^\zeta \circ \Delta^\zeta \sigma = \Delta^\eta \sigma \). Hence \( (\Delta^\eta \sigma)(x) = T_\eta^\zeta((\Delta^\zeta \sigma)(x)) = T_\eta^\zeta((\Delta^\zeta \sigma)(y)) = (\Delta^\eta)(y) \), so that \( (x, y) \in E_\eta \).

Since \( \text{Fr}(\omega) \) is countable, there is a smallest \( d(\sigma) < \omega_1 \) such that \( E_\zeta = E_{d(\sigma)} \) for all \( \zeta \geq d(\sigma) \). Hence if \( \zeta \geq d(\sigma) \), \( x, y \in \text{Fr}(\omega) \), and \( \Delta^\zeta \sigma(x) = \Delta^\zeta(y) \), then \( \Delta((\Delta^\zeta \sigma)(x) = (\Delta^\zeta^1 \sigma)(x) = (\Delta^\zeta^1 \sigma)(y) = \Delta((\Delta^\zeta \sigma)(y)) \). \[\square\]

\( d(\sigma) \) is the depth of \( \sigma \).

Proposition 15.2. If \( M \) is an \( m \)-monoid and \( \sigma, \tau \in \mathcal{M}(M) \), then the following are equivalent:

(i) \( \sigma = \tau \circ k \) for some automorphism \( k \) of \( \text{Fr}(\omega) \).

(ii) \( \forall \zeta < \omega_1 [(\Delta^\zeta \sigma)(1) = (\Delta^\zeta \tau)(1)] \).

(iii) There is a countable ordinal \( \zeta \geq d(\sigma), d(\tau) \) such that \( (\Delta^{\zeta+1} \sigma)(1) = (\Delta^{\zeta+1} \tau)(1) \).

Proof. (i)\( \Rightarrow \) (ii): Assume that \( k \) is an automorphism of \( \text{Fr}(\omega) \) and \( \sigma = \tau \circ k \).

15.2(1) \( \forall \zeta < \omega_1 [(\Delta^\zeta \sigma) = (\Delta^\zeta \circ k)] \).

We prove 15.2(1) by induction on \( \zeta \). \( (\Delta^0 \sigma) = \sigma = \tau \circ k = (\Delta^0 \tau) \circ k \). Assume that \( (\Delta^\xi \sigma) = (\Delta^\xi \tau) \circ k \). Then by Proposition 10.4, \( (\Delta^{\xi+1} \sigma) = (\Delta(\Delta^\xi \sigma)) \circ k = (\Delta(\Delta^\xi \tau)) \circ k \).

Now assume that \( \zeta \) is a limit ordinal, and \( (\Delta^\eta \sigma) = (\Delta^\eta \tau) \circ k \) for every \( \eta < \zeta \). Then for every \( x \in \text{Fr}(\omega) \) and every \( \eta < \zeta \), \( (\Delta^\eta \sigma)(x) = (\Delta^\eta \tau)(k(x)) = (\Delta^\eta \tau)(k(x)) \).

Thus 15.2(1) holds. Hence (ii) follows.

(ii)\( \Rightarrow \) (iii): obvious.

(iii)\( \Rightarrow \) (i): By Proposition 12.2 applied to \( (\Delta(\Delta^\xi \sigma))(1) = (\Delta^{\xi+1} \sigma)(1) = (\Delta(\Delta^\xi \tau))(1) \), there is an automorphism \( k \) of \( \text{Fr}(\omega) \) such that \( (\Delta^\xi \sigma) = (\Delta^\xi \tau) \circ k \). Hence by Proposition 14.2(3), \( \sigma = T_0^\xi \circ \Delta^\xi \sigma = T_0^\xi \circ (\Delta^\xi \tau) \circ k = \tau \circ k \). \[\square\]
Recall that $\mathcal{H}$ is $\omega_1$ with a new $o$ added, and $\mathcal{M}$ is $\mathcal{M}(\mathcal{H})$.

For each ordinal $\zeta$, let $\mathcal{K}^\zeta = \{(\triangle^{\zeta+1}\sigma)(1) : \sigma \in \mathcal{M}, d(\sigma) \leq \zeta\}$. The system of sets $\langle \mathcal{K}^\zeta : \zeta < \omega_1 \rangle$ is called the Boolean hierarchy.

**Proposition 16.1.** Define $\sigma \equiv \tau$ iff $\sigma, \tau \in \mathcal{M}$ and there is an automorphism $k$ of $\text{Fr}(\omega)$ such that $\sigma = \tau \circ k$. Then $\equiv$ is an equivalence relation on $\mathcal{M}$.

**Proposition 16.2.** For all $\sigma, \tau \in \mathcal{M}$ the following are equivalent:

(i) $\sigma \equiv \tau$.

(ii) $\forall \zeta > d(\sigma), d(\tau)[(\triangle^\zeta(\sigma)(1) = (\triangle^\zeta(\tau)(1)].$

**Proof.** By Proposition 15.2.

**Proposition 16.3.** For $\eta \leq \zeta < \omega_1$ there is an injection $\Sigma^\eta_\zeta : \mathcal{H}^\eta \rightarrow \mathcal{K}^\zeta$ such that

16.3(1) If $\xi \leq \eta \leq \zeta < \omega_1$, then $\Sigma^\eta_\zeta \circ \Sigma^\eta_\zeta = \Sigma^\eta_\zeta$.

16.3(2) If $\eta \leq \zeta$, then $T^{\zeta+1}_{\eta+1} \circ \Sigma^\eta_\zeta$ is the identity on $\mathcal{H}^\eta$.

16.3(3) If $\rho \leq \eta \leq \zeta$, then $\text{rng}(\Sigma^\rho_\zeta) \subseteq \text{rng}(\Sigma^\rho_\zeta)$.

**Proof.** Assume that $\eta \leq \zeta < \omega_1$. Take any $\sigma \in \mathcal{M}$ with $d(\sigma) \leq \eta$. Define $\Sigma^\eta_\zeta(\triangle^{\eta+1}\sigma)(1)) = (\triangle^{\zeta+1}\sigma)(1)$. Then $\Sigma^\eta_\zeta$ is well-defined. For, suppose that also $\tau \in \mathcal{M}$, $d(\tau) \leq \eta$, and $(\triangle^{\eta+1}\sigma)(1) = (\triangle^{\eta+1}\tau)(1)$. Then by Proposition 16.2, $(\triangle^{\zeta+1}\sigma)(1) = (\triangle^{\zeta+1}\tau)(1)$.

$\Sigma^\eta_\zeta$ is one-one. For suppose that $\sigma, \tau \in \mathcal{M}$, $d(\sigma), d(\tau) \leq \eta$, and $(\triangle^{\zeta+1}\sigma)(1) = (\triangle^{\zeta+1}\tau)(1)$. Then by Proposition 14.2,

$$(\triangle^{\eta+1}\sigma)(1) = T^{\zeta+1}_{\eta+1}(\triangle^{\zeta+1}\sigma)(1)) = T^{\zeta+1}_{\eta+1}(\triangle^{\zeta+1}\tau)(1)) = (\triangle^{\eta+1}\tau)(1).$$

Now (16.3(1)) is obvious.

For (16.3(2)), suppose that $\eta \leq \zeta$, $\sigma \in \mathcal{M}$, and $d(\sigma) \leq \eta$. Then, using Proposition 14.2,

$$T^{\zeta+1}_{\eta+1}(\Sigma^\eta_\zeta(\triangle^{\eta+1}\sigma)(1)) = T^{\zeta+1}_{\eta+1}(\triangle^{\eta+1}\sigma)(1) = (\eta+1)^{\eta+1}(\sigma)(1).$$

For 16.3(3), $\text{rng}(\Sigma^\rho_\zeta) = \{(\triangle(\sigma)(1) : d(\sigma) \leq \rho) \subseteq \{(\triangle(\sigma)(1) : d(\sigma) \leq \eta) = \text{rng}(\Sigma^\rho_\zeta))\}.$

Now let

$$\mathcal{K} = \{a : \forall \eta < \omega_1[a \text{ is a function with domain } [\eta, \omega_1) \text{ and } \forall \zeta \in [\eta, \omega_1)a(\xi) \in \mathcal{K}^\zeta \text{ and } \forall \xi, \zeta \in [\eta, \omega_1][\xi \leq \zeta \rightarrow \Sigma^\zeta(\alpha) = \alpha]\}$$

For any $\eta < \omega_1$ define $\Sigma^\eta$ with domain $\mathcal{K}^\eta$: for each $a \in \mathcal{K}^\eta$, $\Sigma^\eta(a)$ is the function with domain $[\eta, \omega_1)$ such that for any $\zeta \in [\eta, \omega_1)$, $(\Sigma^\eta(a))(\zeta) = \Sigma^\zeta(a)$.

**Proposition 16.4.** For each $\eta < \omega_1$, $\Sigma^\eta$ is an injection from $\mathcal{K}^\eta$ into $\mathcal{K}$, and $\mathcal{K} = \bigcup_{\eta < \omega_1}\text{rng}(\Sigma^\eta)$. Moreover, if $\xi \leq \eta < \omega_1$ then $\Sigma^\eta \circ \Sigma^\xi = \Sigma^\xi.
Proof. Suppose that \( a \in \mathcal{K}^\eta \). Then \( \Sigma^\eta(a) \) is a function with domain \( \mathcal{K}^\eta \),

\[
\forall \zeta \in [\eta, \omega_1]((\Sigma^\eta(a))(\zeta) = \Sigma^\eta_{\eta}(a) \in \mathcal{K}^\zeta) \quad \text{and} \quad \forall \xi, \rho \in [\eta, \omega_1][\xi \leq \rho \rightarrow \Sigma^\xi_{\zeta}(\Sigma^\eta(a))(\xi) = \Sigma^\xi_{\zeta}(\Sigma^\eta_{\eta}(a)) = \Sigma^\eta_{\zeta}(a) = (\Sigma^\eta(a))(\rho).
\]

Thus \( \Sigma^\eta(a) \in \mathcal{K} \).

If \( a, b \) are distinct elements of \( \mathcal{K}^\eta \), then \( (\Sigma^\eta(a))(\eta) = \Sigma^\eta_{\eta}(a) = a \neq b = \Sigma^\eta_{\eta}(b) = (\Sigma^\eta(a))(\eta) \). So \( \Sigma^\eta(a) \neq \Sigma^\eta(b) \). Thus \( \Sigma^\eta \) is an injection from \( \mathcal{K}^\eta \) into \( \mathcal{K} \). Now suppose that \( a \in \mathcal{K} \). Choose \( \eta < \omega_1 \) such that \( a \) is a function with domain \([\eta, \omega_1] \) and \( \forall \zeta \in [\eta, \omega_1][a_\zeta \in \mathcal{K}^\zeta] \) and \( \forall \xi, \zeta \in [\eta, \omega_1][\xi \leq \zeta \rightarrow \Sigma^\xi_{\zeta}(a_\zeta) = a_\zeta] \). Let \( b = a_\eta \). Thus \( b \in \mathcal{K}^\eta \). We claim that \( a = \Sigma^\eta(b) \). In fact, take any \( \zeta \in [\eta, \omega_1] \). Then \( (\Sigma^\eta(b))(\zeta) = \Sigma^\eta_{\eta}(b) = \Sigma^\eta_{\eta}(a_\eta) = a_\zeta \).

Now we define \( \langle L^\zeta : \zeta < \omega_1 \rangle \) and \( \langle \rho^\zeta : \zeta < \omega_1 \rangle \) by recursion, so that:

(Z1) \( L^0 \subseteq L^1 \subseteq \cdots \);
(Z2) \( \forall \zeta < \omega_1[\rho^\zeta \) is a bijection from \( \mathcal{K}^\zeta \) onto \( L^\zeta \);
(Z3) \( \forall \eta, \zeta \in \omega_1[\eta < \zeta \rightarrow \rho^\zeta \circ \Sigma^\eta = \rho^\eta] \).

Let \( L^0 = \mathcal{K}^0 \) and let \( \rho^0 \) be the identity on \( \mathcal{K}^0 \). Suppose that \( L^\zeta \) and \( \rho^\zeta \) have been defined so that (Z1)–(Z3) hold. Note that \( \rho^\zeta \circ (\Sigma^\zeta_{\zeta})^{-1} \) is a bijection from \( \text{rng}(\Sigma^\zeta_{\zeta}) \) onto \( L^\zeta \). Let \( M = \{ (L^\zeta, x) : x \in \mathcal{K}^{\zeta+1} \setminus \text{rng}(\Sigma^\zeta_{\zeta}) \} \). Thus \( |M| = |\mathcal{K}^{\zeta+1} \setminus \text{rng}(\Sigma^\zeta_{\zeta})| \) and \( M \cap L^\zeta = \emptyset \). Let \( L^{\zeta+1} = L^\zeta \cup M \) and \( \rho^{\zeta+1} = \rho^\zeta \circ (\Sigma^\zeta_{\zeta})^{-1} \cup k \), where \( k \) is a bijection from \( \mathcal{K}^{\zeta+1} \setminus \text{rng}(\Sigma^\zeta_{\zeta}) \) onto \( M \). Thus \( L^\zeta \subseteq L^{\zeta+1} \) and \( \rho^{\zeta+1} \) is a bijection from \( \mathcal{K}^{\zeta+1} \) onto \( L^{\zeta+1} \). Moreover, if \( \eta \leq \zeta \), then

\[
\rho^{\zeta+1} \circ \Sigma^\zeta_{\eta+1} = \rho^{\zeta+1} \circ \Sigma^\zeta_{\zeta} \circ \Sigma^\eta_{\eta} = \rho^\zeta \circ \Sigma^\eta_{\eta} = \rho^\eta.
\]

This is illustrated by the following commutative diagram:

Now suppose that \( \zeta < \omega_1 \) is a limit ordinal. Then \( \mathcal{K}^\zeta = \bigcup_{\eta \leq \zeta} \text{rng}(\Sigma^\eta_{\zeta}) \). In fact, let \( \sigma \in \mathcal{M} \) with \( d(\sigma) \leq \zeta \); so \( (\Delta^{\zeta+1}\sigma)(1) \in \mathcal{K}^\zeta \). Then \( \Sigma^\zeta_{d(\sigma)}((\Delta^{d(\sigma)+1}\sigma)(1)) = (\Delta^{\zeta+1}\sigma)(1) \).

Now let \( N = \bigcup_{\eta < \zeta} L^\eta \), and let \( M = \{(N, x) : x \in \mathcal{K}^\zeta \setminus \bigcup_{\eta < \zeta} \text{rng}(\Sigma^\eta_{\zeta}) \} \). Let \( L^\zeta = N \cup M \). We now define \( \rho^\zeta \) with domain \( \mathcal{K}^\zeta \). Take any \( x \in \mathcal{K}^\zeta \). Say \( x = (\Delta^{\zeta+1}\sigma)(1) \) with \( d(\sigma) \leq \zeta \). If \( d(\sigma) < \zeta \), note that \( \Sigma^\zeta_{d(\sigma)}((\Delta^{d(\sigma)+1}\sigma)(1)) = (\Delta^{\zeta+1}\sigma)(1) = x \), and let
\( \rho^\zeta(x) = \rho^{d(\sigma)}((\Delta^{d(\sigma)+1}\sigma)(1)) \). If \( d(\sigma) = \zeta \) let \( \rho^\zeta(x) = k(x) \), where \( k \) is a bijection from \( \mathcal{K}^\zeta \setminus \bigcup_{\eta<\zeta} \text{rng}(\Sigma^\eta) \) onto \( M \).

Now we check (Z3) for \( \zeta \). Suppose that \( \eta < \zeta \), and let \( x \in \mathcal{K}^\eta \). Say \( x = (\Delta^{\eta+1}\sigma)(1) \) with \( d(\sigma) \leq \eta \). Then

\[
\rho^\zeta(\Sigma^\eta(\zeta)) = \rho^\zeta(\Sigma^\eta((\Delta^{\eta+1}\sigma)(1))) = \rho^\zeta((\Delta^{\zeta+1}\sigma)(1)) = \rho^{d(\sigma)}((\Delta^{d(\sigma)+1}\sigma)(1))
\]

\[
= \rho^\eta(\Sigma^\eta_d((\Delta^{d(\sigma)+1}\sigma)(1))) = \rho^\eta((\Delta^{\eta+1}\sigma)(1)) = \rho^\eta(x).
\]

Next, \( \rho^\zeta \) is one-one. For, suppose that \( x, y \in \mathcal{K}^\zeta \), \( \rho^\zeta(x) = \rho^\zeta(y) \), and \( x \neq y \). Then we must have \( x, y \in \bigcup_{\eta<\zeta} \text{rng}(\Sigma^\eta) \). Say \( x = \Sigma^\eta((\Delta^{\eta+1}\sigma)(1)) \) and \( y = \Sigma^\zeta((\Delta^{\tau+1}\sigma)(1)) \), with \( \eta, \tau < \zeta \). Then \( x = (\Delta^{\zeta+1}\sigma)(1) \) and \( y = (\Delta^{\zeta+1}\tau)(1) \). Hence \( \rho^\zeta(x) = \rho^{d(\sigma)}((\Delta^{d(\sigma)+1}\sigma)(1)) \) and \( \rho^\zeta(y) = \rho^{d(\tau)}((\Delta^{d(\tau)+1}\tau)(1)) \). Say \( d(\sigma) \leq d(\tau) \). Then

\[
\rho^{d(\tau)}(\Sigma^{d(\tau)}_d((\Delta^{d(\sigma)+1}\sigma)(1))) = \rho^{d(\sigma)}((\Delta^{d(\sigma)+1}\sigma)(1))
\]

\[
= \rho^\zeta(x) = \rho^\zeta(y)
\]

\[
= \rho^{d(\tau)}((\Delta^{d(\tau)+1}\tau)(1))
\]

Hence \( (\Delta^{d(\tau)+1}\sigma)(1) = \Sigma^{d(\tau)}_d((\Delta^{d(\sigma)+1}\sigma)(1)) = (\Delta^{d(\tau)+1}\tau)(1) \). So

\[
y = (\Delta^{\zeta+1}\tau)(1) = \Sigma^{\zeta}_d((\Delta^{d(\tau)+1}\tau)(1) = \Sigma^{\zeta}_d((\Delta^{d(\tau)+1}\sigma)(1)) = (\Delta^{\zeta+1}\sigma)(1) = x,
\]

contradiction.

Next, \( \rho^\zeta \) maps onto \( L^\zeta \). In fact, if \( x \in L^\zeta \) then there are two cases.

**Case 1.** \( x \in \mathbb{N} \). Say \( \eta < \zeta \) and \( x \in \mathbb{N}^\eta \). Then there is a \( y \in \mathbb{K}^\eta \) such that \( \rho^\eta(y) = x \). Say \( y = (\Delta^{\eta+1}\sigma)(1) \) with \( d(\sigma) \leq \eta \). Then \( \Sigma^\eta(y) \in \mathcal{K}^\zeta \), and

\[
\rho^\zeta(\Sigma^\eta(y)) = \rho^\zeta(\Sigma^\eta((\Delta^{\eta+1}\sigma)(1))) = \rho^\eta((\Delta^{\eta+1}\sigma)(1)) = \rho^\eta(y) = x.
\]

**Case 2.** \( x \in M \). Obviously there is a \( y \in \mathcal{K}^\zeta \) such that \( \rho^\zeta(y) = x \).

Thus \( \rho^\zeta \) is a bijection \( L^\zeta \).

Hence for \( \zeta \) limit we have the following diagram:

\[
\begin{array}{ccc}
L^\eta & \subseteq & \cdots & \subseteq & L^\zeta \\
\downarrow \rho^\eta & & & & \downarrow \rho^\zeta \\
\mathcal{K}^\eta & \to & \mathcal{K}^\zeta \\
\Sigma^\eta & & \Sigma^\zeta
\end{array}
\]

89
Finally, define \( \tau : \mathcal{K} \to \bigcup_{\eta<\omega} L^\eta \) by defining, for any \( a \in \mathcal{K} \), with \( \text{dmm}(a) = [\eta, \omega_1) \), \( \tau(a) = \rho^n(a_\eta) \). Then for any \( a \in \mathcal{K}^n \) we have

\[
\tau(\Sigma^n(a)) = \tau(\langle \Sigma^z_\eta(a) : z \in [\eta, \omega_1) \rangle) = \rho^n(\Sigma^z_\eta(a)).
\]

This gives the following diagram:

\[
\begin{array}{ccc}
L^n & \subseteq \cdots & \bigcup_{\eta<\omega} L^n \\
\downarrow^{\rho^n} & & \downarrow^{\tau} \\
\mathcal{K}^n & \longrightarrow & \mathcal{K} \\
\Sigma^n & \nearrow &
\end{array}
\]

17. The heirarchy property

**Lemma 17.1.** \( \mathcal{K}^\xi \subseteq \Delta(\Delta^\xi W) \).

**Proof.** Let \( x \in \mathcal{K}^\xi \). Then there is a \( \sigma \in \mathcal{M}(W) \) such that \( d(\sigma) = \zeta \) and \( x = (\Delta^{\xi+1}\sigma)(1) \). By Proposition 14.2, \( x \in \Delta^{\xi+1}W \), and by Proposition 14.2(103), \( \Delta^{\xi+1}W = \Delta(\Delta^\xi W) \).

**Lemma 17.2.** Let \( \alpha \in \mathcal{K}^\xi \), and suppose that \( a \in \Delta^\xi W \) with \( \langle a \rangle \in \Phi(\alpha) \). Also suppose that \( \eta < \zeta \) and \( c \in T_{\eta+1}^\xi(a) \). Then there is a \( b \in \Phi_a(\alpha) \) such that \( c = T_{\eta}^\xi \circ b \).

**Proof.** By definition of \( \mathcal{K}^\xi \), there is a \( \sigma \in \mathcal{M} \) with \( d(\sigma) \leq \zeta \) such that \( \alpha = (\Delta^{\xi+1}\sigma)(1) \). By Proposition 15.1, \( \Delta^\xi \sigma \) is stable. Now by Proposition 17.1 \( \alpha \in \Delta(\Delta^\xi W) \), so by definition \( \alpha \subseteq \ll \omega(\Delta^\xi W) \). Now \( \langle a \rangle \in \Phi(\alpha) \), so there is a \( b \in \ll \omega(\Delta^\xi W) \) such that \( \langle a \rangle \sim b \in \alpha = (\Delta^{\xi+1}\sigma)(1) = (\Delta(\Delta^\xi \sigma))(1) \). So if \( b \) has domain \( m \) we can write \( 1 = x + y_0 + \cdots + y_{m-1} \) with \( (\Delta^\xi \sigma)(x) = a \) and \( (\Delta^\xi \sigma)(y_i) = b_i \) for all \( i < m \). Now \( c \in T_{\eta+1}^\xi(a) = T_{\eta+1}^\xi((\Delta^\xi \sigma)(x)) = (\Delta^{\eta+1}\sigma)(x) = (\Delta^{\eta}\sigma)(x) \). Hence we can write \( x = d_0 + \cdots + d_{n-1} \) with \( c = (\langle (\Delta^\eta \sigma)(d_0), \ldots, (\Delta^\eta \sigma)(d_{n-1}) \rangle) \). Let \( b_i = (\Delta^\eta \sigma)(d_i) \) for all \( i < n \). Then \( T(b) = \sum_{i<n} (\Delta^\xi \sigma)(d_i) = (\Delta^\xi \sigma)(x) = a \). Now \( 1 = d_0 + \cdots + d_{n-1} + y_0 + \cdots + y_{m-1} \), so \( (\langle (\Delta^\xi \sigma)(d_0), \ldots, (\Delta^\xi \sigma)(d_{n-1}), (\Delta^\xi \sigma)(y_0), \ldots, (\Delta^\xi \sigma)(y_{m-1}) \rangle) \subseteq (\Delta(\Delta^\xi \sigma))(1) = (\Delta^{\xi+1}\sigma)(1) = \alpha \). Thus \( b \in \Phi_a(\alpha) \). Finally, for all \( i < n \), \( T_{\eta}^\xi(b_i) = T_{\eta}^\xi((\Delta^\xi \sigma)(d_i)) = (\Delta^\eta \sigma)(d_i) = c_i \).

Now let \( M \) be an \( m \)-monoid and \( \zeta < \omega_1 \). A set \( \alpha \in \Delta(\Delta^\xi M) \) has the heirarchy property (H.P.) iff \( \forall a \in \Delta^\xi M \langle a \rangle = \Phi(\alpha) \rightarrow \forall \eta < \zeta \forall c \in T_{\eta+1}^\xi \exists b \in \Phi_a(\alpha)[c = T_{\eta}^\xi \circ b] \). Note that this condition holds vacuously if \( \zeta = 0 \).

**Proposition 17.3.** If \( \zeta < \omega_1 \) and \( \alpha \in \Delta^{\xi+1} \mathcal{K} \), then the following conditions are equivalent:

\[
90
\]
(i) \( \alpha \in \mathcal{K} \).

(ii) \( \alpha \) satisfies L.P. and H.P.

**Proof.** (i)⇒(ii): Assume (i). Then by the definition of \( \mathcal{K} \) and \( d(\sigma) \) and Proposition 13.8, \( \alpha \) has L.P. By Lemma 17.2, \( \alpha \) has R.P.

(ii)⇒(i): By Proposition 14.1(94) we have \( \alpha \in \triangle(\triangle^\zeta \mathcal{W}) \). Hence by Proposition 13.14 there is a stable \( \triangle^\zeta \mathcal{W} \)-measure \( \tau \) such that \( (\triangle\tau)(1) = \alpha \).

17.3(1) \( \forall x \in \text{Fr}(\omega) \forall \eta < \zeta \forall c \in (T^\zeta_{\eta+1} \circ \tau)(x) \forall m \in \omega \) [\( dmn(c) = m \rightarrow \exists y \in m \text{Fr}(\omega) [x = y_0 + \cdots + y_{m-1} \text{ and } \forall j < m \{ c_j = (T^\zeta_{\eta})(\tau(y_j)) \}] \)].

In fact, suppose that \( x \in \text{Fr}(\omega), \eta < \zeta, c \in (T^\zeta_{\eta+1} \circ \tau)(x), m \in \omega, \) and \( dmn(c) = m \). Now \( x + -x = 1 \), so \( \langle \tau(x), \tau(-x) \rangle \in (\triangle\tau)(1) = \alpha \), and hence \( \langle \tau(x) \rangle \in \Phi(\alpha) \). Since \( c \in T^\zeta_{\eta+1}(\tau(x)) \), we can apply H.P. to \( \tau(x) \) in place of \( a \) to get \( b \in \Phi_{\tau(x)}(\alpha) \) such that \( c = T^\zeta_{\eta} \circ b \). Now there is a \( d \) such that \( b \cdot d = \alpha = (\triangle\tau)(1) \), so we can write \( 1 = y_0 + \cdots + y_{m-1} \) with \( b \cdot d = \langle \tau(y_0), \ldots, \tau(y_{m-1}) \rangle \). Let \( z = y_0 + \cdots + y_{m-1} \). Then \( b = \langle \tau(y_0), \ldots, \tau(y_{m-1}) \rangle \) and so \( b \in (\triangle\tau)(z) \). Then \( \tau(z) = \tau(y_0) + \cdots + \tau(y_{m-1}) = T(b) = \tau(x) \) since \( b \in \Phi_{\tau(x)}(\alpha) \).

Since \( \tau \) is stable it follows that \( \triangle\tau(z) = \triangle\tau(x) \), so \( b \in \triangle\tau(x) \). Hence we can write \( x = d_0 + \cdots + d_{m-1} \) with \( b = \langle \tau(d_0), \ldots, \tau(d_{m-1}) \rangle \). So for any \( j < m, c_j = T^\zeta_{\eta}(b_j) = T^\zeta_{\eta}(\tau(d_j)) \).

This proves 17.3(1).

Now let \( \sigma = T^\zeta_0 \circ \tau \). Note that \( \tau : \text{Fr}(\omega) \rightarrow \triangle^\zeta \mathcal{W} \) and \( T^\zeta_0 : \triangle^\zeta \mathcal{W} \rightarrow \mathcal{W} \). So \( \sigma : \text{Fr}(\omega) \rightarrow \mathcal{W} \), i.e., \( \sigma \in \mathcal{M} \). Now we claim

17.3(2) \( \forall \eta \leq \zeta[T^\zeta_{\eta} \circ \tau = \triangle^\eta \sigma] \).

Note that with \( \eta = \zeta \) in 17.3(2) we get \( \tau = \triangle^\zeta \sigma \). Since \( (\triangle\tau)(1) = \alpha \), this gives \( \alpha = (\triangle^\zeta+1 \sigma)(1) \). Since \( \tau \) is stable and \( \tau = \triangle^\zeta \sigma \), we have \( d(\sigma) \leq \zeta \). Hence \( \alpha \in \mathcal{K} \), finishing the proof.

We prove 17.3(2) by induction on \( \eta \). It is obvious for \( \eta = 0 \). Now suppose that it holds for \( \eta \). Suppose that \( c \in (T^\zeta_{\eta+1} \circ \tau)(x) \). By (114) write \( x = y_0 + \cdots + y_{m-1} \) with \( \forall j < m \{ c_j = T^\zeta_{\eta}(\tau(x_j)) \} \). By the inductive hypothesis \( \forall j < m[T^\zeta_{\eta}(\tau(x_j)) = (\triangle^\eta \sigma)(x_j) \). So \( c \in (\triangle(\triangle^\eta \sigma)(x) \). This proves \( \subseteq \) in (115) for \( \eta + 1 \).

Conversely suppose that \( c \in (\triangle^\eta+1 \sigma)(x) = (\triangle(\triangle^\eta \sigma))(x) \). Say \( c \) has length \( m \). Then we can write \( x = y_0 + \cdots + y_{m-1} \) with \( \forall j < m \{ c_j = (\triangle^\eta \sigma)(y_j) \} \). By the inductive hypothesis, \( (\triangle^\eta \sigma)(y_j) = T^\zeta_{\eta}(\tau(y_j)) \). Now note by Proposition 14.1 that \( T^\eta_{\eta+1} = T^\eta_{\eta} \circ T = T \). Hence by Proposition 14.1, \( T \circ T^\zeta_{\eta+1} = T^\eta_{\eta+1} \circ T^\zeta_{\eta+1} = T^\zeta_{\eta} \). So \( c_j = T(T^\zeta_{\eta+1}(\tau(y_j))) \). Now \( T^\zeta_{\eta+1}(\tau(y_j)) \in \triangle^\eta+1 W = (\triangle(\triangle^\eta)) W \), so by Proposition 8.12, \( c_j \in T^\zeta_{\eta+1}(\tau(y_j)) \). Hence

\[
\begin{align*}
  c &= (c_0, 0, 0, \ldots, 0) + (0, c_1, 0, 0, \ldots, 0) + \cdots + (0, 0, \ldots, c_{m-1}) \\
  &= T^\zeta_{\eta+1}(\tau(x)).
\end{align*}
\]

This proves (17.3(2) for \( \eta + 1 \).

Now suppose that \( \xi < \zeta \) is a limit ordinal and 17.3(2) holds for all \( \eta < \xi \). If \( \eta < \xi \), then \( T^\zeta_{\eta} \circ \triangle^\xi \sigma = \triangle^\eta \sigma \) (by Proposition 14.2) = \( T^\zeta_{\eta} \circ \tau \) (induction hypothesis) = \( T^\zeta_{\eta} \circ T^\zeta_{\xi} \circ \tau \) (by Proposition 14.1). Hence by Proposition 14.1, \( \triangle^\xi \sigma = T^\zeta_{\xi} \circ \tau \). \qed
18. The monoid of isomorphism types

Recall from page 29 the definition of $BA$ and $SBA$, which are $m$-monoids.

**Proposition 18.1.** $BA$ has the refinement property. That is, if $m, n \in \omega \setminus \{0\}$, $a \in {}^mBA$, $b \in {}^nBA$, and $\sum_{i \leq m} a_i = \sum_{j < n} b_j$, then there exist $c_{ij} \in BA$ for $i < m$ and $j < n$ such that $\forall i < m [a_i = \sum_{j < n} c_{ij}]$ and $\forall j < n [b_j = \sum_{i < m} c_{ij}]$.

**Proof.** Assume the hypotheses. Say $\forall i < m [a_i = [A_i]]$ and $\forall j < n [b_j = [B_j]]$. Let $D = \prod_{i \leq m} A_i$ and $E = \prod_{j < n} B_j$. Let $C_{ij} = A_i \times B_j$ for all $i < m$ and $j < n$. For each $i < m$ define $c_i \in \prod_{i \leq m} A_i$ by

$$c_i(k) = \begin{cases} 1 & \text{if } k = i, \\ 0 & \text{otherwise} \end{cases}$$

Similarly, for each $j < n$ define $d_j \in \prod_{j < n} B_j$ by

$$d_j(k) = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{otherwise} \end{cases}$$

Let $f$ be an isomorphism from $D$ onto $E$. Then for any $i < m$,

$$f(c_i) = f(c_i) \cdot \sum_{j < n} d_j = \sum_{j < n} (f(c_i) \cdot d_j).$$

Hence

$$A_i \cong (D \upharpoonright c_i) \cong (E \upharpoonright f(c_i)) \cong \prod_{j < n} (E \upharpoonright (f(c_i) \cdot d_j)) \cong \prod_{j < n} C_{ij}.$$  

Hence $a_i = \sum_{j < n} [C_{ij}]$. Similarly, $\forall j < n [b_j = \sum_{i < m} [C_{ij}]]$.

$NBA$ is the set of isomorphism classes of normalized BAs. $MBA$ is the set of isomorphism classes of BAs which are neither superatomic nor normalized.

**Proposition 18.2.** $BA = SBA \cup NBA \cup MBA$.

**Proposition 18.3.** $SBA$ is a submonoid of $BA$.

**Proof.** By Proposition 5.13.

**Proposition 18.4.** $NBA$ is a subsemigroup of $BA$.

**Proof.** By Proposition 6.29.

19. Refinement monoids

Let $M$ be an $m$-monoid. Define $a \leq b$ iff $\exists c \in M [b = a + c]$.

**Proposition 19.1.** Let $M$ be an $m$-monoid.

(i) $\leq$ is reflexive and transitive.
Proposition 19.3. BA is locally countable.

Proof. If $a \in \text{BA}$, then $\text{BA} 

\begin{align*}
(i) & \forall a \in M[0 \leq a]. \\
(ii) & \forall a \in M[a \leq b \rightarrow a + c \leq b + c]. \\
(iii) & \forall a, b, c \in M[a \leq b \rightarrow a + c \leq b + c]. \\
\end{align*}

Proof. (i): $a = a + 0$.

(ii): Suppose that $a \leq b \leq c$. Choose $d, e$ so that $b = a + d$ and $c = b + e$. Then $c = (a + d) + e = a + (d + e)$.

(iii): Assume that $a, b, c \in M$ and $a \leq b$. Choose $d$ so that $b = a + d$. Then $b + c = a + d + c$.

An element $a$ of an $m$-monoid $M$ has the Schröder-Bernstein property (S.-B.) iff $\forall b \in M[a \leq b \leq a \rightarrow a = b]$.

Proposition 19.2. $a$ has the Schröder-Bernstein property iff $\forall c, d \in M[a = a + c + d \rightarrow a = a + c]$.

Proof. $\Rightarrow$: Assume that $a$ has the Schröder-Bernstein property, and suppose that $c, d \in M$ with $a = a + c + d$. Then $a \leq a + c \leq a$, so $a = a + c$.

$\Leftarrow$: Assume that $\forall c, d \in M[a = a + c + d \rightarrow a = a + c]$ and $a \leq b \leq a$. Choose $c, d$ so that $b = a + c$ and $a = b + d$. Then $a = a + c + d$, so $a = a + c = b$.

If $a \in M$ we define $M \upharpoonright a = \{b \in M : b \leq a\}$. $M$ is locally countable iff $\forall a \in M[M \upharpoonright a$ is countable$]$.

Proposition 19.3. $\text{BA}$ is locally countable.

Proof. If $a \in \text{BA}$, then $\text{BA} \upharpoonright a = \{b \in \text{BA} : b \leq a\} = \{b \in \text{BA} : \exists c[a = b + c]\}$.

A submonoid $N$ of $M$ is a hereditary submonoid of $M$ iff $\forall a \in M \forall b \in N[a \leq b \rightarrow a \in N]$.

Proposition 19.4. $N$ is a hereditary submonoid of $M$ iff $\forall a \in N[M \upharpoonright a = N \upharpoonright a]$.

Proposition 19.5. If $N$ is a submonoid of $M$ and $M$ is locally countable, then so is $N$.

Proof. For $a, b \in N$, $a \leq b$ in the sense of $N$ implies that $a \leq b$ in the sense of $M$. Hence $\forall a \in N[N \upharpoonright a \subseteq M \upharpoonright a]$.

$M$ is a refinement monoid, r-monoid, iff $M$ is an $m$-monoid, and $\forall m, n \in \omega \setminus \{0\} \forall a \in M \forall b \in nM[\sum_{i \leq m} a_i = \sum_{j < n} b_j \rightarrow \exists c \in m^nM[\forall i < m[a_i = \sum_{j < n} c_{ij}]$ and $\forall j < n[b_j = \sum_{i < m} c_{ij}]].$

Proposition 19.6. Every hereditary submonoid of a refinement monoid is a refinement monoid.

Proof. Suppose that $N$ is a hereditary submonoid of a refinement monoid $M$. Suppose that $m, n \in \omega, \ a \in mN, \ b \in nN,$ and $\sum_{i \leq m} a_i = \sum_{j < n} b_j$. Choose $c \in m^nM$ such that $\forall i < m[a_i = \sum_{j < n} c_{ij}]$ and $\forall j < n[b_j = \sum_{i < m} c_{ij}]$. Each $c_{ij}$ is $\leq a_i$, and hence $c_{ij} \in N$.

$M$ has the $(2, 2)$-refinement property iff the above definition holds for $m = n = 2$. 

93
Lemma 19.8. Let $M$ be an $r$-monoid, and suppose that $a \leq \sum_{j<n} b_j$. Then there is a $c \in nM$ such that $a = \sum_{j<n} c_j$ and $\forall j < n[c_j \leq b_j]$.

**Proof.** By the definition of $\leq$, there is an $a_1 \in M$ such that $\sum_{j<n} b_j = a_0 + a_1$, where $a_0 = a$. By the refinement property choose $c \in 2^{\times n}M$ such that $\forall i < 2[a_i = \sum_{j<n} c_{ij}]$ and $\forall j < n[b_j = \sum_{i<j} c_{ij}]$. In particular, $a = a_0 = \sum_{j<n} c_{0j}$ and $\forall j < n[c_{0j} \leq b_j]$. An $m$-monoid $M$ is atomless iff $\forall a \in M^* \exists b, c \in M^*[a = b + c]$. 

94
Proposition 19.9. \( \text{BA is not atomless.} \)

Proof. If \( A \) is a 2-element BA, then \( a = [A] \) is a counterexample. \( \square \)

If \( M \) is an \( m \)-monoid and \( a \in M \), we let

\[
\delta(a) = \delta_M(a) = \left\{ b : \exists n \in \omega \setminus \{0\} \left[ b \in ^n M \text{ and } a = \sum_{j<n} b_j \right] \right\}.
\]

Proposition 19.10. If \( M \) is a locally countable \( r \)-monoid and \( a \in M \), then \( \delta_M(a) \) is a countable subset of \( <^\omega M \) that satisfies C.P., R.P. and L.P.

Proof. Assume that \( M \) is a locally countable \( r \)-monoid and \( a \in M \). Clearly \( \delta_M(a) \) is a countable subset of \( <^\omega M \).

C.P.: Suppose that \( b, c \in <^\omega M \), \( b \in \delta_M(a) \), and \( b < c \). Say \( b \in ^m M \) and \( c \in ^n M \). Let \( \lambda : m \to n \) be such that \( \forall j < n[c_j = \sum\{b_i : i < m, \lambda(i) = j\}] \). Then \( a = \sum_{i<m} b_i = \sum_{j<n} c_j \). Hence \( c \in \delta_M(a) \).

R.P.: Assume that \( m, n \in \omega \setminus \{0\} \), \( b \in ^m M \), \( c \in ^n M \), and \( b, c \in \delta_M(a) \). Clearly then \( \sum_{i<m} b_i = \sum_{j<n} c_j \). The desired conclusion follows since \( M \) is an \( r \)-monoid.

L.P.: First note that if \( a \in M \) and \( b \in \Phi_a(\delta_M(a)) \), then, with \( n = \text{dmm}(b) \) we have \( \sum_{i<n} b_i = a \) and \( b \in \Delta(a) \). Thus \( \Phi_a(\delta_M(a)) \subseteq \delta_M(a) \). The converse holds by Proposition 13.2(i), so \( \Phi_a(\delta_M(a)) = \delta_M(a) \). Hence (79) follows from the above. For (80), suppose that \( b \in M \), \( c \in <^\omega M \), and \( |b| \leq c \in \delta_M(a) \). Say \( c \in ^n M \). Then \( b + \sum_{i<n} c_i = a \).

If \( d \in \Phi_b(\delta_M(a)) \), clearly \( d \leq c \in \delta_M(a) \).

Proposition 19.11. If \( M \) is an atomless locally countable \( r \)-monoid and \( a \in M \), then \( \delta_M(a) \) has S.P.

Proof. Suppose that \( M \) is an atomless locally countable \( r \)-monoid and \( a \in M \), and that \( m \in \omega \setminus \{0\} \), \( b \in ^m M \), \( b \in \delta_M(a) \), and \( b_0 \not= 0 \). Since \( M \) is atomless, there are \( c, d \in M^* \) such that \( b_0 = c + d \). Clearly then \( \langle c, d, b_1, \ldots, b_{m-1} \rangle \in \delta_M(a) \).

Proposition 19.12. Let \( M \) be an atomless locally countable \( r \)-monoid. Define \( \delta M = \{\delta_M(a) : a \in M\} \). Then \( \delta M \) is a submonoid of \( \Delta M \). Moreover, \( \delta M \) is an isomorphism of \( M \) onto \( \delta M \) with inverse \( T \upharpoonright \delta M \).

Proof. By Propositions 19.10 and 19.11, \( \delta M \subseteq \Delta M \). If \( a, b \in M \), \( c \in \delta_M(a) \), \( d \in \delta_M(b) \), and \( \text{dmm}(c) = \text{dmm}(d) \), then clearly \( c + d \in \delta_M(a + b) \). Thus \( \delta_M(a) + \delta_M(b) \subseteq \delta_M(a + b) \). Now suppose that \( c \in \delta_M(a + b) \). Say \( \text{dmm}(c) = m \in \omega \setminus \{0\} \). Thus \( \sum_{i<m} c_i = a + b \). Let \( d_0 = a \), \( d_1 = b \). By the refinement property there is an \( e \in \text{dmm}(c) \) such that \( \forall i < m[c_i = \sum_{j<e} e_{ij}] \) and \( \forall j < 2[d_j = \sum_{i<m} e_{ij}] \). Let \( u = \langle e_{i0} : i < m \rangle \) and \( v = \langle e_{i1} : i < m \rangle \). Then \( \sum_{i<m} e_{i0} = d_0 = a \) and \( \sum_{i<m} e_{i1} = d_1 = b \). So \( u \in \delta_M(a) \) and \( v \in \delta_M(b) \) and so \( a + b = \sum_{i<m} c_i = \sum_{i<m, j<e} e_{ij} = u + v \) and hence \( c \in \delta_M(a) + \delta_M(b) \).

Therefore \( \delta_M(a) + \delta_M(b) \subseteq \delta_M(a + b) \).

Clearly \( \delta_M(a) \not= \delta_M(b) \) if \( a \not= b \). Obviously \( \delta M \) maps \( M \) onto \( \delta M \). If \( x \in \delta M \), say \( x = \delta_M(a) \). Then \( T(x) = a \).

95
20. The V-radical

Let $M$ and $N$ be $m$-monoids. A $V$-m-relation between $M$ and $N$ is a subset $R \subseteq M \times N$ such that the following conditions hold:

(Vm1) $\forall a \in M \forall b \in N [a R b \rightarrow (a = 0 \iff b = 0)]$.

(Vm2) $\forall a \in M \forall b \in N [a R b \rightarrow$

(Vm2a) $\forall a_0, a_1 \in M [a = a_0 + a_1 \rightarrow \exists b_0, b_1 \in N [b = b_0 + b_1 \text{ and } a_0 R b_0 \text{ and } a_1 R b_1]]$

and

(Vm2b) $\forall b_0, b_1 \in N [b = b_0 + b_1 \rightarrow \exists a_0, a_1 \in M [a = a_0 + a_1 \text{ and } a_0 R b_0 \text{ and } a_1 R b_1]]$.

A weak $V$-m-relation is a subset of $M \times N$ which satisfies (Vm2) but not necessarily (Vm1).

**Proposition 20.1.** If $M$ and $N$ are $m$-monoids and $R \subseteq M \times N$, then $R$ is a $V$-relation between $M$ and $N$ iff the following conditions hold:

(Vm1) As above

20.1(1) $\forall a \in M \forall b \in N [a R b \rightarrow$

20.1(1a) $\forall m \in \omega \setminus \{0\} \forall c \in \mathbb{M} [a = \sum_{i < m} c_i \rightarrow \exists d \in \mathbb{M} [b = \sum_{i < m} d_i \text{ and } \forall i < m [c_i R d_i]]]$ and

20.1(1b) $\forall m \in \omega \setminus \{0\} \forall d \in \mathbb{M} [b = \sum_{i < m} d_i \rightarrow \exists c \in \mathbb{M} [a = \sum_{i < m} c_i \text{ and } \forall i < m [c_i R d_i]]]$.

**Proof.** 20.1(1) $\Rightarrow$ (Vm2) is obvious. Now assume (Vm2). We prove 20.1(1) by induction on $m$. It is trivial for $m = 1$, and (Vm2) gives the case $m = 2$. Now assume 20.1(1) for $m$. By symmetry it suffices to prove 20.1(1a) for $m + 1$. Suppose that $c \in \mathbb{M}^{m+1} \text{ and } a = \sum_{i < m} c_i$. By (Vm2a) there are $b_0, b_1 \in N$ such that $b = b_0 + b_1$ and $(\sum_{i < m} c_i) R b_0$ and $c_m R b_1$. By the induction hypothesis there is a $d \in \mathbb{M}$ such that $b_0 = \sum_{i < m} d_i$ and $\forall i < m [c_i R d_i]$. Since also $c_m R b_1$, this proves 20.1(1a) for $m + 1$. \(\square\)

If $R \subseteq M \times N$ is a weak $V$-m-relation and is also a morphism, then it is called a $V$-morphism.

**Proposition 20.2.** $R \subseteq M \times N$ is a $V$-m-morphism iff $R$ is a morphism and $\forall a \in M \forall b_0, b_1 \in N$, if $R(a) = b_0 + b_1$ then there exist $a_0, a_1 \in M$ such that $a = a_0 + a_1$, $R(a_0) = b_0$, and $R(a_1) = b_1$.

**Proof.** $\Rightarrow$: suppose that $R \subseteq M \times N$ is a $V$-morphism. Then it is a morphism. Suppose that $a \in M$, $b_0, b_1 \in N$, and $R(a) = b_0 + b_1$. Thus $a R (b_0 + b_1)$, so by (119b) there are $a_0, a_1 \in M$ such that $a = a_0 + a_1$, $a_0 R b_0$, and $a_1 R b_1$. So $R(a_0) = b_0$ and $R(a_1) = b_1$.

$\Leftarrow$: Assume the indicated conditions. In particular, $R$ is a morphism. To check (Vm2a), suppose that $a R b$ and $a = a_0 + a_1$. Then $b = R(a) = R(a_0) + R(a_1)$. Let $b_0 = R(a_0)$ and $b_1 = R(a_1)$. Then $b = b_0 + b_1$ and $a_0 R b_0$ and $a_1 R b_1$. So (Vm2a) holds.

To check (Vm2b), suppose that $a R b$ and $b = b_0 + b_1$. Then $R(a) = b = b_0 + b_1$, so there exist $a_0, a_1 \in M$ such that $a = a_0 + a_1$, $R(a_0) = b_0$, and $R(a_1) = b_1$. \(\square\)

A $V$-m-congruence on $M$ is a congruence $R$ on $M$ which is a weak $V$-m-relation.
Proposition 20.3. If \( R : M \to N \) is a \( V \)-morphism, then \( \ker(R) = \{(a_0, a_1) \in 2^M : R(a_0) = R(a_1)\} \) is a \( V \)-congruence on \( M \).

**Proof.** Assume that \( R : M \to N \) is a \( V \)-morphism. To check (Vm2a) for \( \ker(R) \), suppose that \( a,a' \in M \), \( a(\ker(R))a' \), and \( a = a_0 + a_1 \). Thus \( R(a) = R(a') \). Hence \( R(a_0) + R(a_1) = R(a') \). Let \( b_i = R(a_i) \) for \( i < 2 \). Then \( a'R(b_0 + b_1) \). Hence there exist \( c_0, c_1 \in M \) such that \( a' = c_0 + c_1, Rb_0 \), and \( c_1Rb_1 \). Then \( R(c_0) = b_0 = R(a_0) \) and \( R(c_1) = b_1 = R(a_1) \), hence \( a_0(\ker(R))c_0 \) and \( a_1(\ker(R))c_1 \). This checks (Vm2a for \( \ker(R) \)).

To check (Vm2b) for \( \ker(R) \), suppose that \( a,a' \in M \), \( a(\ker(R))a' \), and \( a' = a_0 + a_1 \). Thus \( R(a) = R(a') \). Hence \( R(a_0) + R(a_1) = R(a) \). Let \( b_i = R(a_i) \) for \( i < 2 \). Then \( aR(b_0 + b_1) \). Hence there exist \( c_0, c_1 \in M \) such that \( a = c_0 + c_1, Rb_0 \), and \( c_1Rb_1 \). Then \( R(c_0) = b_0 = R(a_0) \) and \( R(c_1) = b_1 = R(a_1) \), hence \( a_0(\ker(R))c_0 \) and \( a_1(\ker(R))c_1 \). This checks (Vm2b) for \( \ker(R) \).

\( \Box \)

Proposition 20.4. If \( R : M \to N \) is a surjective morphism and \( \ker(R) \) is a \( V \)-congruence, then \( R \) is a \( V \)-morphism.

**Proof.** Suppose that \( R : M \to N \) is a surjective morphism and \( \ker(R) \) is a \( V \)-congruence. To check (Vm2a) for \( R \), suppose that \( aRb, a_0, a_1 \in M \), and \( a = a_0 + a_1 \). Let \( b_0 = R(a_0) \) and \( b_1 = R(a_1) \). Then \( b = R(a) = R(a_0) + R(a_1) = b_0 + b_1 \), as desired.

To check (Vm2b) for \( R \), suppose that \( aRb, b_0, b_1 \in N \), and \( b = b_0 + b_1 \). Since \( R \) is surjective, choose \( a, a_1 \in M \) such that \( R(a_0) = b_0 \) and \( R(a_1) = b_1 \). Then \( R(a) = b = b_0 + b_1 = R(a_0) + R(a_1) = R(a_0 + a_1) \). Thus \( a(\ker(R))(a_0 + a_1) \). Hence there are \( c_0, c_1 \in M \) such that \( a = c_0 + c_1, c_0(\ker(R))a_0 \), and \( c_1(\ker(R))a_1 \). Thus \( R(c_0) = R(a_0) = b_0 \) and \( R(c_1) = R(a_1) = b_1 \), as desired.

\( \Box \)

Proposition 20.5. Suppose that \( R \) is a (weak) \( V \)-relation between \( M \) and \( N \) and \( S \) is a (weak) \( V \)-relation between \( N \) and \( P \). Then \( R|S \) is a (weak) \( V \)-relation between \( M \) and \( P \).

**Proof.** Assume that \( R \) is a \( V \)-relation between \( M \) and \( N \) and \( S \) is a \( V \)-relation between \( N \) and \( P \). Then \( R \subseteq M \times N \) and \( S \subseteq N \times P \), so \( R|S \subseteq M \times P \). For (Vm1), suppose that \( a(R|S)c \) and \( a = 0 \). Say \( aRbSc \). Then \( b = 0 \) and so \( c = 0 \). The converse of (Vm1) is similar.

For (Vm2a), suppose that \( a(R|S)c, a_0, a_1 \in M \), and \( a = a_0 + a_1 \). Say \( aRbSc \). Choose \( b_0, b_1 \in N \) such that \( b = b_0 + b_1, a_0Rb_0 \), and \( a_1Rb_1 \). Then choose \( c_0, c_1 \in P \) such that \( c = c_0 + c_1, b_0Sc_0 \), and \( b_1Sc_1 \). Then \( a_0(R|S)c_0 \) and \( a_1(R|S)c_1 \), as desired.

(Vm2b) is treated similarly.

\( \Box \)

Proposition 20.6. If \( R \) is a (weak) \( V \)-relation between \( M \) and \( N \), then \( R^{-1} \) is a (weak) \( V \)-relation between \( N \) and \( M \).

\( \Box \)

Proposition 20.7. If \( M \) is an \( m \)-monoid, then \( \text{Id} \upharpoonright M \) is a \( V \)-relation.

\( \Box \)

Proposition 20.8. If \( \mathcal{A} \) is a collection of (weak) \( V \)-relations between \( M \) and \( N \), then \( \bigcup \mathcal{A} \) is a (weak) \( V \)-relation between \( M \) and \( N \).

\( \Box \)
For any $m$-monoid $M$

$$Y(M) = \bigcup \{ R \subseteq M \times M : R \text{ is a weak } V\text{-relation} \}.$$ 

An $m$-monoid $M$ is $V$-simple iff $Y(M) = \text{Id} \upharpoonright M$.

**Proposition 20.9.** If $M$ is an $r$-monoid, then $Y(M)$ is a $V$-congruence on $M$.

**Proof.** By Proposition 20.8, $Y(M)$ is a weak $V$-relation. So it suffices to show that it is a congruence. By Propositions 20.5, 20.6, and 20.7 it is an equivalence relation on $M$. Now let

$$R = \{(a_0 + a_1, b_0 + b_1) : a_0(Y(M))b_0 \text{ and } a_1(Y(M))b_1 \}.$$ 

We claim that $R$ is a weak $V$-relation. (Hence $Y(M)$ is a congruence.) To prove this, by symmetry it suffices to prove (Vm2a). So suppose that $aRb$ and $a = c_0 + c_1$. Say $a = a_0 + a_1$ and $b = b_0 + b_1$ with $a_0(Y(M))b_0$ and $a_1(Y(M))b_1$. By the refinement property let $d \in 2 \times 2$ $M$ be such that $\forall i < 2[c_i = d_{i0} + d_{i1}]$ and $\forall j < 2[a_i = d_{0j} + d_{1j}]$. Thus $d_{00} + d_{10} = a_0(Y(M))b_0$ so, since $Y(M)$ is a $V$-relation, there are $u_0, v_1 \in M$ such that $b_0 = u_0 + u_1$, $d_{00}(Y(M))u_0$ and $d_{10}(Y(M))u_1$. Similarly, $d_{01} + d_{11} = a_1(Y(M))b_1$, so there are $v_0, v_1 \in M$ such that $b_1 = v_0 + v_1$, $d_{01}(Y(M))v_0$, and $d_{11}(Y(M))v_1$. Then $b = b_0 + b_1 = u_0 + u_1 + v_0 + v_1 = u_0 + v_0 + u_1 + v_1$. Since $d_{00}(Y(M))u_0$ and $d_{01}(Y(M))v_0$ we have $(d_{00} + d_{01})R(u_0 + v_0)$. Since $d_{10}(Y(M))u_1$ and $d_{11}(Y(M))v_1$ we have $(d_{10} + d_{11})R(u_1 + v_1)$. Now $c_0 = d_{00} + d_{01}$ and $c_1 = d_{10} + d_{11}$, so this proves (Vm2a).

**Proposition 20.10.** If $M$ is an $r$-monoid and $R$ is a $V$-congruence on $M$, then $M/R$ is an $r$-monoid.

**Proof.** Assume that $M$ is an $r$-monoid and $R$ is a $V$-congruence on $M$. Suppose that $\sum_{i<m}[a_i] = \sum_{j<n}[b_j]$. Thus $[\sum_{i<m}a_i] = [\sum_{j<n}b_j]$, so $(\sum_{i<m}a_i)R(\sum_{j<n}b_j)$. By Proposition 20.1, choose $d \in mM$ so that $\sum_{j<n}b_j = \sum_{i<m}d_i$ and $\forall i < m[a_i Rd_i]$. Now by the refinement property choose $e \in m \times n M$ so that $\forall j < n[b_j = \sum_{i<m}e_{ij}]$ and $\forall i < m[d_i = \sum_{j<n}e_{ij}]$. Now for each $i < m$ we have $a_iRd_i = \sum_{j<n}e_{ij}$, so by Proposition 20.1 choose $u_i \in nM$ so that $a_i = \sum_{j<n}u_{ij}$ and $\forall j < n[u_{ij}Re_{ij}]$. Then $\forall i < m[a_i] = \sum_{j<n}[u_{ij}] = \sum_{j<n}[e_{ij}]$ and $\forall j < n[b_j] = \sum_{i<m}[e_{ij}]$.

**Proposition 20.11.** If $M$ is an $r$-monoid, $R$ is a $V$-congruence on $M$ and $S$ is a $V$-relation on $M/R$, let $T = \{(a, b) \in M \times M : [a]S[b] \}$. Then $T$ is a $V$-relation on $M$.

**Proof.** To check (Vm2a), suppose that $aTb$ and $a = a_0 + a_1$. Then $[a]S[b]$. Now $[a] = [a_0] + [a_1]$ so, since $S$ is a weak relation on $M/R$, there exist $c, d \in M$ such that $[b] = [c + d]$, $[a_0]S[c]$, and $[a_1]S[d]$. Thus $[b] = [c + d]$, so $bR(c + d)$. Since $R$ is a $V$-relation, there are $u, v \in M$ such that $b = u + v$, $uRc$, and $vRd$. Then $[a_0]S[c] = [u]$ and $[a_1]S[d] = [v]$, so $a_0Tu$ and $a_1Tv$. This proves (Vm2a).

(Vm2b) is similar.

**Proposition 20.12.** If $M$ is an $r$-monoid, then $M/Y(M)$ is a $V$-simple $r$-monoid.
Proposition 20.13. If M is an r-monoid, then the natural map from M onto \( M/Y(M) \) is a V-morphism.

**Proof.** By Propositions 20.9 and 20.4.

The congruence relation \( Y(M) \) on an r-monoid M is called the V-radical of M.

Proposition 20.14. If \( R : M \to N \) is a V-morphism of r-monoids, then \( \text{rng}(R) \) is a hereditary submonoid of N.

**Proof.** Clearly \( \text{rng}(R) \) is a submonoid of N. Suppose that \( a \in N, b \in \text{rng}(R) \), and \( a \leq b \). Choose \( c \in N \) so that \( b = a + c \). Say \( b = R(d) \) with \( d \in M \). Thus \( R(d) = a + c \), i.e., \( dR(a + c) \). By (Vm2b) there are \( u, v \in M \) such that \( b = u + v, uRa \), and \( vRc \). Thus \( R(v) = a \), as desired.

Proposition 20.15. If \( R : M \to N \) is a V-morphism of r-monoids, then \( \ker(R) \subseteq Y(M) \).

**Proof.** By Proposition 20.3.

Proposition 20.16. If M is an hereditary submodel of N and R is a weak V-relation on M, then R is a weak V-relation on N.

**Proof.** Assume the hypotheses. Suppose that \( aRb, a_0, a_1 \in N \), and \( a = a_0 + a_1 \). Then \( a_0, a_1 \leq a \), so \( a_0, a_1 \in M \). Hence the desired conclusion follows.

Proposition 20.17. If M is an hereditary submodel of N and N is V-simple, then M is V-simple.

**Proof.** Assume the hypotheses. Since \( Y(M) \) is a V-relation on M, by Proposition 20.16 it is a weak V-relation on N. Hence \( Y(M) \subseteq Y(N) \), and the desired conclusion follows.

Proposition 20.18. If \( R : M \to N \) is a V-morphism of r-monoids, then there is an isomorphism \( T \) of \( M/\ker(R) \) onto \( \text{rng}(R) \) such that \( T([a]) = R(a) \) for all \( a \in M \).

Proposition 20.19. Suppose that \( R \) is a V-congruence on M. Let \( S = ([a]_R, [b]_R) : aY(M)b \). Then S is a weak V-relation on M/R.

**Proof.** Suppose that \( u, v \in M/R, uSv, w_0, w_1 \in M/R, \) and \( u = w_0 + w_1 \). Say \( u = [a] \) and \( v = [b] \), with \( aY(M)b \), and \( w_0 = [a_0], w_1 = [a_1] \). Then \( [a] = [a_0] + [a_1] \), so \( (a_0 + a_1, a) \in R \subseteq Y(M) \). Now by Proposition 20.5, \( Y(M) \) is closed under \( | \). Hence \( (a_0 + a_1, b) \in Y(M) \). Since \( Y(M) \) is a V-relation, there are \( c_0, c_1 \in M \) such that \( b = c_0 + c_1 \),
c_0 Y(M)a_0, and c_1 Y(M)a_1. So a_0 Y(M)c_0 and a_1 Y(M)c_1. So v = [c_0] + [c_1], w_0 S[c_0], and w_1 S[c_1]. This proves (Vm2a) for S.

Now suppose that u, v ∈ M/R, uSv, w_0, w_1 ∈ M/R, and v = w_0 + w_1. Say u = [a] and v = [b], with aY(M)b, and w_0 = [a_0], w_1 = [a_1]. Then [b] = [a_0] + [a_1], so (b, a_0 + a_1) ∈ R ⊆ Y(M). Now by Proposition 20.5, Y(M) is closed under |. Hence (a, a_0 + a_1) ∈ Y(M).

Since Y(M) is a V-relation, there are c_0, c_1 ∈ M such that a = c_0 + c_1, c_0 Y(M)a_0, and c_1 Y(M)a_1. So u = [c_0] + [c_1], [c_0]Sw_0, and [c_1]Sw_1. This proves (Vm2b) for S.

**Proposition 20.20.** If R is a V-congruence on M, then Y(M/R) = {([a], [b]) : aY(M)b}.

**Proof.** Let T = {(a, b) ∈ M × M : [a]Y(M/R)[b]}. By Proposition 20.11, T is a V-relation on M, so T ⊆ Y(M). So if (u, v) ∈ Y(M/R) we can choose a, b so that u = [a] and v = [b]. Then (a, b) ∈ T, so aY(M)b. Thus ⊆ holds. Conversely, suppose that aY(M)b. Then [a]S[b], where S is as in Proposition 20.19. Hence ([a], [b]) ∈ Y(M/R), proving ⊇.

**Proposition 20.21.** If R : M → N is a V-morphism of r-monoids and N is V-simple, then ker(R) = Y(M).

**Proof.** By Proposition 20.14, rng(R) is an hereditary submonoid of N. Hence by Proposition 20.17, rng(R) is V-simple. So by Proposition 20.18, M/ker(R) is V-simple. Now ker(R) ⊆ Y(M) by Proposition 20.15. Now suppose that aY(M)b. Then [a]Y(M/ker(R))[b] by Proposition 20.20, so [a] = [b], i.e. R(a) = R(b), i.e. (a, b) ∈ ker(R).

**Proposition 20.22.** If R : M → N is a V-morphism of r-monoids and N is V-simple, then there is only one V-morphism from M to N.

**Proof.** Assume that R, S : M → N are V-morphisms of r-monoids and N is V-simple. Now S^{−1}R is a V-relation on N by Propositions 20.5 and 20.6. Hence S^{−1}R ⊆ Y(N) = (id | N). Now if R(a) = b and S(a) = c, then cS^{−1}aRb, hence c(S^{−1}R)b, hence b = c.

**Proposition 20.23.** If M and N are r-monoids and M is a hereditary submonoid of N, then the inclusion map from M to N is a V-morphism.

**Proof.** By Proposition 20.2.

### 21. Dobbertin’s theorem

**Lemma 21.1.** Let M be a locally countable r-monoid and a ∈ M. Then there exist a countable BA A uniquely determined up to isomorphism and a relation R ⊆ A × (M | a) such that:

(i) 1_A Ra.

(ii) ∀x ∈ A[xR0 → x = 0] and ∀b ∈ (M | a)[0Rb → b = 0].

(iii) ∀x ∈ A∀b_0, b_1 ∈ (M | a)[xR(b_0 + b_1) → ∃x_0.x_1 ∈ A[x = x_0 + x_1 and x_0Rb_0 and x_1Rb_1]] and ∀x_0, x_1 ∈ A∀b ∈ (M | a)[(x_0 + x_1)Rb → ∃b_0, b_1 ∈ (M | a)[b = b_0 + b_1 and x_0Rb_0 and x_1Rb_1]].
Proof. Assume that $M$ is a locally countable $r$-monoid and $a \in M$. If $a = 0$, clearly $A = \{0\}$ and $R = \{0, 0\}$ works, and $A$ is unique up to isomorphism. So assume that $a \neq 0$. Recall the definition of $\delta_M$ after Proposition 19.9. By Proposition 19.10, $\delta_M(a)$ satisfies C.P., R.P., and L.P. Hence by Proposition 13.12 there is a uniformly dense $\delta_M(a)$-tree $b$. Thus $b : D \to M$. For each $f \in D$ let $x(f) = \{p \in \omega^2 : f \subseteq p\}$. Let $J$ be the ideal in $\text{clop}(\omega^2)$ generated by $\{x(f) : f \in D, b(f) = 0\}$.

21.1(1) $\forall f \in \mathcal{D}[x(f) \in J \iff b(f) = 0]$. 

In fact, $\Leftarrow$ is clear. Now suppose that $f \in \mathcal{D}$ and $x(f) \in J$. Then there is a finite $F \subseteq \mathcal{D}$ such that $\forall g \in F[b(g) = 0]$ and $x(f) \subseteq \bigcup_{g \in F} x(g)$. Hence $x(f) = \bigcup\{x(f \cup g) : g \in F, f \cup g$ is a function$\}$. For $g \in F$ and $f \cup g$ a function we have $b(f \cup g) = 0$, since $b(g) = 0$. Hence $b(f) = 0$, proving 21.1(1).

Let $A = \text{clop}(\omega^2)/J$ and

$$R = \left\{ \left( \left[ \bigcup_{f \in G} x(f) \right]_J, \sum_{f \in G} b(f) \right) : G \subseteq \mathcal{D}_n, n \in \omega \right\}.$$ 

We claim that $R \subseteq A \times (M \upharpoonright a)$. For, suppose that $G \subseteq \mathcal{D}_n$ with $n \in \omega$. Then obviously $|\bigcup_{f \in G} x(f)|_J \in A$. By (t1), $\forall m \in \omega[b_m \in \delta_M(a)]$, where $b_m = b(\mathcal{D}_m) = \{b(\text{lex}(p)) : p < |\mathcal{D}_m|\}$. By the definition of $\delta_M(a)$ after Proposition 19.9, $a = \sum\{b(\text{lex}(p)) : p < |\mathcal{D}_m|\}$. Hence $\sum_{f \in G} b(f) \leq a$. This proves the claim.

Now we check (i)–(iii). For (i), let $\mathcal{D}_1 = \{f_0, f_1\}$. Then $[\bigcup_{f \in \mathcal{D}_n} x(f)] = 1_A$. Also, $b_1 \in \delta_M(a)$, so $\sum_{f \in \mathcal{D}_1} b(f) = a$. This proves (i).

For (ii), first suppose that $y \in A$ and $yR0$. Say $y = [\bigcup_{f \in G} x(f)]$ and $\sum_{f \in G} b(f) = 0$, where $G \subseteq \mathcal{D}_n$ with $n \in \omega$. Then $\forall f \in G[b(f) = 0]$, so by 21.1(1), $\forall f \in G[x(f) \in J]$. Hence $y = 0$.

Second, suppose that $c \in M \upharpoonright a$ and $0Rc$. Say $[\bigcup_{f \in G} x(f)] = 0$ and $c = \sum_{f \in G} b(f)$, where $G \subseteq \mathcal{D}_n$ with $n \in \omega$. Then by 21.1(1), $c = 0$.

For (iii), first suppose that $y \in A$, $c_0, c_1 \in (M \upharpoonright a)$, and $yR(c_0 + c_1)$. Note that $c_0 + c_1 \leq a$ by the above claim. Say $n \in \omega$, $G \subseteq \mathcal{D}_n$, $y = [\bigcup_{f \in G} x(f)]$, and $c_0 + c_1 = \sum_{f \in G} b(f)$. Then $\sum_{f \in G} b(f) = \sum_{j < |G|} b(\text{lex}(j)) = b(G)$. Thus $\langle c_0, c_1 \rangle \in \Phi_T(b(G)) (\delta_M(a))$. Since $b$ is a uniformly dense $\delta_M(a)$-tree, it follows that there is an $m \geq n$ such that $b(G \upharpoonright m) \prec \langle c_0, c_1 \rangle$. Thus $\langle b(\text{lex}(i)) : i < |G \upharpoonright m| \prec \langle c_0, c_1 \rangle \rangle$. Let $\lambda : |G \upharpoonright m| \to 2$ be such that $\forall j < 2^{c_j} = \sum\{b(\text{lex}(i)) : i < |G \upharpoonright m|, \lambda(i) = j\}$. Let $H_j = \{\text{lex}(i) : i < |G \upharpoonright m|, \lambda(i) = j\}$ for $j < 2$. Thus $(G \upharpoonright m) = H_0 \cup H_1$ and $\forall j < 2^{c_j} = T(b(H_j))$. For each $j < 2$ let $x_j = [\sum_{f \in H_j} x(f)]$. Then $y = x_0 + x_1, x_0 R c_0$, and $x_1 R c_1$.

Second suppose that $y_0, y_1 \in A$, $c \in (M \upharpoonright a)$, and $(y_0 + y_1) R c$. Then there exist $n \in \omega$ and $G \subseteq \mathcal{D}_n$ such that $y_0 + y_1 = [\bigcup_{f \in G} x(f)]$ and $c = \sum_{f \in G} b(f)$. Then there exist $m \geq n$ and subsets $H_i \subseteq \mathcal{D}_m$ for $i < 2$ such that $y_i = [\bigcup_{g \in H_i} x(g)]$. Let $G_i = (G \upharpoonright m) \cap H_i$ for $i < 2$. Then $G_0 \cup G_1 = (G \upharpoonright m)$ and $y_i = [\bigcup_{g \in G_i} x(g)]$ for $i < 2$. Let $d_i = \sum_{f \in G_i} b(g)$ for $i < 2$. Then $c = d_0 + d_1, y_0 R d_0$, and $y_1 R d_1$. \[\square\]

**Theorem 21.2.** $\text{BA}$ is a locally countable $V$-simple $r$-monoid. 

101
**Proof.** BA is locally countable by Proposition 19.3. By Proposition 18.1 it is an r-monoid. Now let $S = Y(BA)$. Then $S$ is a $V$-congruence on BA by Proposition 20.9. Define $ARB$ iff $A, B \in BA$ and $[A]S[B]$. Then $R$ satisfies the conditions of Proposition 3.4, so $ARB$ implies that $A$ is isomorphic to $B$. Thus $[A]S[B]$ implies that $[A] = [B]$. So BA is $V$-simple.

**Theorem 21.3.** If $M$ is a locally countable $r$-monoid, then there is a unique $V$-morphism $S : M \to BA$; $\ker(S) = Y(M)$, and $\text{rng}(S)$ is a hereditary submonoid of BA.

**Proof.** Assume that $M$ is a locally countable $r$-monoid. For each $a \in M$ let $A_a$ and $R_a$ be as in Lemma 21.1. For each $a \in M$ let $S(a) = [A_a]$. So $S : M \to BA$. If $S(a) = 0$, then $A_a$ is a one-element BA, hence by Lemma 21.1(i), $0R_a a$, and so by Lemma 21.1(ii), $a = 0$. On the other hand, if $a = 0$ then by Lemma 21.1(i), $1_A a R_0$, and hence by Lemma 21.1(ii), $1_A, 0 = 0$, and so $A_a$ is a one-element BA, and $S(a) = 0$.

To show that $S$ is a morphism, suppose that $a_0, a_1 \in M$. Let $a = a_0 + a_1$. Then $S(a)R_a a = a_0 + a_1$, so there are $x_0, x_1 \in A_a$ such that $1_A a = x_0 + x_1$, $x_0 R_a a_0$, and $x_1 R_a a_1$.

21.3(1) $a_0, M \ni a_0, A_a \ni x_0, R_a \cap ((A \ni x_0) \times (M \ni a_0))$ satisfy the conditions of Lemma 21.1, with these entries replacing $a, M, A, R$ respectively.

For, let $T = R_a \cap ((A \ni x_0) \times (M \ni a_0))$. For (i), $1_A a \ni x_0 = x_0 T a_0$, as desired.

For (ii), if $x \in (A_a \ni x_0)$ and $x T 0$, then $x R_a 0$ and hence $x = 0$. If $b \in (M \ni a_0)$ and $0 T b$, then $b \in (M \ni a)$ and $0 R_b b$, so $b = 0$.

For (iii), first suppose that $x \in (A_a \ni x_0), c_0, c_1 \in (M \ni a_0)$, and $x T (c_0 + c_1)$. Then $x \in A_a, c_0, c_1 \in (M \ni a)$, and $x R_a (c_0 + c_1)$. Hence there are $y_0, y_1 \in A_a$ such that $x = y_0 + y_1, y_0 R_a c_0$, and $y_1 R_a c_1$. Then $y_0, y_1 \in (A_a \ni x_0), x = y_0 + y_1, y_0 T c_0$, and $y_1 T c_1$.

Second, suppose that $y_0, y_1 \in (A_a \ni x_0), c \in (M \ni a_0), \text{ and } (y_0 + y_1) T c$. Then $y_0, y_1 \in A_a, c \in (M \ni a)$, and $(y_0 + y_1) R_a c$. Hence there exist $d_0, d_1 \in (M \ni a)$ such that $c = d_0 + d_1, y_0 R_a d_0$, and $y_1 R_a d_1$. Then $d_0, d_1 \in (M \ni a_0), c = d_0 + d_1, y_0 T d_0$, and $y_1 T d_1$. This proves 21.3(1).

Then by the uniqueness assertion of Lemma 21.1 we have $A_{a_0} \cong (A_a \ni x_0)$.

By symmetry we get

21.3(2) $a_1, M \ni a_1, A_a \ni x_1, R_a \cap ((A \ni x_1) \times (M \ni a_1))$ satisfy the conditions of Lemma 21.1, with these entries replacing $a, M, A, R$ respectively.

By the uniqueness assertion of Lemma 21.1 we have $A_{a_1} \cong (A_a \ni x_1)$.

Hence $S(a_0) + S(a_1) = [A_{a_0}] + [A_{a_1}] = [A_a \ni x_0] + [A_a \ni x_1] = [(A_a \ni x_0) \times (A_a \ni x_1)] = [A_a] = S(a) = S(a_0 + a_1)$. This proves that $S$ is a morphism.

To check (Vm2a) for $S$, suppose that $a \in M, b \in BA, S(a) = b$, and $a = a_0 + a_1$. Then since $S$ is a morphism, $b = S(a) = S(a_0) + S(a_1)$, as desired.

To check (Vm2b) for $S$, suppose that $a \in M, b \in BA, S(a) = b, b_0, b_1 \in BA$, and $b = b_0 + b_1$. Say $b_0 = [C]$ and $b_1 = [D]$. Then $[A_a] = S(a) = b = [C] + [D]$, so that $A_a \cong C \times D$. Then there is an $x \in A_a$ such that $(A_a \ni x) \cong C$ and $(A_a \ni (x)) \cong D$. Now $1 = (x + -x) R_a a$, so there are $c_0, c_1 \leq a$ such that $a = c_0 + c_1, x R_a c_0$, and $(x - a) R_a c_1$. Now by 21.3(1) and 21.3(2), $A_{c_0} \cong (A_a \ni x)$ and $A_{c_1} \cong (A_a \ni (x))$. Hence $S(c_0) = [A_{c_0}] = [A_a \ni x] = [C] = b_0$ and similarly $S(c_1) = b_1$, as desired.
It follows that $S$ is a $V$-morphism. Now by Theorem 21.2, $\text{BA}$ is $V$-simple. Hence by Proposition 20.22, $S$ is unique. By Proposition 20.21, $\ker(S) = Y(M)$. By Proposition 20.14, $\text{rng}(S)$ is a hereditary submonoid of $\text{BA}$.

**Proposition 21.4.** Suppose that $N$ is a monoid satisfying the following:

(i) $N$ is a locally countable $V$-simple $r$-monoid.

(ii) If $M$ is any locally countable $r$-monoid, then there is a unique $V$-morphism $S : M \to N$ such that $\ker(S) = Y(M)$ and $\text{rng}(S)$ is a hereditary submonoid of $M$.

Then $N$ is isomorphic to $\text{BA}$.

**Proof.** Assume the hypotheses. By Theorem 21.3, let $S : N \to \text{BA}$ be a $V$-morphism such that $\ker(S) = Y(N)$ and $\text{rng}(S)$ is a hereditary submonoid of $N$. Since $N$ is $V$-simple, $S$ is an injection. By the hypotheses of the proposition, let $T : \text{BA} \to N$ be a $V$-morphism such that $\ker(T) = Y(\text{BA})$ and $\text{rng}(T)$ is a hereditary submonoid of $N$. Since $\text{BA}$ is $V$-simple, $T$ is an injection. Then $S \circ T$ is a $V$-morphism from $\text{BA}$ to $\text{BA}$ such that $\ker(S \circ T) = Y(\text{BA})$ and $\text{rng}(S \circ T)$ is a hereditary submonoid of $\text{BA}$. The identity on $\text{BA}$ also has these properties, so by the uniqueness assertion of Theorem 21.3, $S \circ T$ is the identity. Similarly, $T \circ S$ is the identity. □

22. Ketonen’s theorem

**Theorem 22.1.** Any countable commutative semigroup is isomorphic to a subsemigroup of $\text{BA}$.

The proof of Theorem 22.1 will come after a series of lemmas.

**Corollary 22.2.** $\mathbb{Z}_2$ can be isomorphically embedded into $\text{BA}$.

**Corollary 22.3.** There is a countable $\text{BA} A$ such that $A \cong A \times A \times A$ but $A \not\cong A \times A$.

**Proof.** In $\mathbb{Z}_2$, $1 + 1 + 1 = 1$ but $1 + 1 \neq 1$. □

**Corollary 22.4.** There are countable BAs $A, B$ such that $A \cong A \times B \times B$ but $A \not\cong A \times B$.

**Corollary 22.5.** There are countable BAs $A, B, C$ such that $A \cong A \times B \times C$ but $A \not\cong A \times B$.

23. Products of measures

Recall the definition of $\mathcal{M}(M)$ from just before Proposition 7.2. Suppose that $k : \text{Fr}(\omega) \to \text{Fr}(\omega) \times \text{Fr}(\omega)$ is an isomorphism. For $\sigma, \tau \in \mathcal{M}(M)$ and $x \in \text{Fr}(\omega)$ define

$$(\sigma \oplus_k \tau)(x) = \sigma(\pi_0(k(x))) + \tau(\pi_1(k(x))).$$

Thus $(\sigma \oplus_k \tau) : \text{Fr}(\omega) \to M$.
Proposition 23.1. If \( k : \Fr(\omega) \to \Fr(\omega) \times \Fr(\omega) \) is an isomorphism and \( \sigma, \tau \in \mathcal{M}(M) \), then \((\sigma \oplus_k \tau) \in \mathcal{M}(M)\).

Proof. Assume that \( k : \Fr(\omega) \to \Fr(\omega) \times \Fr(\omega) \) is an isomorphism and \( \sigma, \tau \in \mathcal{M}(M) \). If \( x, y \) are disjoint elements of \( \Fr(\omega) \), then \( k(x), k(y) \) are disjoint. \( \pi_0(k(x)) \) and \( \pi_0(k(y)) \) are disjoint, and \( \pi_1(k(x)) \) and \( \pi_1(k(y)) \) are disjoint, hence

\[
(\sigma \oplus_k \tau)(x + y) = \sigma(\pi_0(k(x))) + \tau(\pi_1(k(x)))
\]

Clearly \((\sigma \oplus_k \tau)(x) = 0 \iff x = 0\). \(\Box\)

Proposition 23.2. Assume:

(i) \( k_0 \) and \( k_1 \) are automorphisms of \( \Fr(\omega) \);
(ii) \( \sigma, \sigma', \tau, \tau' \in \mathcal{M}(M) \);
(iii) \( \forall x \in \Fr(\omega)[\sigma(x) = \sigma'(k_0(x))] \);
(iv) \( \forall x \in \Fr(\omega)[\tau(x) = \tau'(k_1(x))] \);
(v) \( l_0, l_1 : \Fr(\omega) \to \Fr(\omega) \times \Fr(\omega) \) are isomorphisms;
(vi) \( \forall x, y \in \Fr(\omega)[s(x, y) = (k_0(x), k_1(y))] \);
(vii) \( \forall x \in \Fr(\omega)[k_2(x) = l_1^{-1}(s(l_0(x)))] \);

Then:

(viii) \( s : \Fr(\omega) \times \Fr(\omega) \to \Fr(\omega) \times \Fr(\omega) \) is an isomorphism;
(ix) \( k_2 \) is an automorphism of \( \Fr(\omega) \);
(x) \( \forall x \in \Fr(\omega)[(\sigma \oplus_{l_0} \tau)(x) = (\sigma' \oplus_{l_1} \tau')(k_2(x))] \).

Proof. Assume (i)–(vii). Then (viii) and (ix) are clear. For (x), suppose that \( x \in \Fr(\omega) \). Then

\[
(\sigma' \oplus_{l_1} \tau')(k_2(x)) = \sigma'(\pi_0(l_1(k_2(x)))) + \tau'(\pi_1(l_1(k_2(x))))
\]

Proposition 23.3. If \( k : \Fr(\omega) \to \Fr(\omega) \times \Fr(\omega) \) is an isomorphism and \( \xi < \omega_1 \), then \( \Delta^\xi(\sigma \oplus_k \tau) = \Delta^\xi \sigma \oplus_k \Delta^\xi \tau \).

Proof. Assume that \( k : \Fr(\omega) \to \Fr(\omega) \times \Fr(\omega) \) is an isomorphism and \( \xi < \omega_1 \). We prove the indicated conclusion by induction on \( \xi \). It is trivial for \( \xi = 0 \). Now assume
that it holds for $\xi$. Suppose that $a \in (\triangle^{\xi+1}(\sigma \oplus_k \tau))(x)$. By Proposition 14.2 (103), $a \in (\triangle(\triangle^{\xi}(\sigma \oplus_k \tau)))(x)$. Say $x = y_0 + \cdots + y_{n-1}$ and

$$
a = ((\triangle^{\xi}(\sigma \oplus_k \tau))(y_0), \ldots, (\triangle^{\xi}(\sigma \oplus_k \tau))(y_{n-1})
= ((\triangle^{\xi}\sigma)(\pi_0(k(y_0))), \ldots, (\triangle^{\xi}\sigma)(\pi_0(k(y_{n-1}))))
= ((\triangle^{\xi}\sigma)(\pi_0(k(y_0))), \ldots, (\triangle^{\xi}\sigma)(\pi_0(k(y_{n-1}))))
= \langle (\triangle^{\xi}\sigma)(\pi_0(k(y_0))), \ldots, (\triangle^{\xi}\sigma)(\pi_0(k(y_{n-1}))) + (\triangle^{\xi}\tau)(\pi_1(k(y_{n-1}))))
= (\triangle^{\xi+1}\sigma \oplus_k \triangle^{\xi+1}\tau)(x).
\]

Thus $(\triangle^{\xi+1}(\sigma \oplus_k \tau))(x) \subseteq (\triangle^{\xi+1}\sigma \oplus_k \triangle^{\xi+1}\tau)(x)$.

Now suppose that $a \in (\triangle^{\xi+1}\sigma \oplus_k \triangle^{\xi+1}\tau)(x)$. Thus

$$
a \in (\triangle^{\xi+1}\sigma)(\pi_0(k(x))) + (\triangle^{\xi+1}\tau)(\pi_1(k(x))).
\]

Say $a$ has domain $n$. Then we can write $\pi_0(k(x)) = y_0 + \cdots + y_{n-1}$ and $\pi_1(k(x)) = z_0 + \cdots + z_{n-1}$ with

$$
a = ((\triangle^{\xi}\sigma)(y_0), \ldots, (\triangle^{\xi}\sigma)(y_{n-1})) + ((\triangle^{\xi}\tau)(z_0), \ldots, (\triangle^{\xi}\tau)(z_{n-1}))
= ((\triangle^{\xi}\sigma)(y_0) + (\triangle^{\xi}\tau)(z_0), \ldots, (\triangle^{\xi}\sigma)(y_{n-1}) + (\triangle^{\xi}\tau)(z_{n-1})).
\]

Now let $x_i = k^{-1}(y_i, z_i)$ for all $i < n$. Then

$$
k(x) = (y_0 + \cdots + y_{n-1}, z_0 + \cdots + z_{n-1}) = (y_0, z_0) + \cdots + (y_{n-1}, z_{n-1});
\]

hence $x = x_0 + \cdots + x_{n-1}$. By the above,

$$
a = ((\triangle^{\xi}\sigma)(y_0) + (\triangle^{\xi}\tau)(z_0), \ldots, (\triangle^{\xi}\sigma)(y_{n-1} + (\triangle^{\xi}\tau)(z_{n-1}))
= ((\triangle^{\xi}\sigma)(\pi_0(k(x_0))), \ldots, (\triangle^{\xi}\sigma)(\pi_0(k(x_{n-1})))) + (\triangle^{\xi}\tau)(\pi_1(k(x_{n-1}))))
= ((\triangle^{\xi}\sigma \oplus_k \triangle^{\xi}\tau)(x_0), \ldots, (\triangle^{\xi}\sigma \oplus_k \triangle^{\xi}\tau)(x_{n-1}))
= (\triangle^{\xi}(\sigma \oplus_k \tau))(x_0), \ldots, (\triangle^{\xi}(\sigma \oplus_k \tau))(x_{n-1})
\in (\triangle^{\xi+1}(\sigma \oplus_k \tau))(x).
\]

This gives the conclusion for $\xi + 1$.

Now suppose the conclusion holds for all $\xi < \zeta$, with $\zeta$ a limit ordinal. Then

$$
(\triangle^{\xi}(\sigma \oplus_k \tau))(x) = ((\triangle^{\eta}(\sigma \oplus_k \tau))(x) : \eta < \zeta)
= ((\triangle^{\eta}(\sigma \oplus_k \triangle^{\eta}\tau))(x) : \eta < \zeta)
= ((\triangle^{\eta}\sigma)(\pi_0(k(x))) + (\triangle^{\eta}\tau)(\pi_1(k(x))) : \eta < \zeta)
= ((\triangle^{\eta}\sigma)(\pi_0(k(x))) : \eta < \zeta) + ((\triangle^{\eta}\tau)(\pi_1(k(x))) : \eta < \zeta)
= (\triangle^{\xi}(\sigma \oplus_k \tau))(x)
= (\triangle^{\xi}(\sigma \oplus_k \triangle^{\xi}\tau))(x).
\]

\[ 105 \]
Proposition 23.6. Suppose that $0 < x < 1$ in $\text{Fr}(\omega)$, and $k_1 : \text{Fr}(\omega) \to \text{Fr}(\omega) \upharpoonright x$ and $k_2 : \text{Fr}(\omega) \to \text{Fr}(\omega) \upharpoonright (-x)$ are isomorphisms. Suppose that $\rho \in \mathcal{M}(M)$. Define $\sigma = \rho \circ k_1$, $\tau = \rho \circ k_2$, and $\forall y \in \text{Fr}(\omega)[k(y) = (k^{-1}_1(x \cdot y), k^{-1}_2((-x) \cdot y))].$

Then $k$ is an isomorphism of $\text{Fr}(\omega)$ onto $\text{Fr}(\omega) \times \text{Fr}(\omega)$, $\sigma, \tau \in \mathcal{M}(M)$, and $\rho = \sigma \oplus k \tau$.

Proof. Clearly $k$ is an isomorphism of $\text{Fr}(\omega)$ onto $\text{Fr}(\omega) \times \text{Fr}(\omega)$ and $\sigma, \tau \in \mathcal{M}(M)$. Now suppose that $y \in \text{Fr}(\omega)$. Then

$$(\sigma \oplus k \tau)(y) = \sigma(\pi_0(k(y))) + \tau(\pi_1(k(y)))$$

$$= \sigma(k^{-1}_1(x \cdot y)) + \tau(k^{-1}_2((-x) \cdot y))$$

$$= \rho(x \cdot y) + \rho((-x) \cdot y) = \rho(y).$$

Proposition 23.5. Assume the hypotheses of Proposition 23.4, and also assume that $\rho$ is stable. Then $\sigma$ and $\tau$ are stable.

Proof. By symmetry it suffices to show that $\sigma$ is stable. So assume that $y, z \in \text{Fr}(\omega)$ and $\sigma(y) = \sigma(z)$. Thus $\rho(k_1(y)) = \sigma(y) = \sigma(z) = \rho(k_1(z))$. Since $\rho$ is stable, $((\triangle \rho)(k_1(y)) = ((\triangle \rho)(k_1(z))$. Hence

$$(\triangle \rho)(y) = \{(\sigma(\pi_0, \ldots, \sigma(\pi_{n-1})) : y = u_0 + \cdots + u_{n-1}\}$$

$$= \{(\rho(\pi_0, \ldots, \rho(\pi_{n-1})) : y = u_0 + \cdots + u_{n-1}\}$$

$$= \{(\rho(v_0, \ldots, \rho(v_{n-1})) : k_1(y) = v_0 + \cdots + v_{n-1}\}$$

To see the step $(*),

$$\{(\rho(k_1(u_0), \ldots, \rho(k_1(u_{n-1}) : k_1(y) = v_0 + \cdots + v_{n-1}\}$$

On the other hand, suppose that $k_1(y) = v_0 + \cdots + v_{n-1}$. For each $i < n$ choose $u_i \in \text{Fr}(\omega)$ such that $k_1(u_i) = v_i$. Then $k_1(u_0 + \cdots + u_{n-1}) = v_0 + \cdots + v_{n-1} = k_1(y)$, so $u_0 + \cdots + u_{n-1} = y$. This proves the other inclusion, so that $(*$) holds.

Now by symmetry, $((\triangle \sigma))(z) = ((\triangle \rho)(k_1(z))$, so $((\triangle \sigma))(y) = ((\triangle \sigma))(z)$.

For the following proposition, recall the definition of $B_s$ and $g_s$ from just before Proposition 5.22.

Proposition 23.6. If $s$ and $t$ are additive functions from $\text{Fr}(\omega)$ into $\omega_1$ and $k : \text{Fr}(\omega) \to \text{Fr}(\omega) \times \text{Fr}(\omega)$ is an isomorphism, then $B_s \times B_t \cong B_s \oplus B_t$.

Proof. Recall $g_s$ is an isomorphism from $B_s/I_{\text{ar}(B_s)}(B_s)$ onto $\text{Fr}(\omega)$ such that $r_{B_s g_s} = s$ and $g_t$ is an isomorphism from $B_t/I_{\text{ar}(B_t)}(B_t)$ onto $\text{Fr}(\omega)$ such that $r_{B_t g_t} = t$. Now $I_{\text{ar}(B_s)}(B_s) \times I_{\text{ar}(B_t)}(B_t)$ is an ideal in $B_s \times B_t$. Clearly

23.6(1) There is an isomorphism $h$ from $(B_s \times B_t)/(A \times A')$ onto $\text{Fr}(\omega) \times \text{Fr}(\omega)$ such that $\forall x \in B_s \forall y \in B_t[h([x, y]) = (g_s([x]), g_t([y]))].$
Then by definition, for any \( u \in B_s \) and \( v \in B_t \),
\[
r_{B_s \times B_t, k^{-1} \circ h}(k^{-1}(h([u, v]))) = \alpha_*((B_s \times B_t) \upharpoonright (u, v)).
\]

Suppose that \( w \in \text{Fr}(\omega) \). Say \( k(w) = (x, y) \). Choose \( u \in B_s \) and \( v \in B_t \) such that \( g_s([u]) = x \) and \( g_t([v]) = y \). Then
\[
r_{B_s \oplus k_t, s \oplus k_t}(w) = (s \oplus k_t)(w)
= (s \oplus k_t)(k^{-1}(x, y))
= s(x) + t(y) = r_{B_s, g_s}(g_s([u])) + r_{B_t, g_t}(g_t([v]))
= \alpha_*((B_s \upharpoonright u) + \alpha_*((B_t \upharpoonright v) = \alpha_*((B_s \times B_t) \upharpoonright (u, v))
= r_{B_s \times B_t, k^{-1} \circ h}(k^{-1}(h([u, v])))
= r_{B_s \times B_t, k^{-1} \circ h}(k^{-1}(g_s([u]), g_t([v])))
= r_{B_s \times B_t, k^{-1} \circ h}(w).
\]

By Proposition 5.8, \( B_{s \oplus k_t} \cong B_s \times B_t \). \( \square \)

24. The strict hierarchy property

Let \( M \) be an \( m \)-monoid and \( \xi \) a countable ordinal. We say that \( \alpha \) has the strict \( \xi \)-heirarchy property if and only if \( \alpha \in \Delta^{\xi+1} M \) and for all \( \eta < \xi \), and all \( a \) and \( b \) with \( \langle a \rangle \preceq b \in \alpha \), for all \( n \in \omega \), and all \( e \) with domain \( n \), if \( e \in T_{\eta+1}(a) \) then there is a \( c \) with domain \( n \) such that \( \sum_{i<n} c_i = a \), \( c \preceq b \in \alpha \), and \( T_{\eta}^\xi(c_i) = e_i \) for all \( i < n \).

**Proposition 24.1.** If \( \alpha \) has the strict \( \xi \)-heirarchy property, then it has the heirarchy property with respect to \( \xi \).

**Proof.** Suppose that \( \alpha \) has the strict \( \xi \)-heirarchy property, \( a \in \Delta^\xi M \), \( \langle a \rangle \in \Phi(\alpha) \), \( \eta < \xi \), and \( e \in T_{\eta+1}^\xi(a) \). Choose \( b \) so that \( \langle a \rangle \preceq b \in \alpha \). Say \( e \) has domain \( n \). Choose \( c \) with domain \( n \) such that \( \sum_{i<n} c_i = a \), \( c \preceq b \in \alpha \), and \( T_{\eta}^\xi(c_i) = e_i \) for all \( i < n \). Thus \( c \in \Phi_a(\alpha) \) and \( e = T_{\eta}^\xi \circ c \). \( \square \)

**Proposition 24.2.** Let \( M \) be an \( m \)-monoid and \( \xi \) a countable ordinal. If \( \alpha \in \Delta^{\xi+1} M \) has L.P. and H.P. then it has the strict \( \xi \)-heirarchy property.

**Proof.** Assume that \( M \) is an \( m \)-monoid, \( \xi \) a countable ordinal, \( \alpha \in \Delta^{\xi+1} M \) has L.P. and H.P., \( \eta < \xi \), \( a, b \) are such that \( \langle a \rangle \preceq b \in \alpha \), \( n \in \omega \), \( e \) has domain \( n \), and \( e \in T_{\eta+1}^\xi(a) \). Then \( a \in \Delta^\xi M \) and \( \langle a \rangle \in \Phi(\alpha) \), so by H.P. there is a \( c \in \Phi_a(\alpha) \) such that \( e = T_{\eta}^\xi \circ c \). By L.P., \( c \preceq b \in \alpha \). \( \square \)

For every \( \xi < \omega_1 \) let
\[
\mathcal{L}^\xi = \{ \alpha \in \Delta^{\xi+1} M : \alpha \text{ has the strict } \xi \text{-heirarchy property} \}.
\]

**Proposition 24.3.** \( \forall \xi < \omega_1 [ \mathcal{K}^\xi \subseteq \mathcal{L}^\xi ] \).
Proposition 24.4. For all $\xi < \omega_1$, $\mathcal{L}^\xi$ is a submonoid of $\triangle^{\xi+1} \mathcal{W}$.

Proof. First we deal with 0. Note:

24.4(1) For each $\xi < \omega_1$ let $0_\xi$ be the zero of $\triangle^{\xi} \mathcal{W}$. Then

24.4(2a) $0_{\xi+1} = \{0_\xi, 0_\xi, 0_\xi, \ldots\}$.

24.4(2b) For $\xi$ limit, $0_\xi$ is the function with domain $\xi$ such that $\forall \eta < \xi [0_\xi(\eta) = 0_\eta]$.

24.4(3) If $\eta \leq \xi$, then $T^\xi_\eta(0_\xi) = 0_\eta$.

24.4(4) $0_{\xi+1} \in \mathcal{L}_\xi$.

For, suppose that $\eta < \xi$, $\langle a \rangle^\eta \cdot b \in 0_{\xi+1}$, $n \in \omega$, $e$ has domain $n$, and $e \in T^\xi_{\eta+1}(a)$. Then $a = 0_\xi$ and $b = \langle 0_\xi, 0_\xi, \ldots \rangle$. Also, $T^\xi_{\eta+1}(a) = 0_{\eta+1}$, so $e = \langle 0_\eta, 0_\eta, \ldots \rangle$. Let $c_i = 0_\xi$ for all $i < n$. Then $\sum_{i<n} c_i = a$, $c^\eta \cdot b \in \alpha$, and $T^\xi_\eta(c_i) = 0_\eta = e_i$. So 24.4(3) holds.

Now suppose that $\alpha$ and $\beta$ are nonzero elements of $\mathcal{L}^\xi$. Suppose that $\eta < \xi$, $\langle a \rangle^\eta \cdot b \in \alpha + \beta$, $n \in \omega$, $e$ has domain $n$, and $e \in T^\xi_{\eta+1}(a)$. Then there exist $a_0, a_1, b_0, b_1$ such that $a = a_0 + a_1$, $b = b_0 + b_1$, $\langle a_0 \rangle^\eta \cdot b_0 \in \alpha$, and $\langle a_1 \rangle^\eta \cdot b_1 \in \beta$. Then $e \in T^\xi_{\eta+1}(a) = T^\xi_{\eta}(a_0) + T^\xi_{\eta}(a_1)$, so there exist $e_0 \in T^\xi_{\eta+1}(a_0)$ and $e_1 \in T^\xi_{\eta+1}(a_1)$ such that $e = e_0 + e_1$. Since $\alpha$ and $\beta$ have the strict $\xi$-hierarchy property, we get $c^0, c^1$ with domain $n$ such that $\sum_{i<n} c_i = a_j$, $c^\eta \cdot b_0 \in \alpha, c^\eta \cdot b_1 \in \beta$, and $T^\xi_{\eta} \circ c^j = e_j$ for $j < 2$. Let $c = c^0 + c^1$. Then $\sum_{i<n} c_i = a_0 + a_1 = a$, $c^\eta \cdot b_0 \in \alpha + \beta$, and $T^\xi_{\eta} \circ c = e$.

Proposition 24.5. Let $M$ be a locally countable submonoid of $\triangle^{\xi+1} \mathcal{W}$ such that

(i) $M$ has the refinement property.

(ii) $\forall \alpha \in M^\omega \alpha \in \alpha \forall n \in \omega [\text{dmn}(a) = n \rightarrow \exists \beta \in n M^\omega [\forall i < n (T^\xi_{\beta_i} = a_i) \text{ and } \sum_{i<n} \beta_i = \alpha]]$.

Then $M^\ast$ is isomorphic to a subsemigroup of $\text{NBA}$.

Proof. First we note:

24.5(1) $M$ is atomless.

For, $M \subseteq \triangle^{\xi+1} \mathcal{W}$, so $M$ satisfies S.P. Hence $M$ is atomless. By Proposition 19.12, $\delta_M$ is an isomorphism of $M$ into $\triangle^{\xi+1} \mathcal{W}$. By Proposition 19.10, if $\alpha \in M$ then $\delta_M(\alpha)$ satisfies R.P. and L.P., and so by Proposition 24.2 $\delta_M(\alpha)$ has the strict $(\xi+1)$-hierarchy property. Then by Propositions 17.3, $\delta_M(\alpha) \in \mathcal{W}^{\xi+1}$. Hence by the definition of $\mathcal{W}^{\xi+1}$ there is a $\sigma_\alpha \in \mathcal{W}$ such that $\delta_M(\alpha) = (\triangle^{\xi+2} \sigma_\alpha)(1)$ and $d(\sigma_\alpha) \leq \xi + 1$.

24.5(2) $\forall \alpha \in M^\omega [(\triangle^{\xi+1} \sigma_\alpha)(1) = \alpha]$.

In fact, if $a \in \delta_M(\alpha)$ then we can write $1 = x_0 \dot{+} \cdots \dot{+} x_{n-1}$ with $\forall i < n [a_i = (\triangle^{\xi+1} \sigma_\alpha)(x_i)]$. Then $\alpha = \sum_{i<n} a_i = (\triangle^{\xi+1} \sigma_\alpha)(1)$.

If $k$ is an automorphism of $\text{Fr}(\omega)$ such that $\sigma_\alpha = \sigma_\beta \circ k$, then, using Proposition 14.3, $\delta_M(\alpha) = (\triangle^{\xi+2} \sigma_\alpha)(1) = ((\triangle^{\xi+2} (\sigma_\beta) \circ k))(1) = (\triangle^{\xi+2} \sigma_\beta)(1) = \delta_M(\beta)$, hence $\alpha = \beta$, since by Proposition 19.12 $\delta_M$ is one-one.
Suppose that $\alpha, \beta \in M^*$. Then
\[
\langle (\triangle^{\xi+1}\sigma_\alpha)(1), (\triangle^{\xi+1}\sigma_\beta)(1) \rangle = \langle (\triangle^{\xi+1}\sigma_\alpha)(1), 0 \rangle + \langle 0, (\triangle^{\xi+1}\sigma_\beta)(1) \rangle \\
\in (\triangle^{\xi+2}\sigma_\alpha)(1) + (\triangle^{\xi+2}\sigma_\beta)(1)
\]
\[
= \delta_M(\alpha) + \delta_M(\beta) = \delta_M(\alpha + \beta) = \\
= (\triangle^{\xi+2}\sigma_{\alpha+\beta})(1) = (\triangle(\triangle^{\xi+1}\sigma_{\alpha+\beta}))(1).
\]
Hence by the definition of $\triangle\sigma$ before Proposition 10.1, we can write $1 = x + (-x)$ with
\[
\langle (\triangle^{\xi+1}\sigma_\alpha)(1), (\triangle^{\xi+1}\sigma_\beta)(1) \rangle = \langle (\triangle^{\xi+1}\sigma_{\alpha+\beta})(x), (\triangle^{\xi+1}\sigma_{\alpha+\beta})(-x) \rangle.
\]
So $(\triangle^{\xi+1}\sigma_\alpha)(1) = (\triangle^{\xi+1}\sigma_{\alpha+\beta})(x)$ and $(\triangle^{\xi+1}\sigma_\beta)(1) = (\triangle^{\xi+1}\sigma_{\alpha+\beta})(-x)$. Let $k_1 : \text{Fr}(\omega) \to \text{Fr}(\omega) \upharpoonright x$ and $k_2 : \text{Fr}(\omega) \to \text{Fr}(\omega) \upharpoonright (-x)$ be isomorphisms. Then $(\triangle^{\xi+1}\sigma_{\alpha+\beta})(k_1(1)) = (\triangle^{\xi+1}\sigma_\alpha)(1)$ and $(\triangle^{\xi+1}\sigma_{\alpha+\beta})(k_2(1)) = (\triangle^{\xi+1}\sigma_\beta)(1)$. Now $\triangle^{\xi+1}\sigma_{\alpha+\beta}$ is stable, so by Proposition 23.5, also $(\triangle^{\xi+1}\sigma_{\alpha+\beta}) \circ k_1$ and $(\triangle^{\xi+1}\sigma_{\alpha+\beta}) \circ k_2$ are stable.

24.5(3) $(\triangle^{\xi+2}\sigma_\alpha)(1) = (\triangle^{\xi+2}\sigma_{\alpha+\beta})(x)$.

In fact, suppose that $1 = u_0 + \cdots + u_{n-1} = v_0 + \cdots + v_{m-1}$. Then
\[
\langle (\triangle^{\xi+1}\sigma_\alpha)(u_0), \ldots, (\triangle^{\xi+1}\sigma_\alpha)(u_{n-1}) \rangle \in (\triangle^{\xi+2}\sigma_\alpha)(1)
\]
and
\[
\langle (\triangle^{\xi+1}\sigma_\beta)(v_0), \ldots, (\triangle^{\xi+1}\sigma_\beta)(v_{m-1}) \rangle \in (\triangle^{\xi+2}\sigma_\beta)(1),
\]
so
\[
\langle (\triangle^{\xi+1}\sigma_\alpha)(u_0), \ldots, (\triangle^{\xi+1}\sigma_\alpha)(u_{n-1}), 0, 0, \ldots, 0 \rangle + \\
\langle 0, 0, \ldots, 0, (\triangle^{\xi+1}\sigma_\beta)(v_0), \ldots, (\triangle^{\xi+1}\sigma_\beta)(v_{m-1}) \rangle \\
\in (\triangle^{\xi+2}\sigma_\alpha)(1) + (\triangle^{\xi+2}\sigma_\beta)(1) = \delta_M(\alpha) + \delta_M(\beta) = \delta_M(\alpha + \beta).
\]
Hence we can write $1 = w_0 + \cdots + w_{n-1} + w_m + \cdots + w_{n+m-1}$ so that $\forall i < n, [(\triangle^{\xi+1}\sigma_\alpha)(u_i) = (\triangle^{\xi+1}\sigma_{\alpha+\beta}(w_i)]$ and $\forall i < m, [(\triangle^{\xi+1}\sigma_\beta)(v_i) = (\triangle^{\xi+1}\sigma_{\alpha+\beta})(w_{n+i})]$. Thus
\[
(\triangle^{\xi+1}\sigma_{\alpha+\beta})(x) = (\triangle^{\xi+1}\sigma_\alpha)(1) = \sum_{i<n} (\triangle^{\xi+1}\sigma_\alpha)(u_i)
\]
\[
= \sum_{i<n} (\triangle^{\xi+1}\sigma_{\alpha+\beta})(w_i) = (\triangle^{\xi+1}\sigma_{\alpha+\beta}) \left( \sum_{i<n} w_i \right).
\]
Since $\triangle^{\xi+1}\sigma_{\alpha+\beta}$ is stable, it follows that $(\triangle^{\xi+2}\sigma_{\alpha+\beta})(x) = (\triangle^{\xi+2}\sigma_{\alpha+\beta}) \left( \sum_{i<n} w_i \right)$. Now
\[
\langle (\triangle^{\xi+1}\sigma_\alpha)(u_0), \ldots, (\triangle^{\xi+1}\sigma_\alpha)(u_{n-1}) \rangle \in (\triangle^{\xi+1}\sigma_{\alpha+\beta}) \left( \sum_{i<n} w_i \right) = (\triangle^{\xi+2}\sigma_{\alpha+\beta})(x).
\]
It follows that \((\Delta^\xi + 2\sigma_\alpha)(1) \subseteq (\Delta^\xi + 2\sigma_{\alpha + \beta})(x)\).

Now suppose that \(a \in (\Delta^\xi + 2\sigma_{\alpha + \beta})(x)\). Say \(a_i = (\Delta^\xi + 1\sigma_{\alpha + \beta})(y_i)\) for all \(i < n\) with \(x = y_0 + \cdots + y_{n-1}\). Then \(\sum_{i<n} a_i = (\Delta^\xi + 1\sigma_{\alpha + \beta})(x) = (\Delta^\xi + 1\sigma_{\alpha})(1) + \alpha\), and so \(a \in \delta_M(\alpha) = (\Delta^\xi + 2\sigma_\alpha)(1)\). Thus \((\Delta^\xi + 2\sigma_\alpha)(1) = (\Delta^\xi + 2\sigma_{\alpha + \beta})(x)\). Thus (130) holds.

24.5(4) There is an automorphism \(l\) of \(\text{Fr}(\omega)\) such that \(\Delta^\xi + 1\sigma_{\alpha + \beta} \circ k_1 = \Delta^\xi + 1\sigma_\alpha \circ l\).

In fact, by 24.5(3) we have \((\Delta^\xi + 2\sigma_\alpha)(1) = (\Delta^\xi + 2\sigma_{\alpha + \beta}) \circ k_1\). Hence 24.5(4) holds by Proposition 12.2.

By symmetry we have

24.5(5) There is an automorphism \(l'\) of \(\text{Fr}(\omega)\) such that \(\Delta^\xi + 1\sigma_{\alpha + \beta} \circ k_2 = \Delta^\xi + 1\sigma_\alpha \circ l'\).

Now by definition of \(\text{Fr}(\omega)\) we get an isomorphism \(s\) of \(\text{Fr}(\omega)\) onto \(\text{Fr}(\omega) \times \text{Fr}(\omega)\) such that \(\Delta^\xi + 1\sigma_{\alpha + \beta} = (\Delta^\xi + 1\sigma_\alpha \circ l) \oplus (\Delta^\xi + 1\sigma_\beta \circ l')\). Now we apply Proposition 23.2(x) with \(k_0, k_1, \sigma, \sigma', \tau, \tau', l_0, l_1\) replaced by \(l, l', \Delta^\xi + 1\sigma_\alpha \circ l, \Delta^\xi + 1\sigma_\alpha, \Delta^\xi + 1\sigma_\beta \circ l', \Delta^\xi + 1\sigma_{\alpha + \beta}, s, s\); we get an automorphism \(k_2\) of \(\text{Fr}(\omega)\) such that \(\Delta^\xi + 1\sigma_{\alpha + \beta} = ((\Delta^\xi + 1\sigma_\alpha) \oplus_s (\Delta^\xi + 1\sigma_\beta)) \circ k_2\). By Proposition 23.3 we then have \(\Delta^\xi + 1\sigma_{\alpha + \beta} = (\Delta^\xi + 1\sigma_\alpha + \Delta^\xi + 1\sigma_\beta) \circ k_2\). Then \((\Delta^\xi + 1\sigma_{\alpha + \beta})(1) = (\Delta^\xi + 1\sigma_{\alpha + \beta})\). By Proposition 15.2 there is an automorphism \(k_3\) of \(\text{Fr}(\omega)\) such that \(\sigma_{\alpha + \beta} = (\sigma_{\alpha + \beta}) \circ k_3\).

Now for the proof of the proposition, for each \(\alpha \in M\) let \(f(\alpha) = B_{\sigma_\alpha}\); see Proposition 6.24. Then \(f(\alpha + \beta) = B_{\sigma_{\alpha + \beta}} \cong B_{\sigma_\alpha \oplus \sigma_\beta}\) (by Proposition 16.1) \(\cong B_{\sigma_\alpha} \times B_{\sigma_\beta}\) (by Proposition 23.6) \(= f(\alpha) \times f(\beta)\). So \(f\) is a homomorphism. If \(f(\alpha) \cong f(\beta)\), then by Proposition 16.1 there is an automorphism \(k_4\) of \(\text{Fr}(\omega)\) such that \(\sigma_{\alpha} = \sigma_{\beta} \circ k_4\), and so by a remark early in this proof, \(\alpha = \beta\).

25. Shifting monoids

Recall the definition of \(N\) from just before Proposition 9.3. For any \(\theta \in N\) let

\[
\text{supp}(\theta) = \{\xi < \omega_1 : \theta(\xi) > 0\}.
\]

**Proposition 25.1.** Suppose that \(\theta \in N\).

(i) \(\text{supp}(\theta)\) has a largest element.

(ii) \(\text{supp}(\theta)\) is countable and nonempty.

(iii) \(\theta(\min(\text{supp}(\theta))) = \omega\).

Now if \(a \in \langle \alpha \rangle^\omega\) \(W\) we define \(\varphi_a : \omega_1 \to \omega\) by \(\varphi_a(\xi) = |\{i < \text{dmn}(a) : a_i = \xi\}|\). Then \(\text{supp}(\varphi_a) = \{\xi < \omega_1 : \varphi_a(\xi) > 0\}\). Then for any \(\theta \in N^*\) we let \(\Psi(\theta) = g(\theta)\) with \(g\) as in the proof of Proposition 9.6. We also set \(\Psi(o) = o\).

Two mappings \(\theta, \chi : \omega + 1 \to \omega + 1\) are equal almost everywhere, in symbols \(\theta \equiv_{ae} \chi\), iff \(\{k \leq \omega : \theta(k) \neq \chi(k)\}\) is finite. \(0^c\) and \(\omega^c\) are the constant mappings: \(0^c(k) = 0\) and \(\omega^c(k) = \omega\) for all \(k \in \omega + 1\). \(N^0^*\) is the set of all \(\theta : \omega + 1 \to \omega + 1\) such that one of the following holds:

(AE1) \(\theta \equiv_{ae} 0^c\).

(AE2) \(\theta \equiv_{ae} \omega^c\) and \(\theta(\omega) = \omega\).

110
Proposition 25.2. \((\omega + 1, +)\) is an r-monoid.

\textbf{Proof.} Clearly \((\omega + 1, +)\) is an \(m\)-monoid. To show that it is an \(r\)-monoid we apply Proposition 19.7. So, suppose that \(a, b \in 2^{(\omega + 1)}\) and \(a_0 + a_1 = b_0 + b_1\). We want to find \(c \in 2^{\omega + 1}\) such that \(\forall i < 2[a_i = \sum_{j < 2} c_{ij}]\) and \(\forall j < 2[b_j = \sum_{i < 2} c_{ij}]\).

Case 1. \(a_0 = a_1 = b_0 = b_1 = \omega\). Let \(c_{ij} = \omega\) for all \(i, j < 2\).

Case 2. \(a_0 = a_1 = \omega = b_0 \) and \(b_1 < \omega\). Let \(c_{01} = 0, c_{11} = b_1,\) and \(c_{ij} = \omega\) otherwise.

Case 3. \(a_0 = b_0 = b_1 = \omega\). Let \(c_{10} = a_0, c_{11} = 0, c_{01} = b_1,\) and \(c_{ij} = \omega\) otherwise.

Case 4. \(a_0 = b_0 = \omega\) and \(a_1, b_1 < \omega\). Let \(c_{10} = a_1, c_{11} = 0, c_{01} = b_1,\) and \(c_{ij} = \omega\) otherwise.

Case 5. Other cases with one or more of \(a_0, a_1, b_0, b_1\) equal to \(\omega\). This is symmetric to above cases.

Case 6. \(a_0, a_1, b_0, b_1 < \omega\). Wlog \(a_0 \leq b_0\). Let \(c_{00} = a_0, c_{01} = 0, c_{10} = b_0 - a_0, c_{11} = b_1\). Then \(c_{00} + c_{01} = a_0, c_{10} + c_{11} = b_0 - a_0 + b_1 = a_1, c_{00} + c_{10} = a_0 + b_0 - a_0 = b_0, c_{01} + c_{11} = b_1\). □

Proposition 25.3. \(N^0\) is an \(r\)-monoid.

\textbf{Proof.} First we check that \(N^0\) is closed under \(\omega\). Suppose that \(x, y \in N^0\). Obviously \(x + y \in N^0\) if one of \(x, y\) is \(o\). Now suppose that \(x, y \neq o\). So \(x, y : \omega + 1 \to \omega + 1\).

Case 1. \(x, y \equiv_{ae} 0^c\). Clearly \(x + y \equiv_{ae} 0^c\).

Case 2. \(x \equiv_{ae} 0^c\), \(y \equiv_{ae} \omega^c\), and \(y(\omega) = \omega\). Clearly \(x + y \equiv_{ae} \omega\) and \((x + y)(\omega) = \omega\).

Case 3. \(x \equiv_{ae} 0^c\). \(\{k \in \omega + 1 : y(k) = \omega\}\) is finite, \(\lim_{k \to \omega} y(k) = \omega = y(\omega)\). Then clearly \(\{k \in \omega + 1 : (x + y)(k) = \omega\}\) is finite, \(\lim_{k \to \omega} (x + y)(k) = \omega = (x + y)(\omega)\).

Case 4. \(x \equiv_{ae} \omega^c\), \(y \equiv_{ae} 0^c\). Symmetric to Case 2.

Case 5. \(x \equiv_{ae} \omega^c\), \(y \equiv_{ae} \omega^c\). \(x(\omega) = \omega, y(\omega) = \omega\). Clearly \(x + y \equiv_{ae} \omega^c\) and \((x + y)(\omega) = \omega\).
Case 6. \(x \equiv_{ae} \omega^c\), \(\{k \in \omega + 1 : y(k) = \omega\}\) is finite, \(\lim_{k \to \infty} y(k) = \omega = y(\omega)\). Clearly \(x + y \equiv_{ae} \omega^c\) and \((x + y)(\omega) = \omega\).

Case 7. \(\{k \in \omega + 1 : x(k) = \omega\}\) is finite, \(\lim_{k \to \infty} x(k) = \omega = x(\omega)\), \(y \equiv_{ae} 0^c\). Symmetric to Case 3.

Case 8. \(\{k \in \omega + 1 : x(k) = \omega\}\) is finite, \(\lim_{k \to \infty} x(k) = \omega = x(\omega)\), \(y \equiv_{ae} \omega^c\). Symmetric to Case 6.

Case 9. \(\{k \in \omega + 1 : x(k) = \omega\}\) is finite, \(\lim_{k \to \infty} x(k) = \omega = x(\omega)\), \(\{k \in \omega + 1 : y(k) = \omega\}\) is finite, \(\lim_{k \to \infty} y(k) = \omega = y(\omega)\). Clearly \(\{k \in \omega + 1 : (x + y)(k) = \omega\}\) is finite, \(\lim_{k \to \infty} (x + y)(k) = \omega = (x + y)(\omega)\).

So \(\mathcal{N}^0\) is closed under +.

Clearly now \(\mathcal{N}^0\) is an \(m\)-monoid.

To prove that \(\mathcal{N}^0\) is an \(r\)-monoid we again apply Proposition 19.7. So, suppose that \(a_0, a_1, b_0, b_1 \in \mathcal{N}^0\) and \(a_0 + a_1 = b_0 + b_1\). We want to define \(\theta \in 2 \times 2 \mathcal{N}^0\) so that \(\forall i < 2[a_i = \sum_{j < 2} \theta_{ij}]\) and \(\forall j < 2[b_j = \sum_{i < 2} \theta_{ij}]\).

Case 1. One or more of \(a_0, a_1, b_0, b_1\) is equal to 0. By symmetry say \(a_0 = 0\). Let \(\theta_{00} = 0, \theta_{10} = b_0, \theta_{11} = b_1\).

Case 2. All of \(a_0, a_1, b_0, b_1\) are different from 0. For each \(k \leq \omega\), by Proposition 25.2 there is a \(\{\theta_{ij}(k) : i, j < 2\}\) such that each \(\theta_{ij}(k) \in \omega + 1\), \(\forall i < 2[a_i(k) = \sum_{j < 2} \theta_{ij}(k)]\), and \(\forall j < 2[b_j(k) = \sum_{i < 2} \theta_{ij}(k)]\).

Subcase 2.1. \(a_0 \equiv_{ae} 0^c\) and \(a_1 \equiv_{ae} 0^c\). Clearly then \(b_0 \equiv_{ae} 0^c\) and \(b_1 \equiv_{ae} 0^c\).

Clearly also \(\theta_{ij} \equiv_{ae} 0^c\) for all \(i, j < 2\).

Subcase 2.2. \(a_0 \equiv_{ae} 0^c\), \(a_1 \equiv_{ae} \omega^c\), and \(a_1(\omega) = \omega\). Then \(a_0 + a_1 \equiv_{ae} \omega^c\), so also \(b_0 + b_1 \equiv_{ae} \omega^c\).

Subsubcase 2.2.1. \(b_0 \equiv_{ae} 0^c\). Then \(b_1 \equiv_{ae} \omega^c\) and \(b_1(\omega) = \omega\) since \(b_0 + b_1 \equiv_{ae} \omega^c\).

Now the set \(M \stackrel{\text{def}}{=} \{k \in \omega + 1 : a_0(k) \neq 0 \text{ or } a_1(k) \neq \omega \text{ or } b_0(k) \neq \omega \text{ or } b_1(k) \neq \omega\}\) is finite.

For \(k \in (\omega + 1) \setminus M\) we have \(0 = a_0(k) = \sum_{j < 2} \theta_{0j}(k)\), hence \(\theta_{00}(k) = 0\) and \(\theta_{01}(k) = 0\).

Also, \(0 = b_0(k) = \sum_{i < 2} \theta_{0i}(k)\), hence \(\theta_{00}(k) = 0\) and \(\theta_{10}(k) = 0\). Thus \(\theta_{00}, \theta_{01}, \theta_{10}\) are \(\in \mathcal{N}^0\) by (133). Also \(\omega = a_1(k) = \sum_{j < 2} \theta_{1j}(k)\) while \(\theta_{10}(k) = 0\), so \(\theta_{11}(k) = \omega\).

Let \(\theta_{11}'\) be equal to \(\theta_{11}\) except that \(\theta_{11}(\omega) = \omega\). Then \(\theta_{11}' \in \mathcal{N}^0\), \(a_1(\omega) = \sum_{j < 2} \theta_{1j}'(\omega)\), and \(b_1(\omega) = \sum_{i < 2} \theta_{1i}'(\omega), \text{ with } \theta_{01}' = \theta_{01}\).

Subsubcase 2.2.2. \(b_1 \equiv_{ae} 0^c\). This is symmetric to Subsubcase 2.2.1.

Subsubcase 2.2.3. \(b_0 \equiv_{ae} \omega^c \equiv_{ae} b_1\) and \(b_0(\omega) = b_1(\omega) = \omega\). Then the set \(M \stackrel{\text{def}}{=} \{k \in \omega + 1 : a_0(k) \neq 0 \text{ or } a_1(k) \neq \omega \text{ or } b_0(k) \neq \omega \text{ or } b_1(k) \neq \omega\}\) is finite.

For \(k \in (\omega + 1) \setminus M\) we have \(0 = a_0(k) = \sum_{j < 2} \theta_{0j}(k)\), hence \(\theta_{00}(k) = 0\) and \(\theta_{01}(k) = 0\).

Also, \(\omega = b_0(k) = \sum_{i < 2} \theta_{0i}(k)\) and \(\theta_{00}(k) = 0\), so \(\theta_{10}(k) = \omega\). Also \(\omega = b_1(k) = \sum_{i < 2} \theta_{1i}(k)\) and \(\theta_{01}(k) = \omega\), so \(\theta_{11}(k) = \omega\). Let \(\theta_{10}'\) be like \(\theta_{10}\) except that \(\theta_{10}(\omega) = \omega\), and let \(\theta_{11}'\) be like \(\theta_{11}\) except that \(\theta_{11}(\omega) = \omega\). Then \(\theta_{10}, \theta_{11} \in \mathcal{N}^0\), \(a_1 = \sum_{j < 2} \theta_{1j}'(k)\), \(b_0 = \sum_{i < 2} \theta_{0i}'(k)\), and \(b_1 = \sum_{i < 2} \theta_{1i}'(k), \text{ with } \theta_{00}' = \theta_{00} \text{ and } \theta_{01}' = \theta_{01}\).

Subsubcase 2.2.4. \(b_0 \equiv_{ae} \omega^c\), \(b_0(\omega) = \omega\), \(\{k \in \omega + 1 : b_1(k) = \omega\}\) is finite, and \(\lim_{k \to \infty} b_1(k) = \omega = b_1(\omega)\). Then \(M \stackrel{\text{def}}{=} \{k \in \omega + 1 : a_0(k) \neq 0 \text{ or } a_1(k) \neq \omega \text{ or } b_0(k) \neq \omega \text{ or } b_1(k) = \omega\}\) is finite. Take any \(k \in (\omega + 1) \setminus M\). Then as in Subsubcase 2.2.3, \(\theta_{00}(k) = \theta_{01}(k) = 0\) and \(\theta_{10}(k) = \omega\). Hence \(\theta_{00}, \theta_{01} \in \mathcal{N}^0\). Also \(b_1(k) = \sum_{i < 1} \theta_{1i}(k)\) and \(\theta_{01}(k) = 0\), so \(b_1(k) = \theta_{11}(k)\). Let \(\theta_{10}\) be like \(\theta_{10}\) except that \(\theta_{10}(\omega) = \omega\), and let \(\theta_{11}'\) be
like \( \theta_{11} \) except that \( \theta_{11}'(\omega) = \omega \). Then \( \theta_{10}', \theta_{11}' \in \mathcal{N}^0 \). \( a_1 = \sum_{j < q} \theta_{1j}' \), \( b_0 = \sum_{i < 2} \theta_{i0}' \), and \( b_1 = \sum_{i < 2} \theta_{i1}' \), with \( \theta_{00}' = \theta_{00} \) and \( \theta_{01}' = \theta_{01} \).

Subcase 2.2.5. \( b_1 \equiv_{ae} \omega^c \), \( b_1(\omega) = \omega \), \( \{ k \in \omega + 1 : b_0(k) = \omega \} \) is finite, and \( \lim_{k \to \infty} b_0(k) = \omega = b_0(\omega) \). This is symmetric to Subcase 2.2.4.

Subcase 2.3. \( a_0 \equiv_{ae} \omega^c \), \( \{ k \in \omega + 1 : a_1(k) = \omega \} \) is finite, and \( \lim_{k \to \infty} a_1(k) = \omega = a_1(\omega) \). Then (134) does not hold for \( b_0 \) or \( b_1 \). Also \( b_0 \) and \( b_1 \) are not both \( \equiv_{ae} \omega^c \).

Subcase 2.3.1. \( b_0 \equiv_{ae} \omega^c \), \( \{ k \in \omega + 1 : b_1(k) = \omega \} \) is finite, and \( \lim_{k \to \infty} b_1(k) = \omega = b_1(\omega) \). Then \( M \overset{\text{def}}{=} \{ k \in \omega + 1 : a_0(k) \neq 0 \text{ or } a_1(k) = \omega \text{ or } b_0(k) \neq 0 \text{ or } b_1(k) = \omega \} \) is finite. Suppose that \( k \in (\omega + 1) \setminus M \). Then \( \theta_{00}(k) = \theta_{01}(k) = 0 \) as in Subcase 2.3.1. Hence \( \theta_{00}, \theta_{01} \in \mathcal{N}^0 \). \( b_0(k) = \sum_{i < 2} \theta_{i0}(k) \), \( b_1(k) = \sum_{i < 2} \theta_{i1}(k) \). Let \( \theta_{10}' \) be like \( \theta_{10} \) except that \( \theta_{10}'(\omega) = \omega \). Let \( \theta_{11}' \) be like \( \theta_{11} \) except that \( \theta_{11}'(\omega) = \omega \). Then \( \theta_{10}'(k) = \omega \), and \( \theta_{11}'(k) = \omega \). The desired conclusion follows.

Subcase 2.3.2. \( \{ k \in \omega + 1 : b_0(k) = \omega \} \) is finite, \( \lim_{k \to \infty} b_0(k) = \omega = b_0(\omega) \), \( \{ k \in \omega + 1 : b_1(k) = \omega \} \) is finite, \( \lim_{k \to \infty} b_1(k) = \omega = b_1(\omega) \). Then \( M = \{ k \in \omega + 1 : a_0(k) \neq 0 \text{ or } a_1(k) = \omega \text{ or } b_0(k) \neq 0 \text{ or } b_1(k) = \omega \} \) is finite. For \( k \in (\omega + 1) \setminus M \) we have \( \theta_{ij}(k) = \omega \) for all \( i, j < 2 \). Defining \( \theta_{ij}' \) as above, the desired conclusion follows.

Subcase 2.4. \( a_0 \equiv_{ae} \omega^c \equiv_{ae} a_1 \) and \( a_0(\omega) = \omega = a_1(\omega) \). Then \( b_0 \equiv_{ae} \omega^c \) and \( b_0(\omega) = \omega \), or \( b_1 \equiv_{ae} \omega^c \) and \( b_1(\omega) = \omega \).

Subcase 2.4.1 \( b_0 \equiv_{ae} \omega^c \equiv_{ae} b_1 \), \( b_0 = \omega \), and \( b_1(\omega) = \omega \). Then \( M = \{ k \in \omega + 1 : a_0(k) \neq 0 \text{ or } a_1(k) = \omega \text{ or } b_0(k) \neq 0 \text{ or } b_1(k) = \omega \} \) is finite. For \( k \in (\omega + 1) \setminus M \) we have \( \theta_{ij}(k) = \omega \) for all \( i, j < 2 \). Defining \( \theta_{ij}' \) as above, the desired conclusion follows.

Subcase 2.4.2 \( b_0 \equiv_{ae} \omega^c \), \( b(\omega) = \omega \), \( \{ k \in \omega + 1 : b_1(k) = \omega \} \) is finite, and \( \lim_{k \to \infty} b_1(k) = \omega = b_1(\omega) \). Then \( M = \{ k \in \omega + 1 : a_0(k) \neq 0 \text{ or } a_1(k) = \omega \text{ or } b_0(k) \neq 0 \text{ or } b_1(k) = \omega \} \) is finite. Suppose that \( k \in (\omega + 1) \setminus M \). Then \( b_1(k) = \sum_{i < 2} \theta_{i1}(k) \), so \( \theta_{01}(k) \neq \omega = \theta_{11}(k) \). \( a_0(k) = \sum_{j < 2} \theta_{0j}(k) \), so \( \theta_{00}(k) = \omega \). \( a_1(k) = \sum_{j < 2} \theta_{1j}(k) \), so \( \theta_{10}(k) = \omega \). Thus \( \theta_{00}, \theta_{10} \in \mathcal{N}^0 \). Now we define for \( k \in \omega + 1 \)

\[
\theta_{00}'(k) = \begin{cases} 
\theta_{00}(k) & \text{if } k < \omega, \\
\omega & \text{if } k = \omega,
\end{cases}
\quad \theta_{10}'(k) = \begin{cases} 
\theta_{10}(k) & \text{if } k < \omega, \\
\omega & \text{if } k = \omega.
\end{cases}
\]

\[
\theta_{01}'(k) = \begin{cases} 
\theta_{01}(k) & \text{if } k \in M, k \neq \omega, \\
b_1(k) & \text{if } k \notin M, k \neq \omega, \\
\omega & \text{if } k = \omega,
\end{cases}
\quad \theta_{11}'(k) = \begin{cases} 
\theta_{11}(k) & \text{if } k \in M, \\
0 & \text{if } k \notin M \text{ or } k = \omega.
\end{cases}
\]

Then each \( \theta_{ij}' \in \mathcal{N}^0 \), \( \forall i < 2[a_i = \sum_{j < 2} \theta_{ij}'] \), and \( \forall j < 2[b_j = \sum_{i < 2} \theta_{ij}'] \). Subcase 2.5. \( a_0 \equiv_{ae} \omega^c \), \( a_0(\omega) = \omega \), \( \{ k \in \omega + 1 : a_1(k) = \omega \} \) is finite, and \( \lim_{k \to \infty} a_1(k) = \omega = a_1(\omega) \). Then (134) holds for \( b_0 \) or \( b_1 \). If it holds for both, we have a subcase symmetric to Subcase 2.4; and if (133) holds for one and (134) for the other we have a subcase symmetric to Subcase 2.2. So we may assume that \( b_0 \equiv_{ae} \omega^c \), \( b_0(\omega) = \omega \), \( \{ k \in \omega + 1 : b_1(k) = \omega \} \) is finite, and \( \lim_{k \to \infty} b_1(k) = \omega = b_1(\omega) \). Then \( M = \{ k \in \omega + 1 : a_0(k) \neq \omega \text{ or } a_1(k) = \omega \text{ or } b_0(k) \neq \omega \text{ or } b_1(k) = \omega \} \) is finite. Suppose
that \( k \in (\omega + 1) \setminus M \). Now \( a_1(k) = \sum_{j < 2} \theta_{1j}(k) \), so \( \theta_{10}(k) \neq \omega \neq \theta_{11}(k) \). Also \( b_1(k) = \sum_{i < 2} \theta_{i1}(k) \), so \( \theta_{01}(k) \neq \omega \neq \theta_{11}(k) \). Since \( a_0(k) = \sum_{j < 2} \theta_{0j}(k) \), we have \( \theta_{00}(k) = \omega \). Define

\[
\theta'_{00}(k) = \begin{cases} 
\theta_{00}(k) & \text{if } k < \omega, \\
\omega & \text{if } k = \omega,
\end{cases}
\]

\[
\theta'_{11}(k) = \begin{cases} 
\theta_{11}(k) & \text{if } k 
\in \omega, k \neq \omega, \\
\lfloor \min(a_1(k), b_1(k))/2 \rfloor & \text{if } k \notin M, k \neq \omega, \\
\omega & \text{if } k = \omega,
\end{cases}
\]

\[
\theta'_{10}(k) = \begin{cases} 
\frac{a_1(k) - \theta'_{11}(k)}{\omega} & \text{if } k \notin M, k \neq \omega, \\
\theta_{10}(k) & \text{if } k \in M, k \neq \omega, \\
\omega & \text{if } k = \omega,
\end{cases}
\]

\[
\theta'_{01}(k) = \begin{cases} 
\frac{b_1(k) - \theta'_{11}(k)}{\omega} & \text{if } k \notin M, k \neq \omega, \\
\theta_{01}(k) & \text{if } k \in M, k \neq \omega, \\
\omega & \text{if } k = \omega.
\end{cases}
\]

Clearly \( \forall i < 2 [a_i = \sum_{j < 2} \theta'_{ij}] \) and \( \forall j < 2 [b_j = \sum_{i < 2} \theta'_{ij}] \); and also each \( \theta'_{ij} \in \mathbb{N}^0 \).

**Subcase 2.6.** \( \{ k \in \omega + 1 : a_0(k) = \omega \} \) is finite, \( \lim_{k \to \infty} a_0(k) = \omega = a_0(\omega) \), \( \{ k \in \omega + 1 : a_1(k) = \omega \} \) is finite, \( \lim_{k \to \infty} a_1(k) = \omega = a_1(\omega) \), \( \{ k \in \omega + 1 : b_0(k) = \omega \} \) is finite, \( \lim_{k \to \infty} b_0(k) = \omega = b_0(\omega) \), \( \{ k \in \omega + 1 : b_1(k) = \omega \} \) is finite, and \( \lim_{k \to \infty} b_1(k) = \omega = b_1(\omega) \). Then \( M \overset{\text{def}}{=} \{ k \in \omega + 1 : a_0(k) = \omega \text{ or } a_1(k) = \omega \text{ or } b_0(k) = \omega \text{ or } b_1(k) = \omega \} \) is finite.

(*) Suppose that \( k \notin M \cup \{ \omega \} \) and \( a_0(k) \min(a_0(k), a_1(k), b_0(k), b_1(k)) \). Then there are \( c^k_{ij} \) for \( i, j < 2 \) such that \( \forall i < 2 [a_i(k) = \sum_{j < 2} c^k_{ij}] \), \( \forall j < 2 [b_j(k) = \sum_{i < 2} c^k_{ij}] \), and \( \forall i, j < 2 [[a_0(k)/2] \leq c^k_{ij}] \).

For, let \( c^k_{01} = [a_0(k)/2] \), \( c^k_{00} = a_0(k) - c^k_{01} \), \( c^k_{10} = b_0(k) - c^k_{00} \), \( c^k_{11} = b_1(k) - c^k_{01} \).

Then \( c^k_{00} + c^k_{01} = a_0(k), c^k_{00} + c^k_{10} = b_0(k), c^k_{01} + c^k_{11} = b_1(k), \) and \( c^k_{10} + c^k_{11} = b_0(k) - c^k_{00} + b_1(k) - c^k_{01} = a_0(k) - c^k_{00} + a_1(k) - c^k_{01} = a_1(k) \).

Moreover,

\[
c^k_{00} = a_0(k)/2 + a(0)/2 - [a_0(k)/2] \geq [a_0(k)/2];
\]

\[
c^k_{10} \geq a_0(k) - (a_0(k) - c^k_{01}) = c^k_{01} = [a_0(k)/2];
\]

\[
c^k_{11} \geq a_0(k) - [a_0(k)/2] \geq [a_0(k)/2].
\]

Thus (*) holds.

We obtain \( c^k_{ij} \), similarly for other cases of \( \min(a_0(k), a_1(k), b_0(k), b_1(k)) \). Now for each \( i, j < 2 \) and \( k \in \omega + 1 \) define

\[
\theta'_{ij}(k) = \begin{cases} 
\theta_{ij}(k) & \text{if } k \in M \setminus \{ \omega \}, \\
c^k_{ij} & \text{if } k \notin M \cup \{ \omega \}, \\
\omega & \text{if } k = \omega,
\end{cases}
\]

Clearly the desired conditions hold.

\[\Box\]

**Proposition 25.4.** Suppose that \( a_0, a_1, b_0, b_1 \in \omega \), one of them is 0, and \( a_0 + a_1 = b_0 + b_1 \). Then there exists \( c \in 2^{\times 2} \omega \) such that \( \forall i < 2 [a_i = \sum_{j < 2} c_{ij}] \), and \( \forall j < 2 [b_j = \sum_{i < 2} c_{ij}] \).
Proof. Say wlog \( a_0 = 0 \). Let \( c_{00} = c_{01} = 0, c_{10} = b_0, \) and \( c_{11} = b_1. \) Then \( a_0 = c_{00} + c_{01}, a_1 = c_{10} + c_{11}, b_0 = c_{00} + c_{10}, \) and \( b_1 = c_{01} + c_{11}. \)

Proposition 25.5. Suppose that \( a_0, a_1, b_0, b_1 \in \omega \setminus \{0\}, 0 < m < \min\{a_0, a_1, b_0, b_1\}/2, \) \( a_0 + a_1 = b_0 + b_1. \) Then there exists \( c \in 2^{\times \times \omega} \) such that \( \min\{c_{ij} : i, j < 2\} = m, \) \( c_{00} = m \) or \( c_{11} = m, \) \( \forall i < 2[a_i = \sum_{j < 2} c_{ij}], \) and \( \forall j < 2[b_j = \sum_{i < 2} c_{ij}]. \)

Proof. Assume the hypotheses.

Case 1. \( \min\{a_0, a_1, b_0, b_1\} = b_0. \) Let \( c_{00} = m, c_{01} = a_0 - m, c_{10} = b_0 - m, c_{11} = a_1 - b_0 + m. \) So \( c_{00} + c_{01} = a_0, c_{10} + c_{11} = a_1, c_{00} + c_{10} = b_0, c_{01} + c_{11} = b_1. \)

Case 2. \( \min\{a_0, a_1, b_0, b_1\} = b_1. \) Let \( c_{11} = m, c_{01} = b_1 - m, c_{10} = a_1 - m, c_{00} = a_0 - b_1 + m. \) So \( c_{00} + c_{01} = a_0, c_{10} + c_{11} = a_1, c_{00} + c_{10} = b_0, c_{01} + c_{11} = b_1. \)

Case 3. \( \min\{a_0, a_1, b_0, b_1\} = a_0. \) Let \( c_{00} = m, c_{01} = a_0 - m, c_{10} = b_0 - m, c_{11} = b_1 - a_0 + m. \) So \( c_{00} + c_{01} = a_0, c_{10} + c_{11} = a_1, c_{00} + c_{10} = b_0, c_{01} + c_{11} = b_1. \)

Case 4. \( \min\{a_0, a_1, b_0, b_1\} = a_1. \) Let \( c_{11} = m, c_{01} = b_1 - m, c_{10} = a_1 - m, c_{00} = b_0 - a_1 + m. \) So \( c_{00} + c_{01} = a_0, c_{10} + c_{11} = a_1, c_{00} + c_{10} = b_0, c_{01} + c_{11} = b_1. \)

Proposition 25.6. If \( \theta_0 + \theta_1 = \psi_0 + \psi_1 \) in \( N^0^* \), \( \theta_0 \neq_{ae} 0^c, \theta_1 \neq_{ae} 0^c, \psi_0 \neq_{ae} 0^c, \psi_1 \neq_{ae} 0^c, \) and \( \psi_1 \neq_{ae} 0^c, \) and \( \not(\theta_0 \equiv_{ae} \theta_1 \equiv_{ae} \psi_0 \equiv_{ae} \psi_1 \equiv_{ae} \omega^c) \), then there are uncountably many \( \chi \in 2^{\times \times (N^0^*)} \) such that \( \forall i < 2[\theta_i = \sum_{j < 2} \chi_{ij}] \) and \( \forall j < 2[\psi_j = \sum_{i < 2} \chi_{ij}] \).

Proof. For each \( k \in \omega + 1 \) let \( \mu(k) = \min(\theta_0(k), \theta_1(k), \psi_0(k), \psi_1(k)). \) Then \( \mu \in N^0^* \). In fact, this is clear since one of \( \theta_0, \theta_1, \psi_0, \psi_1 \) satisfies (135) and so clearly \( \mu \) does also. Clearly also \( \mu \neq_{ae} 0^c, \omega^c \). Let \( F \subseteq \omega \) be a finite set such that \( \forall k \in \omega \setminus F \) \( 0 < \mu(k) < \omega \).

For each \( E \subseteq \omega \setminus F \) we define \( \chi_E^F \in 2^{\times \times (N^0^*)} \). For \( k \in F \) take \( \chi_{ij}^E(k) \) by Proposition 25.2 so that \( \forall i < 2[\theta_i(k) = \sum_{j < 2} \chi_{ij}^E(k)] \) and \( \forall j < 2[\psi_j = \sum_{i < 2} \chi_{ij}^E(k)]. \) For \( k \in E \) take \( \chi_{ij}^E(k) \) by Proposition 25.5 so that \( \forall i < 2[\theta_i(k) = \sum_{j < 2} \chi_{ij}^E(k)], \forall j < 2[\psi_j = \sum_{i < 2} \chi_{ij}^E(k)], \) and \( \min\{\chi_{ij}^E(k) : i, j < 2\} = \mu(k)/3. \) For \( k \in \omega \setminus (F \cup E) \) take \( \chi_{ij}^E(k) \) by Proposition 25.5 so that \( \forall i < 2[\theta_i(k) = \sum_{j < 2} \chi_{ij}^E(k)], \forall j < 2[\psi_j = \sum_{i < 2} \chi_{ij}^E(k)], \) and \( \min\{\chi_{ij}^E(k) : i, j < 2\} = \mu(k)/4. \) Finally, for any \( i, j < 2 \) we let \( \chi_{ij}^E(k) = \omega. \)

For each \( \theta \in N^0^* \) define \( (\Omega\theta) : \omega_1 \rightarrow \omega + 1 \) by:

\[
(\Omega\theta)(\alpha) = \begin{cases} 
\omega & \text{if } \alpha = 0, \\
\theta(\alpha - 1) & \text{if } 0 < \alpha < \omega, \\
\theta(\omega) & \text{if } \alpha = \omega, \\
0 & \text{if } \omega < \alpha < \omega_1.
\end{cases}
\]

Proposition 25.7. \((\Omega\theta) \in N^* \).

We also define \((\Omega o) = o \) So \((\Omega x) \in N \) for any \( x \in N^0 \).

Proposition 25.8. \(\Omega : N^0 \rightarrow N\) is a morphism of m-monoids.

Proof. \((\Omega(\theta + o)) = (\Omega\theta) + o = (\Omega\theta) + (\Omega o)\). For any \( \theta \in N^0^* \) we have \((\Omega(\theta + o)) = (\Omega\theta) = (\Omega\theta) + o = (\Omega\theta) + (\Omega o)\) and similarly for \( o = \theta \). Finally, suppose that
\( \theta, \psi \in \mathcal{N}^0 \). Then for any \( \alpha \in \omega_1 \),

\[
(\Omega(\theta + \psi))(\alpha) = \begin{cases} 
\omega & \text{if } \alpha = 0, \\
(\theta + \psi)(\alpha - 1) & \text{if } 0 < \alpha < \omega \\
(\theta + \psi)(\omega) & \text{if } \alpha = \omega \\
0 & \text{if } \omega < \alpha < \omega_1.
\end{cases}
\]

\[
= \begin{cases} 
\omega + \omega & \text{if } \alpha = 0, \\
\theta(\alpha - 1) + \psi(\alpha - 1) & \text{of } 0 < \alpha < \omega \\
\theta(\omega) + \psi(\omega) & \text{if } \alpha = \omega \\
0 + 0 & \text{if } \omega < \omega_1
\end{cases}
\]

\[
= (\Omega\theta)(\alpha) + (\Omega\psi)(\alpha).
\]

\[
(\Omega^0c)(\alpha) = \begin{cases} 
\omega & \text{if } \alpha = 0, \\
0 & \text{if } 0 < \alpha < \omega \\
0 & \text{if } \alpha = \omega \\
0 & \text{if } \omega < \alpha < \omega_1.
\end{cases}
\]

Thus \( \text{supp}(\Omega^0c) = \{0\} \). So \( \Psi((\Omega^0)) = \{a \in <\omega^\omega \mid T(a) = 0 \text{ and } \forall \alpha \in \omega_1 \{0, \omega\}[a_\alpha = 0]\} \). Hence \( T(\Psi((\Omega^0))) = 0 \).

(iii): Suppose that \( \theta \in \mathcal{N}^* \setminus \{0^c\} \) and \( \theta \equiv ae 0^c \). Then there is a \( k \leq \omega \) such that \( \theta(k) \neq 0 \). Let \( k = \max(\text{supp}(\theta)) \). Then \( k + 1 \) is the "\( \eta \)" for \( (\Omega\theta) \), and so \( T(\Psi((\Omega\theta))) = 1 + \max(\text{supp}(\theta)) \).

(iv): Suppose that \( \theta \in \mathcal{N}^* \) and \( \theta \neq ae 0^c \). Then \( T(\omega) = \omega \), and so \( \omega \) is the "\( \eta \)" for \( (\Omega\theta) \). Hence \( T(\Psi((\Omega\theta))) = \omega \). \( \square \)

We extend \( \equiv^a' \) to \( << \omega \mathcal{N}^0 \) by defining \( a \equiv^a' b \) iff \( \text{dmn}(a) = \text{dmn}(b) \) and \( \forall i < \text{dmn}(a) [a_i \equiv^a' b_i] \).

\( \mathcal{N}^{1^*} \) is the set of all \( \alpha \in \triangle \mathcal{N}^0 \) such that:

\( \forall n \in \omega \forall a, b[\text{dmn}(a) = \text{dmn}(b) = n \text{ and } a \equiv^a' b \in \alpha \Rightarrow \sum_{i < n} a_i = \sum_{i < n} b_i \rightarrow a \in \alpha] \).

\( \forall a, b[a \rightarrow b \in \alpha \text{ and } a \equiv ae 0^c \rightarrow \exists c, d[a = c + d \in \mathcal{N}^0 \text{ and } c, d \neq ae 0^c \text{ and } c - d \rightarrow b \in \alpha] \].

Further, we let \( \mathcal{N}^{1} = \mathcal{N}^{1^*} \cup \{0\} \), where 0 is the 0 of \( \triangle \mathcal{N}^0 \).

**Proposition 25.10.** In \( \omega + 1 \), if \( \sum_{i < n} u_i = \sum_{i < n} (v_i + w_i) \), then there exist \( v', w' \in n(\omega + 1) \) such that \( \forall i < n [u_i = v'_i + w'_i] \sum_{i < n} v'_i = \sum_{i < n} v_i \), and \( \sum_{i < n} w'_i = \sum_{i < n} w_i \).

116
Proof. We may assume that $n > 1$. For $i < 2n$ let
\[
b_i = \begin{cases} v_i & \text{if } i < n, \\ w_{i-n} & \text{if } n \leq i < 2n. \end{cases}
\]
Then let $c \in n \times 2^n (\omega + 1)$ be such that $\forall i < n [u_i = \sum_{j < 2n} c_{ij}]$ and $\forall j < 2n [b_j = \sum_{i < n} c_{ij}]$. Then $\forall j < n [v_j = \sum_{i < n} c_{ij}]$ and $\forall j < n [w_j = \sum_{i < n} c_{i,j+n}]$. For each $i < n$ let $v'_i = \sum_{j < n} c_{i,j}$ and $w'_i = \sum_{j < n} c_{i,n+j}$. Then $\forall i < n [u_i = v'_i + w'_i]$,
\[
\sum_{i < n} v'_i = \sum_{i < n} \sum_{j < n} c_{ij} = \sum_{j < n} \sum_{i < n} c_{ij} = \sum_{j < n} v_j,
\]
\[
\sum_{i < n} w'_i = \sum_{i < n} \sum_{j < n} c_{i,n+j} = \sum_{j < n} \sum_{i < n} c_{i,n+j} = \sum_{j < n} w_j.
\]
\[\square\]

Proposition 25.11. $N^1$ is a submonoid of $\Delta N^0$.

Proof. Clearly $0 + 0 = 0$ and $\alpha + 0 = 0 + \alpha = \alpha$ for any $\alpha \in N^1$. Now suppose that $\alpha, \beta \in N^1^*$. We want to prove that $\alpha + \beta \in N^1^*$. Suppose that $n \in \omega$, $a, b$ have domain $n$, $a \equiv_{ae} b \in (\alpha + \beta)$, and $\sum_{j < n} a_i = \sum_{i < n} b_i$. Choose $c, d$ with domain $n$ such that $\forall i < n [b_i = c_i + d_i]$, $c \in \alpha$, and $d \in \beta$. Let $F = \{ k \in \omega + 1 : \exists i < n [a_i(k) \neq b_i(k)] \}$. Thus $F$ is a finite subset of $\omega + 1$. For any $k \in (\omega + 1) \setminus F$ and any $i < n$ let $c'_i(k) = c_i(k)$ and $d'_i(k) = d_i(k)$. Now for any $k \in F$ we apply Proposition 25.10 with $a_i(k), c_i(k), d_i(k)$ in place of $u_i, v_i, w_i$ to obtain $c'_i(k)$ and $d'_i(k)$ such that $\forall i < n [a_i(k) = c'_i(k) + d'_i(k)]$, $\sum_{i<n} c'_i(k) = \sum_{i<n} c_i(k)$, and $\sum_{i<n} d'_i(k) = \sum_{i<n} d_i(k)$. Then $\forall i < n [a_i = c'_i + d'_i]$. Now $c' \equiv_{ae} c \in \alpha$ and $d' \equiv_{ae} d \in \beta$, so $a \equiv_{ae} c \in \alpha + \beta$, as desired in (AE8).

For (AE9), suppose that $a \sim b \in (\alpha + \beta)$ and $a \not\equiv_{ae} 0^c$. Write $a \sim b = (a' \sim b') + (a'' \sim b'')$ with $a' \sim b' \in \alpha$ and $a'' \sim b'' \in \beta$. Then choose $c', d', c'', d''$ so that $a' = c' + d'$, $a'' = c'' + d''$, $c', d', c'', d'' \not\equiv_{ae} 0^c$, and $a' \sim b' = a'' \sim b'' \in \beta$. Let $c = c' + c''$ and $d = d' + d''$. Then $c \sim d \in (\alpha + \beta)$, as desired.
\[\square\]

Proposition 25.12. If $\Theta : M \rightarrow N$ is an injective morphism of $m$-monoids, for any $\alpha \in \Delta M$ let $(\Delta \Theta)(\alpha) = \{ \Theta \circ a : a \in \alpha \}$. Then $\Delta \Theta$ is an injective morphism from $\Delta M$ to $\Delta N$.

Proof. First we check that $\forall \alpha \in \Delta M[(\Delta \Theta)(\alpha) \in \Delta N]$. So suppose that $\alpha \in \Delta M$.

C.P.: Suppose that $a \in (\Delta \Theta)(\alpha)$ and $a \sim b$. Say $\text{dmm}(a) = m$, $\text{dmm}(b) = n$, $\lambda : m \rightarrow n$, and $\forall j < n [b_j = \sum \{ a_i : i < m, \lambda(i) = j \}]$. Say $a = \Theta \circ c$ with $c \in \alpha$. Define $d_j = \sum \{ c_i : i < m, \lambda(i) = j \}$ for all $j$. Then $c \sim d$, so $d \in \alpha$. Clearly $\Theta \circ d = b$. So $b \in (\Delta \Theta)(\alpha)$.

R.P.: Suppose that $m, n \in \omega \setminus \{ 0 \}$, $a \in m \setminus N$, $b \in n \setminus N$, and $a, b \in (\Delta \Theta)(\alpha)$. Choose $a', b' \in \alpha$ such that $a = \Theta \circ a'$ and $b = \Theta \circ b'$. Then there is a $c \in m \setminus M$ such that $c \in \alpha$, $\forall i < m [a'_i = \sum \{ c_{f(i,j)} : j < n \}]$, and $\forall j < n [b'_j = \sum \{ c_{f(i,j)} : i < m \}]$. Then $(\Theta \circ c) \in (\Delta \Theta)(\alpha)$, $\forall i < m [a_i = \sum \{ (\Theta \circ c)_{f(i,j)} : j < n \}]$, and $\forall j < n [b_j = \sum \{ (\Theta \circ c)_{f(i,j)} : i < m \}]$.

S.P.: Suppose that $m \in \omega \setminus \{ 0 \}$, $a \in m \setminus N$, $a \in (\Delta \Theta)(\alpha)$, and $a_0 \neq 0$. Say $a = \Theta \circ a'$ with $a' \in \alpha$. Then $a'_0 \neq 0$. Hence there are $b, c \in M \setminus \{ 0 \}$ such that
\[a'_0 = b + c \text{ and } \langle b, c, a'_1, \ldots, a'_{m-1} \rangle \in \alpha. \text{ Then since } \Theta \text{ is injective, } \Theta(b), \Theta(c) \neq a, \text{ and } (\Theta \circ \langle b, c, a'_1, \ldots, a'_{m-1} \rangle) \in (\Delta \Theta)(\alpha). \]

Thus \((\Delta \Theta)(\alpha) \in \Delta N.\)

We check that \(\Delta \Theta\) preserves the operation:

\[
(\Delta \Theta)(\alpha + \beta) = \{\Theta \circ a : a \in \alpha + \beta\}
= \{\Theta \circ a : \exists b, c[b \in \alpha \text{ and } c \in \beta \text{ and } a = b + c]\}
= \{\Theta \circ (b + c) : b \in \alpha, c \in \beta\}
= \{\Theta \circ b : b \in \alpha\} + \{\Theta \circ c : c \in \beta\}
= (\Delta \Theta)(\alpha) + (\Delta \Theta)(\beta).
\]

Finally, if \(\alpha, \beta \in \Delta M\) and \(\alpha \neq \beta\), say \(a \in \alpha \setminus \beta\). Then \(\Theta \circ a \in (\Delta \Theta)(\alpha) \setminus (\Delta \Theta)(\beta)\).

\[\square\]

**Proposition 25.13.** If \(\Theta : M \to N\) is an isomorphism, then \(\Delta \Theta : \Delta M \to \Delta N\) is an isomorphism.

**Proof.** For any \(\alpha \in \Delta M\),

\[
(\Delta \Theta^{-1})(\Delta \Theta)(\alpha) = \{\Theta^{-1} \circ a : a \in (\Delta \Theta)(\alpha)\}
= \{\Theta^{-1} \circ a : \exists b \in \alpha[a = \Theta \circ b]\}
= \alpha.
\]

Thus \((\Delta \Theta^{-1}) \circ (\Delta \Theta)\) is the identity. Similarly, \((\Delta \Theta) \circ (\Delta \Theta^{-1})\) is the identity. \[\square\]

**Proposition 25.14.** If \(\Theta : M \to N\) and \(T : N \to P\) are morphisms of \(m\)-monoids, then \(\Delta(T \circ \Theta) = (\Delta T) \circ (\Delta \Theta)\).

**Proof.** Assume that \(\Theta : M \to N\) and \(T : N \to P\) are morphisms of \(m\)-monoids. Let \(\alpha \in \Delta M\). Then

\[
(\Delta(T \circ \Theta))(\alpha) = \{T \circ \Theta \circ a : a \in \alpha\}
= \{T \circ b : \exists a \in \alpha[b = \Theta \circ a]\}
= \{T \circ b : b \in (\Delta \Theta)(\alpha)\}
= (\Delta T)((\Delta \Theta)(\alpha)).
\]

\[\square\]

**Proposition 25.15.** \((\Delta(\Psi \circ \Omega)) \upharpoonright N^1\) is an injective morphism from \(N^1\) into \(L^1\).

**Proof.** By Proposition 25.8, \(\Omega\) is a morphism from \(N^0\) to \(N\), and by the proof of Proposition 9.6, \(\Psi\) is a morphism from \(N\) to \(\triangle W\); see the definition of \(\Psi\) following Proposition 25.1. Clearly both \(\Omega\) and \(\Psi\) are injective, so by Proposition 25.12, \((\Delta(\Psi \circ \Omega)) \upharpoonright N^1\) is injective. Now \(\Psi \circ \Omega\) is a morphism from \(N^0\) to \(\triangle W\). Hence by 25.13, \((\Delta(\Psi \circ \Omega))\) is a morphism from \(\Delta N^0\) to \(\triangle(\Delta W)\). By Proposition 25.11, \(N^1\) is a submonoid of \(\Delta N^0\). By Proposition 24.4, \(L^1\) is a submonoid of \(\Delta(\Delta W)\).
Now let $\alpha \in N^1$; we want to show that $(\triangle(\Psi \circ \Omega))(\alpha) \in L^1$. Thus we want to show that $(\triangle(\Psi \circ \Omega))(\alpha)$ has the strict 1-hierarchy property. Suppose that $\langle a \rangle \triangle b \in (\triangle(\Psi \circ \Omega))(\alpha)$ and $e$, with domain $n$, is in $a$. We want to find $c$ with domain $n$ such that $\sum_{i<n} c_i = a$, $c \triangle b \in (\triangle(\Psi \circ \Omega))(\alpha)$ and $\forall i < n[T(c_i) = e_i]$. By the definition of $(\triangle(\Psi \circ \Omega))(\alpha)$, there exists $\langle a' \rangle \triangle b' \in \alpha$ such that $\langle a \rangle \triangle b = \Psi \circ \Omega \circ (\langle a' \rangle \triangle b')$. Thus $a = \Psi(\Omega(a'))$. Note that $a \subseteq \omega \omega$ and $a \in \triangle \mathcal{W}$. By C.P. for $\triangle \mathcal{W}$ we may assume that $e$ has the form

$$\langle e_0, \ldots, e_{r-1}, e_r, \ldots, e_{s-1}, e_s, \ldots, e_{n-1} \rangle,$$

where

$$e_0 \geq e_1 \geq \cdots \geq e_{r-1} \geq 1 > e_r = \cdots = e_{s-1} = 0 > e_s = \cdots = e_{n-1} = 0.$$

Now for each $i < n$ and $k \in \omega + 1$ we define

$$c_i(k) = \begin{cases} 
1 & \text{if } 1 \leq i < r \text{ and } k = e_i - 1 \\
0 & \text{if } 1 \leq i < r \text{ and } k \neq e_i - 1 \\
0 & \text{if } r \leq i \leq s,
\end{cases}$$

with $c_i = 0$ if $s \leq i < n$. Let $c_0 = a' - \sum_{1 \leq i < n} c_i$, with $c_0 = \omega$ if $a' = \omega$. Then $\sum_{i<n} c_i = a'$. If $i < n$ and $k < \omega_1$, then

$$(\Omega c_i)(k) = \begin{cases} 
\omega & \text{if } k = 0, \\
c_i(k - 1) & \text{if } 0 < k \leq \omega, \\
c_i(\omega) & \text{if } k = \omega, \\
0 & \text{if } \omega < k < \omega_1
\end{cases}$$

$$(\Omega a')(k) = \begin{cases} 
\omega & \text{if } k = 0, \\
a'(k - 1) & \text{if } 0 < k \leq \omega, \\
a'(\omega) & \text{if } k = \omega, \\
0 & \text{if } \omega < k < \omega_1
\end{cases}$$

The "η" for $\Omega c_i$ is the largest $k$ such that $e_i = k$. Thus $\forall i < n[T(c_i) = e_i]$. Also, $\supp(c_i) = \{e_i - 1\}$, so by Proposition 25.9(iii), $T(\Psi(\Omega c_i)) = e_i$. Now for any $k < \omega_1$,

$$(\Omega a')(k) = \begin{cases} 
\omega & \text{if } k = 0, \\
a'(k - 1) & \text{if } 0 < k \leq \omega, \\
a'(\omega) & \text{if } k = \omega, \\
0 & \text{if } \omega < k < \omega_1.
\end{cases}$$

Since $e \in a = \Theta(\Omega(a'))$, it follows that for any $k \in \omega_1$, $|\{i < n : e_i = k\}| \leq (\Omega a')(k)$. Hence $c_0(k) \geq 0$. Thus $\Psi \circ \Omega \circ c$ is as desired.

□
26. The monoid $\mathcal{P}$

Clearly $\equiv_{ae}$ is a congruence relation on $\mathcal{N}^0$. The congruence class of an element $\theta$ of $\mathcal{N}^0$ will be denoted by $[\theta]_{ae}$. $\mathcal{P}$ is the quotient semigroup of all congruence classes. Thus $[\theta]_{ae} + [\psi]_{ae} = [\theta + \psi]_{ae}$. $\Pi$ is the homomorphism of $\mathcal{N}^0 \rightarrow \mathcal{P}$ such that $\Pi(\theta) = [\theta]_{ae}$ for all $\theta \in \mathcal{N}^0$.

**Proposition 26.1.** $\mathcal{P}$ is an $m$-monoid.

**Proof.** It remains only to show that if $[\theta]_{ae} + [\psi]_{ae} = [0^c]_{ae}$, then $[\theta]_{ae} = [0^c]_{ae} = [\psi]_{ae}$. Suppose that $[\theta]_{ae} + [\psi]_{ae} = [0^c]_{ae}$. Then there is a finite set $F$ such that $\forall x \notin F[\theta(x) + \psi(x) = 0]$, hence $\forall x \notin F[\theta(x) = 0]$, so $[\theta]_{ae} = [0^c]_{ae}$. Similarly $[\psi]_{ae} = [0^c]_{ae}$. □

**Proposition 26.2.** $\forall \theta \in \mathcal{N}^0([\theta]_{ae}$ is countable). □

**Proposition 26.3.** The natural ordering of $\mathcal{P}$ is antisymmetric.

**Proof.** Clearly $[\theta]_{ae} \leq [0^c]$ implies that $[\theta]_{ae} = [0^c]_{ae}$. So $\forall [0^c] \leq [\theta]_{ae} \leq [0^c] \rightarrow [0^c]_{ae} = [\theta]_{ae}$.

26.3(1) $\forall \theta([\omega^c]_{ae} \leq [\theta]_{ae} \rightarrow [\omega^c]_{ae} = [\theta]_{ae}$.

In fact, suppose that $[\omega^c]_{ae} \leq [\theta]_{ae}$. Choose $\psi$ so that $[\omega^c]_{ae} + [\psi]_{ae} = [\theta]_{ae}$. Then there is a finite set $F$ such that $\forall x \notin F[\omega + \psi(x) = \theta(x)]$, hence $\forall x \notin F[\omega = \theta(x)]$. Thus $[\omega^c]_{ae} = [\theta]_{ae}$.

By 26.3(1) we have $\forall \theta([\omega^c]_{ae} \leq [\theta]_{ae} \leq [\omega^c]_{ae} \rightarrow [\theta]_{ae} = [\omega^c]_{ae}$.

Hence to prove the proposition it suffices to take $\theta, \psi$ each satisfying (AE3) and show that $[\theta]_{ae} \leq [\psi]_{ae} \leq [\theta]_{ae}$ implies that $[\theta]_{ae} = [\psi]_{ae}$. So, suppose that $\theta, \psi$ each satisfy (AE3) and $[\theta]_{ae} \leq [\psi]_{ae} \leq [\theta]_{ae}$. Say $[\theta]_{ae} + [\mu]_{ae} = [\psi]_{ae}$ and $[\psi]_{ae} + [\nu]_{ae} = [\theta]_{ae}$. Then there is a finite set $F$ such that $\forall x \notin F[\theta(x) \neq \omega \neq \psi(x)$ and $\theta(x) + \mu(x) = \psi(x)$ and $\psi(x) + \nu(x) = \theta(x)]$. So $\forall x \notin F[\omega \neq \theta(x) = \theta(x) + \mu(x) + \nu(x)]$, hence $\forall x \notin F[\mu(x) = \nu(x) = 0]$, so $\forall x \notin F[\theta(x) = \psi(x)]$.

**Proposition 26.4.** $\mathcal{P}\{[\omega^c]_{ae}\}$ is a submonoid of $\mathcal{P}$. □

**Proposition 26.5.** $\mathcal{P}\{[\omega^c]_{ae}\}$ is isomorphic to a submonoid of the additive monoid of a vector space over $\mathbb{Q}$.

**Proof.** Define $f \sim g$ iff $f, g \in \omega \mathbb{Q}$ and $\{n \in \omega : f(n) \neq g(n)\}$ is finite. Clearly $\sim$ is an equivalence relation on $\omega \mathbb{Q}$. Note that $\omega \mathbb{Q}$ can be considered to be a vector space over $\mathbb{Q}$. The following two statements are clear.

26.5(1) If $f, g, f', g' \in \omega \mathbb{Q}$, $f \sim f'$, and $g \sim g'$, then $f + g \sim f' + g'$.

26.5(2) If $f, f' \in \omega \mathbb{Q}$, $q \in \mathbb{Q}$, and $f \sim f'$, then $qf \sim qf'$.

Now let $V$ be the collection of all $\sim$-classes, with $[f]_\sim + [g]_\sim = [f + g]_\sim$ and $q[f]_\sim = [qf]_\sim$; so $V$ is a vector space over $\mathbb{Q}$.

120
For any \( x \in \mathcal{P} \setminus \{[\omega]^ae\} \) choose \( \theta \in \mathcal{N}^0 \) such that \( \forall k \in \omega[\theta(k) < \omega] \) and \( x = [\theta]^ae \), and set \( f(x) = [\theta \restriction \omega] \). Clearly \( f \) is well-defined and one-one and is an isomorphism of \( \mathcal{P} \setminus \{[\omega]^ae\} \) into \( V \). 

**Proposition 26.6.** \( \mathcal{P} \) is an \( r \)-monoid.

**Proof.** We apply Proposition 19.7. Suppose that \( \theta, \psi \in 2^\mathcal{N}^0 \) and \([\theta]^ae + [\theta]^ae = [\psi]^ae + [\psi]^ae\). Thus \([\theta + \theta]^ae = [\psi + \psi]^ae\). Let \( F \) be a finite subset of \( \omega + 1 \) such that \( \forall k \in (\omega + 1) \setminus F[\theta(k) + \theta(k) = \psi(k) + \psi(k)] \). Define \( \theta'_0, \theta'_1, \psi'_0, \psi'_1 \) by:

\[
\begin{align*}
\theta'_0(k) &= \begin{cases} 
\theta(0) & \text{if } k \notin F, \\
0 & \text{otherwise};
\end{cases} \\
\theta'_1(k) &= \begin{cases} 
\theta(1) & \text{if } k \notin F, \\
0 & \text{otherwise};
\end{cases} \\
\psi'_0(k) &= \begin{cases} 
\psi(0) & \text{if } k \notin F, \\
0 & \text{otherwise};
\end{cases} \\
\psi'_1(k) &= \begin{cases} 
\psi(1) & \text{if } k \notin F, \\
0 & \text{otherwise}. 
\end{cases}
\end{align*}
\]

Then \( \theta'_0 + \theta'_1 = \psi'_0 + \psi'_1 \), so by Proposition 25.3 there is a \( c \in 2^{\times 2} \mathcal{N}^0 \) such that \( \forall i < 2[\theta' = \sum_{j < 2} c_{ij}] \) and \( \forall j < 2[\psi' = \sum_{i < 2} c_{ij}] \). Then \( \forall i < 2[\theta'^ae = \sum_{j < 2} c_{ij}] \) and \( \forall j < 2[\psi'^ae = \sum_{i < 2} c_{ij}] \).

**Proposition 26.7.** If \( a \in \mathcal{P} \setminus \{[\omega]^ae\} \), then \( \mathcal{P} \upharpoonright a \) is uncountable.

**Proof.** We may assume that \( a = [\theta]^ae \), where \( \theta \) satisfies (135) and \( \forall k \in \omega[0 < \theta(k) < \omega] \). Define \( \varepsilon \equiv \delta \) iff \( \varepsilon, \delta \in \omega \) and \( \{ i \in \omega : \varepsilon(i) \neq \delta(i) \} \) is finite. This is an equivalence relation, and each equivalence class is countable. Let \( X \) consist of one element from each equivalence class. So \( |X| = 2^\omega \). Let \( \langle k_n : n \in \omega \rangle \) be strictly increasing such that \( \langle \theta(k_n); n \in \omega \rangle \) is also stictly increasing. For each \( \varepsilon \in X \) define \( \psi_\varepsilon \) with domain \( \omega + 1 \) by

\[
\psi_\varepsilon(m) = \begin{cases}
\frac{\theta(k_n)}{2} - \varepsilon(m) & \text{if } m = k_n \text{ for some } n \in \omega, \\
|\theta(m) - \varepsilon(m)| & \text{if } m \in \omega \setminus \text{rng}(k), \\
\omega & \text{if } m = \omega.
\end{cases}
\]

Then \( \lim_{m \to \infty} \psi_\varepsilon(m) = \omega \). For, given \( M > 0 \) choose \( N \) so that \( \forall m \geq N[\theta(m) \geq 2M + 2] \). Then if \( m \geq N \) and \( m = k_n \) for some \( n \), then \( \psi_\varepsilon(m) \geq \frac{\theta(k_n)}{2} - 1 \geq M \), while if \( m \notin \text{rng}(k) \) then \( \psi_\varepsilon(m) = \theta(m) - 1 \geq 2M + 1 \).

Next, note that for any \( n \in \omega \), \( \theta(k_n) \geq \left\lceil \frac{\theta(k_n)}{2} \right\rceil \). Now define \( \chi_\varepsilon \) with domain \( \omega + 1 \) by:

\[
\chi_\varepsilon(m) = \begin{cases}
\theta(k_n) - \frac{\theta(k_n)}{2} & \text{if } m = k_n \text{ for some } n \in \omega, \\
\theta(m) + \varepsilon(m) & \text{if } m \in \omega \setminus \text{rng}(k), \\
\omega & \text{if } m = \omega.
\end{cases}
\]

Clearly \( \lim_{m \to \infty} \chi_\varepsilon(m) = \omega \). It follows that \( [\psi_\varepsilon]^ae \leq [\theta]^ae \).

Clearly \([\psi_\varepsilon]^ae \neq [\psi_\delta]^ae \) for \( \varepsilon \neq \delta \).

**Proposition 26.8.** If \( K \) is a countable subset of \( \mathcal{P} \), and \( a \in \mathcal{P} \), then there exist \( a_0, a_1 \in \mathcal{P} \) such that \( a = a_0 + a_1 \) and \( a_0 \neq [\omega]^ae \) is not in the vector space span of \( K \setminus \{[\omega]^ae\} \).
Proof. If \( a = [\omega^c]_{ae} \), then let \( a_0 \) be any member of \( \mathcal{P}\{[\omega^c]_{ae}\} \) not in the span of \( K\{[\omega^c]_{ae}\} \), using Proposition 26.7; and let \( a_1 = [\omega^c]_{ae} \). If \( a \neq [\omega^c]_{ae} \), then apply Proposition 26.7 to get the desired \( a_0, a_1 \).

\[ \square \]

27. The monoid \( \mathcal{P}^1 \)

Let \( \mathcal{P}^1 = \triangle \mathcal{P} \).

**Proposition 27.1.** Suppose that \( m, n \in \omega \setminus \{0\} \), \( \theta \in m^\aleph_0 \), \( \chi \in n^{\aleph_0} \), \( \psi \in m \times n \mathcal{P} \), \( \forall i < m[\Pi(\theta_i) = \sum_{j<n} \psi_{ij}] \), and \( \forall j < n[\Pi(\chi_j) = \sum_{i<m} \psi_{ij}] \). Then there is a \( \rho \in m \times n \mathcal{P}^\aleph_0 \) such that \( \forall i < m \forall j < n[\Pi(\rho_{ij}) = \psi_{ij}] \), \( \forall i < m[\theta_i = \sum_{j<n} \rho_{ij}] \), and \( \forall j < n[\chi_j = \sum_{i<m} \rho_{ij}] \).

**Proof.** Assume the hypotheses. Since \( \mathcal{P} \) is surjective, there is a \( \psi' \in m \times n \mathcal{P}^\aleph_0 \) such that \( \forall i < m \forall j < n[\Pi(\psi'_{ij}) = \psi_{ij}] \). Thus \( \forall i < m[\Pi(\theta_i) = \sum_{j<n} \Pi(\psi'_{ij})] \), and \( \forall j < n[\Pi(\chi_j) = \sum_{i<m} \Pi(\psi'_{ij})] \). Hence there is a finite subset \( F \) of \( \omega + 1 \) such that \( \forall k \in (\omega + 1) \setminus F[\forall i < m[\theta_i(k) = \sum_{j<n} \psi'_{ij}(k)] \) and \( \forall j < n[\chi_j(k) = \sum_{i<m} \psi'_{ij}(k)] \). Now for each \( k \in F \) let \( \sigma^k \in m \times n(\omega + 1) \) be such that \( \forall i < m[\theta_i(k) = \sum_{j<n} \sigma^k_{ij}] \) and \( \forall j < n[\chi_j(k) = \sum_{i<m} \sigma^k_{ij}] \). Now for any \( k \in \omega + 1 \) and \( i < m \), \( j < n \) define

\[ \rho_{ij}(k) = \begin{cases} \psi'_{ij} & \text{if } k \notin F, \\ \sigma^k_{ij} & \text{if } k \in F. \end{cases} \]

Clearly the desired conclusion holds.

\[ \square \]

For any \( \alpha \in \mathcal{N}^1 \) let

\[ \Pi_1(\alpha) = \{ \langle \Pi(\theta_i) : i < n \rangle : \theta \in \alpha \}. \]

**Proposition 27.2.** \( \Pi_1 \) is a monoid morphism from \( \mathcal{N}^1 \) to \( \mathcal{P}^1 \).

**Proof.** Let \( \alpha \in \mathcal{N}^1 \). Thus by definition, \( \alpha \in \Delta \mathcal{N}^0 \). If \( \theta \in \alpha \), say \( \theta \) has domain \( m \). Then \( \Pi \circ \theta \in m^\aleph_0 \). Hence \( \Pi_1(\alpha) \in [\langle \mathcal{P} \rangle]^\omega \).

C.P.: Suppose that \( a, b \in \mathcal{P} \), \( a \in \Pi_1(\alpha) \), and \( a < b \). Say \( a = \Pi \circ \theta \) with \( m \in \omega \setminus \{0\} \) and \( \theta \in \alpha \), \( \theta \) with domain \( m \). Also say \( b \) has domain \( n \), \( \lambda : m \rightarrow n \), and \( \forall j < n[b_j = \sum \{ \alpha_i : i < m, \lambda(i) = j \}] \). Say \( b = \Pi \circ \psi \). Then \( \forall j < n[\Pi(\psi_j) = \sum \{ \Pi(\theta(i)) : i < m, \lambda(i) = j \}] \). So \( \forall j < n[\psi_j \equiv_{ae} \sum \{ \theta(i) : i < m, \lambda(i) = j \}] \). Hence there is a \( \psi' \) with \( \forall j < n[\psi_j \equiv_{ae} \psi'_j] \) and \( \theta < \psi' \). So \( \psi' \in \alpha \) and hence \( b = \Pi \circ \psi = \Pi \circ \psi' \in \Pi_1(\alpha) \).

R.P.: Suppose that \( m, n \in \omega \setminus \{0\} \), \( a \in m \mathcal{P} \), \( b \in n \mathcal{P} \), and \( a, b \in \Pi_1(\alpha) \). Choose \( \theta, \psi \in \alpha \) such that \( a = \Pi \circ \theta \) and \( b = \Pi \circ \psi \). Then choose \( c \in m \times n \mathcal{N}^0 \) so that \( \forall i < m[\theta_i = \sum_{j<n} c_{ij}] \) and \( \forall j < n[\psi_j = \sum_{i<m} c_{ij}] \). Then \( \forall i < m[\Pi(\theta_i) = \sum_{j<n} \Pi(c_{ij})] \), and \( \forall j < n[\Pi(\psi_j) = \sum_{i<m} \Pi(c_{ij})] \). S.P.: Suppose that \( m \in \omega \setminus \{0\} \), \( a \in m \mathcal{P} \), \( a \in \Pi_1(\alpha) \), \( a_0 \neq a \). Say \( a = \Pi(\theta) \) with \( \theta \in \alpha \). Thus \( \theta_0 \neq a_0 \). By (141) there are \( c, d \) such that \( \theta_0 = c+d \), \( c, d \not\equiv_{ae} 0 \), and \( c \sim d \sim (\theta_1, \ldots) \in \alpha \). Then \( a_0 = \Pi(c) + \Pi(d) \), \( \Pi(c), \Pi(d) \not\equiv a_0 \), and \( \Pi(c) \sim \Pi(d) \sim (a_1, \ldots) \in \Pi(\alpha) \).

This checks that \( \Pi_1(\alpha) \in \mathcal{P}^1 \). To check that it preserves the monoid operation, suppose that \( \alpha, \beta \in \mathcal{N}^1 \). Then

\[ \Pi_1(\alpha + \beta) = \{ \Pi \circ \theta : \theta \in \alpha + \beta \} \]

122
= \{ \Pi \circ (\theta + \psi) : \theta, \psi \text{ have the same domain and } \theta \in \alpha, \psi \in \beta \}
= \{ \mu + \rho : \mu, \rho \text{ have the same domain and } \mu \in \Pi_1(\alpha), \rho \in \Pi_1(\beta) \}
= \Pi_1(\alpha) + \Pi_1(\beta).

\[ \square \]

**Proposition 27.3.** For any \( \alpha \in \mathcal{P}^1 \) and any \( \chi \in \aleph^0 \), if \( T(\alpha) = \Pi(\chi) \), then there is a unique \( \beta \in \aleph^1 \) such that \( \Pi_1(\beta) = \alpha \) and \( T(\beta) = \chi \).

**Proof.** Assume that \( \alpha \in \mathcal{P}^1 \), \( \chi \in \aleph^0 \), and \( T(\alpha) = \Pi(\chi) \). Let

\[
\beta = \left\{ \theta \in \aleph^0 : n \in \omega \setminus \{0\}, \Pi \circ \theta \in \alpha, \sum_{i<n} \theta_i = \chi \right\}.
\]

We claim that \( \beta \in \aleph^1 \). First we show that \( \beta \in \Delta \aleph^0 \). By the definition of \( \mathcal{P}^1 \), \( \alpha \) is countable, and so by Proposition 26.2, \( \beta \) is countable.

C.P.: Suppose that \( a, b \in \aleph^0 \), \( a \in \beta \), and \( a < b \). Say \( a \) has domain \( m \), \( b \) has domain \( n \), \( \lambda : m \to n \), and \( \forall j < n [b_j = \sum\{a_i : i < m, \lambda(i) = j\}] \). Since \( a \in \beta \), we have \( \Pi \circ a \in \alpha \). Now \( \forall j < n [\Pi(b_j) = \sum\{\Pi(a_i) : i < m, \lambda(i) = j\}] \). Hence \( \Pi \circ a < \Pi \circ b \), so \( \Pi \circ b \in \alpha \). Also, \( \sum_{j<n} b_j = \sum_{i<n} a_i = \chi \). So \( b \in \beta \).

R.P.: Suppose that \( m, n \in \omega \setminus \{0\}, a \in m^\aleph^0 \), \( b \in n^\aleph^0 \), and \( a, b \in \beta \). So \( \Pi \circ a, \Pi \circ b \in \alpha \). By R.P. for \( \alpha \), there is a \( c \in m \times n \mathcal{P} \) such that \( \forall i < n [\Pi(c_i) = \sum\{\Pi(b_j) = \sum\{c_{f(i,j)} : j < m\}] \) and \( \forall j < n [\Pi(b_j) = \sum\{c_{f(i,j)} : j < m\}] \). Let \( c(i,j) = c_{f(i,j)} \) for all \( i < m \) and \( j < n \). By Proposition 27.1 there is a \( \rho \in m \times n \aleph^0 \) such that \( \forall j < n [\Pi(\rho) = c(i,j)] \). \( \forall i < m [a_i = \sum_{j<n} \rho_{ij}] \), and \( \forall j < n [b_j = \sum_{i<n} \rho_{ij}] \). Let \( \rho(i,j) = \rho_{ij} \) for all \( i < m \) and \( j < n \). This proves R.P. for \( \beta \).

S.P.: Suppose that \( m \in \omega \setminus \{0\}, a \) has domain \( m \), \( a \in \beta \), and \( a_0 \neq a \). Then also \( \Pi(a_0) \neq a \). By S.P. for \( \alpha \), choose \( b, c \in \mathcal{P} \) such that \( \Pi(a_0) = b + c \) and \( \langle b, c, \Pi(a_1), \ldots, \Pi(a_{m-1}) \rangle \in \alpha \). Say \( b = \Pi(b') \) and \( c = \Pi(c') \). Then \( \Pi \circ \{b', c', a_1, \ldots, a_{m-1} \} \in \alpha \). Hence \( \langle b', c', a_1, \ldots, a_{m-1} \rangle \in \beta \).

This proves that \( \beta \in \Delta \aleph^0 \). Now for \( \beta \in \aleph^1 \), for (AE8), suppose that \( n \in \omega \), \( \text{dmn}(a) = \text{dmn}(b) = n \), \( a \equiv a e b \in \beta \), and \( \sum_{i<n} a_i = \sum_{i<n} b_i \). Then \( \Pi \circ a = \Pi \circ b \in \alpha \) and \( \sum_{i<n} a_i = \sum_{i<n} b_i = \chi \). So \( a \in \beta \).

For (AE9), suppose that \( a \equiv a \neq 0^c \). Then \( \Pi \circ (a \setminus b) \in \alpha \) and \( \sum_{i<m} a_i + \sum_{j<n} b_j = \chi \). Then \( \Pi \circ a \neq 0 \). Wlog \( \Pi(a_0) \neq 0 \). By S.P. for \( \alpha \) choose \( c, d \neq a \) so that \( \Pi(a_0) = c + d \) and \( \langle c, d, \Pi(a_1), \ldots, \Pi(b_{m-1}) \rangle \in \alpha \). Say \( c = \Pi(c') \) and \( d = \Pi(d') \). Then \( \Pi \circ \{c', d', a_1, \ldots, b_{m-1} \} \in \alpha \). We may assume that \( c' + d' = a_0 \). In fact, let \( F = \{i \in \omega + 1 : c(i) + d'(i) \neq a_0(i) \} \); so \( F \) is finite. let

\[
c''(i) = \begin{cases} c'(i) & \text{if } i \notin F, \\
a_0(i) & \text{if } i \in F, \end{cases}
\quad d''(i) = \begin{cases} d'(i) & \text{if } i \notin F, \\
0 & \text{if } i \in F. \end{cases}
\]

Then \( c'' + d'' = a_0 \) and \( \Pi(c'') = \Pi(c'), \Pi(d'') = \Pi(d') \). This proves (141).

So \( \beta \in \aleph^1 \). Clearly \( \Pi_1(\beta) \subseteq \alpha \). If \( c \in \alpha \), say \( c \in n \mathcal{P} \). Then there is a \( b \in n \aleph^0 \) such that \( c = \Pi \circ b \). Then \( [\chi] = \Pi(\chi) = T(\alpha) = T(c) = [\sum_{i<n} b_i] \). Let \( F \) be a finite subset of
\[ \omega + 1 \] such that \( \chi(m) = \sum_{i<n} b_i(m) \) for all \( m \notin F \). Then we define

\[
b'_i(m) = \begin{cases} b_i(m) & \text{if } m \notin F, \\ \chi(m) & \text{if } m \in F \text{ and } i = 0, \\ 0 & \text{if } m \notin F \text{ and } i \neq 0. \end{cases}
\]

Then \( c = \Pi \circ b' \) and \( \sum_{i<n} b'_i = \chi \). Hence \( c \in \Pi_1(\beta) \). So \( \Pi_1(\beta) = \alpha \). Clearly \( T(\beta) = \chi \).

For uniqueness, suppose that \( \Pi_1(\gamma) = \alpha \) and \( T(\gamma) = \chi \). Take any \( \theta \in \gamma \); say \( \theta \) has domain \( n \). Then \( \Pi \circ \theta \in \Pi_1(\gamma) \), so \( \Pi \circ \theta \in \Pi_1(\beta) \). Choose \( \psi \in \beta \) so that \( \Pi \circ \theta = \Pi \circ \psi \).

Now \( T(\gamma) = \sum_{i<n} \theta_i = \sum_{i<n} \psi_i \). Hence by (140), \( \theta \in \beta \). By symmetry, \( \gamma = \beta \).

Let \( \mathcal{Q} = \{ \alpha \in \mathcal{P}^1 : T(\alpha) = \omega^c \} \).

**Proposition 27.4.** \( \mathcal{Q} \) is a subsemigroup of \( \mathcal{P}^1 \).

**Proposition 27.5.** There is a unique morphism \( \Gamma_1 : \mathcal{Q} \to \mathcal{N}^1 \) such that \( \forall \alpha \in \mathcal{Q} [\Pi_1(\Gamma_1(\alpha)) = \alpha] \) and \( T(\Gamma_1(\alpha)) = \omega^c \).

**Proof.** Apply Proposition 27.3 with \( \chi = \omega^c \); for \( \alpha \in \mathcal{Q} \) let \( \Gamma_1(\alpha) \) be the \( \beta \) of 27.3.

**Proposition 27.6.** \( \Pi_1 \) maps \( \mathcal{N}^1 \) onto \( \mathcal{P}^1 \). In fact, if \( \alpha \in \mathcal{P}^1 \) and \( T(\alpha) = \Pi(d) \) then there is a \( \beta \in \mathcal{N}^1 \) such that \( \Pi_1(\beta) = \alpha \) and \( T(\beta) = d \).

**Proof.** By Proposition 27.3.

**28. One step up the hierarchy**

**Proposition 28.1.** The following diagram is commutative:

\[
\begin{array}{ccc}
\Delta^2 \mathcal{W} & \xrightarrow{\Delta(\Psi \circ \Omega)} & \mathcal{N}^1 \\
\downarrow T & & \downarrow T \\
\Delta \mathcal{W} & \xrightarrow{\Psi \circ \Omega} & \mathcal{N}^0 \\
\end{array}
\]

\[
\begin{array}{ccc}
& \xrightarrow{\Pi_1} & \mathcal{P}^1 \\
\Pi & & \downarrow T \\
\mathcal{N}^0 & \xrightarrow{\Pi} & \mathcal{P} \\
\end{array}
\]

**Proof.** By Proposition 25.15, \( \Delta(\Psi \circ \Omega) : \mathcal{N}^1 \to \mathcal{P}^1 \). By Proposition 27.2, \( \Pi_1 : \mathcal{N}^1 \to \mathcal{P}^1 \). By definition, \( \Pi : \mathcal{N}^0 \to \mathcal{P} \). By Proposition 25.8, \( \Omega : \mathcal{N}^0 \to \mathcal{N} \), and by Proposition 9.6, \( \Psi : \mathcal{N} \to \Delta \mathcal{W} \). Hence \( \Psi \circ \Omega : \mathcal{N}^0 \to \Delta \mathcal{W} \). Clearly \( T : \Delta^2 \mathcal{W} \to \mathcal{W} \). By Proposition 25.11, \( T : \mathcal{N}^1 \to \mathcal{N}^0 \). Clearly \( T : \mathcal{P}^1 \to \mathcal{P}^0 \).
To show that $\Psi \circ \Omega \circ T = T \circ \triangle(\Psi \circ \Omega)$, suppose that $\alpha \in N^1$. Then $\alpha \in \triangle N^0$, so $T(\alpha) = T(a)$ for any $a \in \alpha$; say $a$ has domain $n$. Then $\Psi(\Omega(T(\alpha))) = \Psi(\Omega(T(a))) = \sum_{i<n} \Psi(\Omega(a_i))$. On the other hand,

$$(T\triangle(\Psi\circ\Omega))(\alpha) = T((\triangle(\Psi\circ\Omega))(\alpha)) = T(\{\Psi\circ\Omega\circ a : a \in \alpha\} = T(\Psi\circ\Omega\circ a) = \sum_{i<n} \Psi(\Omega(a_i)),$$

as desired.

To show that $\Pi \circ T = T \circ \Pi_1$, suppose that $\alpha \in \mathcal{S}^1 = \triangle \mathcal{S}$. Then $T(\alpha) = T(a)$ for any $a \in \alpha$, so $\Pi(T(\alpha)) = \Pi(T(a))$. On the other hand,

$$T(\Pi_1(\alpha)) = T(\{\{\Pi(a_i) : i < n\} : a \in \alpha, \text{dmm}(a) = n\} = \sum_{i<n} \Pi(a_i) \text{ for any } a \in \alpha = \Pi(T(a)).$$

Let $M$ be an $m$-monoid and $A \in \triangle^2 M$. Then $A$ satisfies the new hierarchy property iff

\begin{itemize}
  \item[(N1)] For all $m, n \in \omega \setminus \{0\}$, all $\langle \alpha, \beta_1, \ldots, \beta_{m-1} \rangle \in A$ and all $\langle a_0, \ldots, a_{n-1} \rangle \in \alpha$ there exist $\alpha_i \in \Delta M$ for $i < n$ such that $\langle \alpha_0, \ldots, \alpha_{n-1}, \beta_1, \ldots, \beta_{m-1} \rangle \in A$, $\sum_{i<n} \alpha_i = \alpha$, and $\forall i < n[T(\alpha_i) = a_i]$. \n
  \item[(N2)] $A$ satisfies the new hierarchy property. \n
  \item[(N3)] If $\forall n \in \omega \setminus \{0\} \forall \alpha, \beta[\alpha$ and $\beta$ have domain $n, \alpha \in A,$ and $\forall i < n[\Pi_1(\beta_i) = \Pi_1(\alpha_i)]$, then $\beta \in A$. \n
  \item[(N4)] $\forall n \in \omega \forall \langle \alpha, \beta_1, \ldots, \beta_{n-1} \rangle \in A[\Pi_1(\alpha) \neq 0 \rightarrow \exists \gamma_0, \gamma_1[\alpha = \gamma_0 + \gamma_1 \in N^1, \Pi_1(\gamma_0), \Pi_1(\gamma_1) \neq 0, \text{and } \langle \gamma_0, \gamma_1, \beta_1, \ldots, \beta_{n-1} \rangle \in A]]$. \n
\end{itemize}

$\mathcal{N}^2$ is the set of all $A \in \triangle \mathcal{S}^1$ such that:

\begin{itemize}
  \item[(N2)] $A$ satisfies the new hierarchy property. \n
  \item[(N3)] If $\forall n \in \omega \setminus \{0\} \forall \alpha, \beta[\alpha$ and $\beta$ have domain $n, \alpha \in A,$ and $\forall i < n[\Pi_1(\beta_i) = \Pi_1(\alpha_i)]$, then $\beta \in A$. \n
  \item[(N4)] $\forall n \in \omega \forall \langle \alpha, \beta_1, \ldots, \beta_{n-1} \rangle \in A[\Pi_1(\alpha) \neq 0 \rightarrow \exists \gamma_0, \gamma_1[\alpha = \gamma_0 + \gamma_1 \in N^1, \Pi_1(\gamma_0), \Pi_1(\gamma_1) \neq 0, \text{and } \langle \gamma_0, \gamma_1, \beta_1, \ldots, \beta_{n-1} \rangle \in A]]$. \n
\end{itemize}

$\mathcal{S}^2$ is the set of all $A \in \triangle \mathcal{S}^1$ which satisfy the new hierarchy property.

**Proposition 28.2.** If $A$ and $B$ satisfy the new hierarchy property, then so does $A + B$.

**Proof.** Suppose that $A$ and $B$ satisfy the new hierarchy property. Suppose that $m, n \in \omega \setminus \{0\}$, $\langle \alpha, \beta_1, \ldots, \beta_{m-1} \rangle \in A + B$, and $\langle a_0, a_1, \ldots, a_{n-1} \rangle \in \alpha$. Now $A, B \in \triangle^2 M$, so there are $\langle \alpha^0, \beta_1^0, \ldots, \beta_{m-1}^0 \rangle \in A$, and $\langle \alpha^1, \beta_1^1, \ldots, \beta_{m-1}^1 \rangle \in B$ such that $\langle \alpha, \beta_1, \ldots, \beta_{m-1} \rangle = \langle \alpha^0, \beta_1^0, \ldots, \beta_{m-1}^0 \rangle + \langle \alpha^1, \beta_1^1, \ldots, \beta_{m-1}^1 \rangle$. Moreover,

$$\langle \alpha^0, \beta_1^0, \ldots, \beta_{m-1}^0 \rangle + \langle \alpha^1, \beta_1^1, \ldots, \beta_{m-1}^1 \rangle = \langle \alpha^0 + \alpha^1, \beta_1^0 + \beta_1^1, \ldots, \beta_{m-1}^0 + \beta_{m-1}^1 \rangle.$$
Similarly, there are $\gamma_0^1, \gamma_1^1, \ldots, \gamma_{n-1}^1 \in \Delta M$ such that
\[
\langle \gamma_0^1, \gamma_1^1, \ldots, \gamma_{n-1}^1, \beta_1^1, \ldots, \beta_{m-1}^1 \rangle \in B, \quad \sum_{i<n} \gamma_i^1 = \alpha^1, \quad \text{and } \forall i < n[T(\gamma_i^1) = a_i^1].
\]

Hence
\[
\langle \gamma_0^0 + \gamma_0^1, \ldots, \gamma_{n-1}^0 + \gamma_{n-1}^1, \beta_1, \ldots, \beta_{m-1} \rangle \in A + B,
\]
\[
\sum_{i<n} (\gamma_i^0 + \gamma_i^1) = \alpha, \quad \forall i < n[T(\gamma_i^0 + \gamma_i^1) = a_i].
\]

**Proposition 28.3.** $\mathcal{P}^2$ is a submonoid of $\triangle \mathcal{P}^1$.

**Proposition 28.4.** Assume that $m, n \in \omega \setminus \{0\}$, $\alpha$ has domain $m$, $\beta$ has domain $n$, \forall i < m[\alpha_i \in N^1]$, \forall j < n[\beta_j \in N^1], $\gamma$ has domain $m \times n$, $\forall i < m \forall j < n[\gamma_{ij} \in \mathcal{P}^1]$, \forall i < m[\Pi_1(\alpha_i) = \sum_{j<n} \gamma_{ij}]$, and \forall j < n[\Pi_1(\beta_j) = \sum_{i<m} \gamma_{ij}]. Then there is a $\delta$ with domain $m \times n$ such that $\forall i < m \forall j < n[\delta_{ij} \in N^1]$ and $\Pi_1(\delta_{ij}) = \gamma_{ij}$, $\forall i < m[\alpha_i = \sum_{j<n} \delta_{ij}]$, and \forall j < n[\beta_j = \sum_{i<m} \delta_{ij}]$.

**Proof.** Assume that $m, n \in \omega \setminus \{0\}$, $\alpha$ has domain $m$, $\beta$ has domain $n$, $\forall i < m[\alpha_i \in N^1]$. \forall j < n[\beta_j \in N^1]$, $\sum_{i<m} \Pi_1(\alpha_i) = \sum_{j<n} \Pi_1(\beta_j)$, $\gamma$ has domain $m \times n$, $\forall i < m \forall j < n[\gamma_{ij} \in \mathcal{P}^1]$, $\forall i < m[\Pi_1(\alpha_i) = \sum_{j<n} \gamma_{ij}]$, and \forall j < n[\Pi_1(\beta_j) = \sum_{i<m} \gamma_{ij}].$ Let $\theta_i = T(\alpha_i)$ for all $i < m$, $\psi_j = T(\beta_j)$ for all $j < n$, and $\rho_{ij} = T(\gamma_{ij})$ for all $i < m, j < n$. Then by Proposition 28.1, $\Pi(\theta_i) = \Pi(T(\alpha_i)) = \Pi(T(\Pi_1(\alpha_i))) = \sum_{j<n} T(\gamma_{ij}) = \sum_{j<n} \rho_{ij}$ for all $i < m$, and similarly $\Pi(\psi_j) = \sum_{i<m} \rho_{ij}$ for all $j < n$. Moreover, $\forall i < m[\theta_i \in N^0]$, $\forall j < n[\psi_j \in N^0]$, and $\forall i < m \forall j < n[\rho_{ij} \in \mathcal{P}]$. Hence by Proposition 27.1 with $m, n, \theta, \psi, \rho$ replaced by $n, m, \theta, \psi, \rho$ we get $\sigma \in m \times n N^{0^*}$ such that $\forall i < m \forall j < n[\Pi_1(\sigma_{ij}) = \rho_{ij}]$, $\forall i < m[\theta_i = \sum_{j<n} \sigma_{ij}]$, and $\forall j < n[\psi_j = \sum_{i<m} \sigma_{ij}]$. Then $T(\gamma_{ij}) = \rho_{ij} = \Pi(\sigma_{ij})$ for all $i < m, j < n$, so by Proposition 27.3 with $\alpha, \chi$ replaced by $\gamma_{ij}, \sigma_{ij}$, there is a $\delta$ with domain $m \times n$ such that $\forall i < m \forall j < n[\delta_{ij} \in N^1]$, $\Pi_1(\delta_{ij}) = \gamma_{ij}$ and $T(\delta_{ij}) = \sigma_{ij}$. Now for any $i < m$, $\Pi_1(\sum \delta_{ij} : j < n) = \sum \delta_{ij} : j < n$ and $T(\sum \delta_{ij} : j < n) = \sum \sigma_{ij} : j < n$. Also, $\Pi_1(\alpha_i) = \sum \gamma_{ij} : j < n$ and $T(\alpha_i) = \theta_i = \sum \sigma_{ij} : j < n$. Hence by the uniqueness assertion in Proposition 27.3, $\alpha_i = \sum \delta_{ij} : j < n$. Similarly, $\forall j < n[\beta_j = \sum_{i<m} \delta_{ij}]$.

**Proposition 28.5.** $N^2$ is a submonoid of $\triangle N^1$.

**Proof.** Suppose that $A, B \in N^2$. By Proposition 28.3, $A + B$ satisfies the new hierarchy property. To prove (N2) for $A + B$, suppose that $n \in \omega \setminus \{0\}$, $\alpha$ and $\beta$ have domain $n$, $\alpha \in A + B$, and $\forall i < n[\Pi_1(\beta_i) = \Pi_1(\alpha_i)]$. Say $\alpha = \gamma + \delta$ with $\gamma \in A$ and $\delta \in B$. Take any $i < n$. Then $\alpha_i, \gamma_i, \delta_i \in N^1$ and $\alpha_i = \gamma_i + \delta_i$. Hence $\Pi_1(\alpha_i) = \Pi_1(\gamma_i) + \Pi_1(\delta_i)$.

Let $e_{f(u,i)} = \gamma_{ij}$ and $e_{f(1,i)} = \delta_{ij}$.

For $i < n$ and $k < 2n$ let
\[
c_{ik} = \begin{cases} 
\Pi_1(\varepsilon_k) & \text{if } k = f(u,i) \text{ for some } u \in 2, \\
0 & \text{otherwise}.
\end{cases}
\]
Then for all \( i < n \),
\[
\Pi_1(\beta_i) = \Pi_1(\alpha_i) = \Pi_1(\gamma_i) + \Pi_1(\delta_i) = \Pi_1(\varepsilon_f(0,i)) + \Pi_1(\varepsilon_f(1,i)) = c_{i_f(0,i)} + c_{i_f(1,i)} = \sum_{k < 2n} c_{ik}
\]
and for all \( k < 2n \), with \( f(u, j) = k \),
\[
\Pi_1(\varepsilon_k) = c_{jk} = \sum_{i < n} c_{ik}.
\]
Now we apply Proposition 28.4 with \( m, n, \alpha, \beta, \gamma \) replaced by \( n, 2n, \beta, \varepsilon, c \); we get \( d \) with domain \( n \times 2n \) such that \( \forall i < n \forall k < 2n [d_{ik} \in \mathcal{N}^1 \text{ and } \Pi_1(d_{ik}) = c_{ik}] \), \( \forall i < n [\beta_i = \sum_{k < 2n} d_{ik}] \), and \( \forall k < 2n [\varepsilon_k = \sum_{i < n} d_{ik}] \). Then
\[
\sum_{i < n} \sum_{i < n} \varepsilon_f(0,i) = \sum_{i < n} \sum_{i < n} d_f(0,i) = \sum_{j < n} \sum_{i < n} d_f(0,i) = \sum_{i < n} \sum_{j < n} d_f(0,j).
\]
Further,
\[
\Pi_1(\gamma_i) = \Pi_1(\varepsilon_f(0,i)) = \Pi_1 \left( \sum_{j < n} d_f(0,j) \right) = \sum_{j < n} c_{jf(0,j)} = \Pi_1(\varepsilon_f(0,i)).
\]
Also, \( \Pi_1(\sum_{j < n} d_f(0,j)) = \sum_{j < n} c_{jf(0,j)} = \Pi_1(\varepsilon_f(0,i)) \). Now we apply (N3) with \( n, \alpha_i, \beta_i \) replaced by \( n, \sum_{j < n} d_f(0,j), \varepsilon_f(0,i) \); we get \( \langle \sum_{j < n} d_f(0,j) : i < n \rangle \in A \). Similarly,
\[
\left\langle \sum_{j < n} d_f(1,j) : i < n \right\rangle \in B.
\]
Now for each \( i < n \), \( \beta_i = \sum_{k < 2n} d_{ik} = \sum_{j < n} d_{f(0,j)} + \sum_{j < n} d_{f(1,j)} \). Hence \( \beta \in A + B \).

To prove (N4) for \( A + B \), suppose that \( n \in \omega \), \( \langle \alpha, \beta_1, \ldots, \beta_{n-1} \rangle \in A + B \), and \( \Pi_1(\alpha) \neq 0 \). Say \( \langle \alpha, \beta_1, \ldots, \beta_{n-1} \rangle = \langle \alpha^0, \beta^0_1, \ldots, \beta^0_{n-1} \rangle + \langle \alpha^1, \beta^1_1, \ldots, \beta^1_{n-1} \rangle \) with \( \alpha^0, \beta^0_1, \ldots, \beta^0_{n-1} \in A \) and \( \alpha^1, \beta^1_1, \ldots, \beta^1_{n-1} \in B \). Wlog \( \Pi_1(\alpha^0) \neq 0 \). Then there are \( \gamma_0, \gamma_1 \) such that \( \alpha^0 = \gamma_0 + \gamma_1 \), \( \Pi_1(\gamma_0), \Pi_1(\gamma_1) \neq 0 \), and \( \langle \gamma_0, \gamma_1, \beta^0_1, \ldots, \beta^0_{n-1} \rangle \in A \). Also, \( \langle 0, \alpha^1, \beta^1_1, \ldots, \beta^1_{n-1} \rangle \in B \), so \( \langle \gamma_0, \gamma_1 + \alpha^1, \beta_1, \ldots, \beta_{n-1} \rangle \in A + B \) and \( \Pi_1(\gamma_0), \Pi_1(\gamma_1 + \alpha^1) \neq 0 \).

Now we define, for any \( A \in \mathcal{N}^2 \),
\[
\Pi_2(A) = \{ \Pi_1 \circ a : a \in A \}.
\]

**Proposition 28.6.** \( \Pi_2 \) is a monoid morphism from \( \mathcal{N}^2 \) to \( \mathcal{P}^2 \).

**Proof.** By Proposition 28.1, \( \Pi_1 \) is a morphism from \( \mathcal{N}^1 \) to \( \mathcal{P}^1 \). Note that \( \Pi_2 = (\triangle \Pi_1) \). Now let \( A \in \mathcal{N}^2 \). Then by the proof of Proposition 25.12, \( \Pi_2(A) \) satisfies C.P. and R.P. For S.P., suppose that \( a \in \Pi_2(A) \) and \( a_0 \neq 0 \). Say \( a = \Pi_1 \circ b \) with \( b \in A \). By (148) there are \( \gamma_0, \gamma_1 \) such that \( b_0 = \gamma_0 + \gamma_1 \), \( \Pi_1(\gamma_0), \Pi_1(\gamma_1) \neq 0 \),
and \( \langle \gamma_0, \gamma_1, a_1, \ldots \rangle \in A \). Hence \( \Pi_1 \circ \langle \gamma_0, \gamma_1, a_1, \ldots \rangle \in \Pi_2(A) \), as desired. So we have shown that \( \Pi_2(A) \in \Delta \mathcal{P}^1 \). To check the new hierarchy property, suppose that \( m \in \omega \setminus \{0\} \), \( \langle \alpha, \beta_1, \ldots, \beta_{m-1} \rangle \in \Pi_2(A) \), and \( \langle a_0, a_1, \ldots, a_{n-1} \rangle \in \alpha \). Say \( \langle \alpha, \beta_1, \ldots, \beta_{m-1} \rangle = \Pi_1 \circ \langle \alpha', \beta'_1, \ldots, \beta'_{m-1} \rangle \) with \( \langle \alpha', \beta'_1, \ldots, \beta'_{m-1} \rangle \in A \). Then \( \langle a_0, a_1, \ldots, a_{n-1} \rangle \in \Pi_1(\alpha') \); say \( \langle a_0, a_1, \ldots, a_{n-1} \rangle = \langle \Pi(a_0'), \Pi(a_1'), \ldots, \Pi(a'_{n-1}) \rangle \) with \( \langle a_0', a_1', \ldots, a'_{n-1} \rangle \in \alpha' \). Then by the new hierarchy property for \( A \), there are \( \alpha_0', \alpha_1', \ldots, a'_{n-1} \) such that \( \alpha' = \sum_{i<n} \alpha'_i \), \( \forall i < n \lceil \tau(\alpha'_i) \rceil = a'_i \), and \( \langle \alpha_0', \alpha_1', \ldots, \alpha_{n-1}', \beta'_1, \ldots, \beta'_{m-1} \rangle \in A \). Then \( \alpha = \Pi_1(\alpha') = \sum_{i<n} \Pi_1(\alpha'_i) \). Using Proposition 28.1, for any \( i < n \), \( T(\Pi_1(\alpha'_i)) = \Pi(T(\alpha'_i)) = \Pi(a'_i) = a_i \). Finally,

\[
\Pi_1 \circ \langle \alpha_0', \alpha_1', \ldots, \alpha_{n-1}', \beta'_1, \ldots, \beta'_{m-1} \rangle \\
= \langle \Pi_1(\alpha_0'), \Pi_1(\alpha_1'), \ldots, \Pi_1(\alpha_{n-1}'), \Pi_1(\beta'_1), \ldots, \Pi_1(b'_{m-1}) \rangle \\
= \langle \Pi_1(\alpha_0'), \Pi_1(\alpha_1'), \ldots, \Pi_1(\alpha_{n-1}'), \beta_1, \ldots, \beta_{m-1} \rangle \in \Pi_2(A).
\]

Thus \( \Pi_2(A) \) has the new hierarchy property. So \( \Pi_2(A) \in \mathcal{P}^2 \).

Now \( \Pi_2 \) preserves +:

\[
\Pi_2(A + B) = \{ \Pi_1 \circ a : a \in A + B \} = \{ \Pi_1 \circ (b + c) : b \in A, c \in B, \text{dom}(a) = \text{dom}(b) \} = \{ (\Pi_1 \circ b) + (\Pi_1 \circ c) : b \in A, c \in B, \text{dom}(a) = \text{dom}(b) \} = \Pi_2(A) + \Pi_2(B).
\]

**Proposition 28.7.** The following diagram is commutative:

![Diagram](image)

**Proof.** \( \Pi_2 \) is a morphism from \( \mathcal{N}^2 \) to \( \mathcal{P}^2 \) by Proposition 28.6. \( T \) is a morphism from \( \mathcal{N}^2 \) to \( \mathcal{N}^1 \) by Proposition 8.11. \( T \) is a morphism from \( \mathcal{P}^2 \) to \( \mathcal{P}^1 \) by Proposition 8.11. The arrow at the bottom holds by Proposition 28.1. For commutativity, for any \( A \in \mathcal{N}^2 \) we have

\[
T(\Pi_2(A)) = T(\{ \Pi_1 \circ a : a \in A \}) \\
= \sum_{i<n} \Pi_1(a_i) \text{ for some } a \in A \\
= \Pi_1 \left( \sum_{i<n} a_i \right) \text{ for some } a \in A \\
= \Pi_1(T(A)).
\]
Proposition 28.8.  (2.11.5) If $A \in \mathcal{P}^2$, $\beta \in \mathcal{N}^1$, and $T(A) = \Pi_1(\beta)$, then there is a unique $B \in \mathcal{N}^2$ such that $\Pi_2(B) = A$ and $\Pi_1(T(B)) = \Pi_1(\beta)$.

Proof. Suppose that $A \in \mathcal{P}^2$, $\beta \in \mathcal{N}^1$, and $T(A) = \Pi_1(\beta)$.

28.8(1) $\forall \alpha \in \mathcal{P}^1[\{\gamma \in \mathcal{N}^1 : \Pi_1(\gamma) = \alpha\}]$ is countable.

For, suppose that $\alpha \in \mathcal{P}^1$, and let $X = \{\gamma \in \mathcal{N}^1 : \Pi_1(\gamma) = \alpha\}$. Now $\alpha \in \mathcal{P}^1 = \Delta \mathcal{P}$. Hence $T(\alpha) = T(c) \in \mathcal{P}$. So $T(\alpha)$ is countable. If $\gamma \in X$, then $T(\alpha) = T(\Pi_1(\gamma)) = \Pi(T(\gamma))$ by Proposition 28.1. Thus $T(\gamma) \in T(\alpha)$. So $X = \bigcup_{x \in T(\alpha)} \{\gamma \in X : T(\gamma) = x\}$.

We claim that for any $x \in T(\alpha)$ there is at most one $\gamma \in X$ such that $T(\gamma) = x$. For, suppose that $\gamma \in X$ and $T(\gamma) = x$ with $x \in T(\alpha)$. Then $T(\alpha) = \Pi(T(\gamma))$, so by Proposition 27.3 there is a unique $\beta \in \mathcal{N}^1$ such that $\Pi_1(\beta) = \alpha$ and $T(\beta) = T(\gamma)$. So $\beta = \gamma$. This proves 28.8(1).

The following is a consequence of 28.8(1):

28.8(1) $\forall m \in \omega \setminus \{0\} \forall \alpha \in \mathcal{P}^1[\{\gamma \in \mathcal{N}^1 : \Pi_1 \circ \gamma = \alpha\}]$ is countable.

Now let

$$B = \{b \in n(\mathcal{N}^1) : n \in \omega, \Pi_1 \circ b \in A \text{ and } \Pi_1 \left( \sum_{i < n} b_i \right) = \Pi_1(\beta)\}$$

Now $A \in \mathcal{P}^2 \subseteq \Delta \mathcal{P}^1$, so $A$ is countable. Hence by 28.8(1), also $B$ is countable. We claim that $B \in \mathcal{N}^2$. First we prove that $B \in \Delta \mathcal{N}^1$. We have $B \in [\mathcal{N}^1]^\omega$.

C.P.: Suppose that $a \in B$ and $a < b$. Say $\text{dnn}(a) = m$, $\text{dnn}(b) = n$, $\lambda : m \to n$, and $\forall j < n [b_j = \sum \{a_i : i < m, \lambda(i) = j\}]$. We have $\Pi_1 \circ a \in A$ and $\forall j < n [\Pi_1(b_j) = \sum \{\Pi_1(a_i) : i < m, \lambda(i) = j\}]$, so $\Pi_1 \circ a < \Pi_1 \circ b$. Hence $\Pi_1 \circ b \in A$. Also, $\sum_{j < n} b_j = \sum_{i < m} a_i$, so $\Pi_1(\sum_{j < n} b_j) = \Pi_1(\sum_{i < n} a_i) = \Pi_1(\beta)$. Hence $b \in B$.

R.P.: Suppose that $m, n \in \omega \setminus \{0\}$, $a \in \mathcal{P}^1(\mathcal{N}^1)$, $b \in \mathcal{N}^1$, and $a, b \in B$. By the definition of $B$, $\Pi_1(\sum_{i < m} a_i) = \Pi_1(\beta) = \Pi_1(\sum_{i < n} b_i)$. Since $a, b \in B$ we have $\Pi_1 \circ a, \Pi_1 \circ b \in A$. Now $A \in \mathcal{P}^2 \subseteq \Delta \mathcal{P}^1$. Hence by R.P. for $A$ there is a $c$ with domain $m \cdot n$ such that $c \in A$, $\forall i < m [\Pi_1(a_i) = \sum_{j < n} c(f(i, j))]$ and $\forall j < n [\Pi_1(b_j) = \sum_{i < m} c(f(i, j))]$. Define $d$ with domain $m \cdot n$ by $d_{ij} = c(f(i, j))$. Then $\forall i < m [\Pi_1(a_i) = \sum_{j < n} d_{ij}]$ and $\forall j < n [\Pi_1(b_j) = \sum_{i < m} d_{ij}]$. Hence by Proposition 28.4 there is an $e$ with domain $m \cdot n$ such that $\forall i < m [\forall j < n [\Pi_1(e_{ij}) = d_{ij}], \forall i < m [a_i = \sum_{j < n} e_{ij}], \text{ and } \forall j < n [b_j = \sum_{i < m} e_{ij}]]$. Now define $g$ with domain $m \cdot n$ by $g_{f(i, j)} = e_{ij}$. Then $\forall i < m [a_i = \sum_{j < n} g_{f(i, j)}]$ and $\forall j < n [b_j = \sum_{i < m} g_{f(i, j)}]$. Now for any $i < m$ and $j < n$, $\Pi_1(g_{f(i, j)}) = \Pi_1(e_{ij}) = d_{ij} = c(f(i, j))$. So $\Pi_1 \circ g = c \in A$. Also,

$$\Pi_1 \left( \sum_{k < m \cdot n} g_k \right) = \Pi_1 \left( \sum_{i < m} \sum_{j < n} e_{ij} \right) = \Pi_1 \left( \sum_{i < m} a_i \right) = \Pi_1(\beta).$$

It follows that $g \in B$. This proves R.P. for $B$.

S.P.: suppose that $m \in \omega \setminus \{0\}$, $a \in \mathcal{P}^1(\mathcal{N}^1)$, $a \in B$ and $a_0 \neq o$. Thus $\Pi_1 \circ a \in A$ and $\Pi_1(a_0) \neq O$. Hence by S.P. for $A$, there exist $u, v \in \mathcal{P}^1(\mathcal{N}^1)$ such that $\Pi_1(a_0) = u + v$ and
\[\left\langle u, v, \Pi_1(a_1), \ldots, \Pi_1(a_{m-1}) \right\rangle \in A. \] Hence by Proposition 27.3 there exist \(u', v' \in \mathcal{N}^1\) such that \(\Pi_1(u') = u\) and \(\Pi_1(v') = v\). Thus \(\Pi_1 \circ \left\langle u', v', a_1, \ldots, a_{m-1} \right\rangle \in A\). Also,
\[
\Pi_1(u' + v' + a_1 + \cdots + a_{m-1}) = u + v + \Pi_1(a_1 + \cdots + a_{m-1})
\]
\[
= \Pi_1(a_0) + \Pi_1(a_1 + \cdots + a_{m-1})
\]
\[
= \Pi_1(a_0 + \cdots + a_{m-1}) = \Pi_1(\beta).
\]

Hence \(\left\langle u', v', a_1, \ldots, a_{m-1} \right\rangle \in B\). So S.P. holds.

Thus we have shown that \(B \in \triangle \mathcal{N}^1\).

To check the new hierarchy property for \(B\), suppose \(m \in \omega \setminus \{0\},\) \(\left\langle \alpha, \gamma_1, \ldots, \gamma_{m-1} \right\rangle \in B\), and \(\langle a_0, a_1, \ldots, a_{n-1} \rangle \in \alpha\). Then
\[
\Pi_1 \left( \alpha + \sum_{i=1}^{m-1} \gamma_i \right) = \Pi_1(\beta) \text{ and } \langle \Pi_1(\alpha), \Pi_1(\gamma_1), \ldots, \Pi_1(\gamma_{m-1}) \rangle \in A,
\]
and \(\langle \Pi(a_0), \Pi(a_1), \ldots, \Pi(a_{n-1}) \rangle \in \Pi(\alpha)\). By the new hierarchy property for \(A\), there are \(\delta_0, \delta_1, \ldots, \delta_{n-1} \in \mathcal{P}^1\) such that \(\Pi_1(a_0) = \sum_{i<n} \delta_i, \forall i < n[T(\delta_i) = \Pi(a_i)]\), and
\[
\left\langle \delta_0, \delta_1, \ldots, \delta_{n-1}, \Pi_1(\gamma_1), \ldots, \Pi_1(\gamma_{m-1}) \right\rangle \in A.
\]

By Proposition 27.3, for each \(i < n\) there is an \(\varepsilon_i\) such that \(\Pi_1(\varepsilon_i) = \delta_i\) and \(T(\varepsilon_i) = a_i\). Now \(T(\sum_{i<n} \varepsilon_i) = \Pi(\sum_{i<n} a_i)\), so by Proposition 27.3 there is a unique \(x\) such that \(\Pi_1(x) = \sum_{i<n} \delta_i\) and \(T(x) = \sum_{i<n} a_i\). Now \(\Pi_1(\alpha) = \sum_{i<n} \delta_i\) and \(T(\alpha) = \sum_{i<n} a_i\). Also \(\Pi_1(\sum_{i<n} \varepsilon_i) = \sum_{i<n} \delta_i\) and \(T(\sum_{i<n} \varepsilon_i) = \sum_{i<n} a_i\). So by the uniqueness assertion in Proposition 27.3 we have \(\alpha = \sum_{i<n} \varepsilon_i\). Now \(\Pi_1 \circ \left\langle \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n-1}, \gamma_1, \ldots, \gamma_{m-1} \right\rangle \in A\) and
\[
\Pi_1 \left( \sum_{i<n} \varepsilon_i + \sum_{i=1}^{m-1} \gamma_i \right) = \Pi_1 \left( \alpha + \sum_{i=1}^{m-1} \gamma_i \right) = \Pi_1(\beta).
\]

Hence \(\left\langle \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n-1}, \gamma_1, \ldots, \gamma_{m-1} \right\rangle \in B\). This proves the new hierarchy property for \(B\).

To check (N3) for \(B\), suppose that \(n \in \omega \setminus \{0\}, a \text{ and } b \text{ have domain } n, a \in B, \text{ and } \forall i < n[\Pi_1(a_i) = \Pi_1(b_i)]\). Then \(\Pi_1 \circ b = \Pi_1 \circ a \in A\) and \(\Pi_1(\sum_{i<n} b_i) = \Pi_1(\sum_{i<n} a_i) = \Pi_1(\beta)\). So \(b \in B\).

For (N4) for \(B\), suppose that \(n \in \omega \text{ and } \langle a, b_1, \ldots, b_{n-1} \rangle \in B, \text{ with } \Pi_1(a) \neq 0\). Then \(\langle \Pi_1(a), \Pi_1(b_1), \ldots, \Pi_1(b_{n-1}) \rangle \in A\), so by S.P. for \(A\), there are nonzero \(c, d\) such that \(\Pi_1(a) = c + d\) and \(\langle c, d, \Pi_1(b_1), \ldots, \Pi_1(b_{n-1}) \rangle \in A\). Say \(T(c) = [c']\) and \(T(d) = [d']\). Now \([c' + d'] = T(c + d) = T(\Pi_1(a)) = \Pi(T(a)) = [T(a)]\). So \(T(a) \equiv_{ae} c' + d'\). Note that \(T(a) \in \mathcal{N}^0\). Let \(F = \{k \in \omega + 1 : (T(a))(k) \neq c'(k) + d'(k)\}\). So \(F\) is finite. Define
\[
c'''(k) = \begin{cases} (T(a))(k) & \text{if } k \in F, \\ c'(k) & \text{otherwise.} \end{cases}
\]
\[
d'''(k) = \begin{cases} (T(a))(k) & \text{if } k \in F, \\ d'(k) & \text{otherwise.} \end{cases}
\]
Thus \(T(a) = c'' + d'', [c'] = [c''],\) and \([d'] = [d'']\). Now \(T(c) = [c''] = [c'''] = \Pi(c'')\), so by Proposition 27.3 there is a \(c'\) such that \(\Pi_1(c') = c\) and \(T(c') = c''\). Similarly there
is a $d'$ such that $\Pi_1(d') = d$ and $T(d') = d''$. Hence $\Pi_1(c' + d') = c + d = \Pi_1(a)$ and $T(c' + d') = c'' + d''' = T(a)$. Now $T(c + d) = T(\Pi_1(a)) = \Pi(T(a))$, so by Proposition 27.3 there is a unique $\beta$ such that $\Pi_1(\beta) = c + d$ and $T(\beta) = T(a)$. $\beta = a$ works, and so does $c' + d'$. Hence $a = c + d'$. Also,

$$
\Pi_1(c' + d' + b_1 + \cdots + b_{m-1}) = \Pi_1(c' + d' + \Pi_1(b_1 + \cdots + b_{m-1})
= \Pi_1(a) + \Pi_1(b_1 + \cdots + b_{m-1})
= \Pi_1(a + b_1 + \cdots + b_{m-1}) = \Pi_1(\beta).
$$

So $\langle c', d', b_1, \ldots, b_{m-1} \rangle \in B$, as desired.

So, we have shown that $B \in \mathcal{N}^2$.

Clearly $\Pi_2(B) \subseteq A$. Now suppose that $c \in A$. Now $A \in \mathcal{P}^2 \subseteq \triangle \mathcal{P}^1$. Say $c \in m \mathcal{P}^1$. For any $i < m$, $c_i \in \mathcal{P}^1$. Then there is a $d_i \in \mathcal{N}^0$ such that $T(c_i) = \Pi(d_i)$. By Proposition 27.3 there is an $e_i \in \mathcal{N}^1$ such that $\Pi_1(e_i) = c_i$ and $T(e_i) = d_i$. So $\Pi_1 \circ e = c$ and $T \circ e = d$. Then

$$
\left\{ \Pi \circ \theta : \theta \in \sum_{i < m} e_i \right\} = \Pi_1 \left( \sum_{i < n} e_i \right) = \sum_{i < n} \Pi_1(e_i) = \sum_{i < n} c_i
= T(c) = T(A) = \Pi_1(\beta) = \{ \Pi \circ \theta : \theta \in \beta \}.
$$

Thus $\Pi_1(\sum_{i < m} e_i) = \Pi_1(\beta)$. Hence $c \in \Pi_2(B)$. So $\Pi_2(B) = A$.

Clearly $\Pi_1(T(B)) = \Pi_1(\beta)$.

Now suppose that also $C \in \mathcal{N}^2$, $\Pi_2(C) = A$, and $\Pi_1(T(C)) = \Pi_1(\beta)$. If $c \in C$, then $\Pi_1 \circ c \in \Pi_2(C) = A = \Pi_2(B)$, so there is a $b \in B$ such that $\Pi_1 \circ c = \Pi_1 \circ b \in A$. Also, $\Pi_1(\sum_{i < m} c_i) = \Pi_1(T(c)) = \Pi_1(T(C))$, so $c \in B$.

If $b \in B$, then $\Pi_1 \circ b \in \Pi_2(B) = A = \Pi_1(C)$, so there is a $c \in C$ such that $\Pi_1 \circ b = \Pi_1 \circ c$.

By (147) for $C$, $b \in C$. Thus $B = C$. \hfill \Box

Let $\mathcal{Q}^2 = \{ A \in \mathcal{P}^2 : T(A) \in \mathcal{Q}^1 \}$.

**Proposition 28.9.** $\mathcal{Q}^2$ is a subsemigroup of $\mathcal{P}^2$.

**Proof.** Let $A, B \in \mathcal{Q}^2$. Then $T(A + B) = T(A) + T(B)$, and this sum is in $\mathcal{Q}^1$ by Proposition 27.4. \hfill \Box

**Proposition 28.10.** There is a homomorphism $\Gamma_2 : \mathcal{Q}^2 \rightarrow \mathcal{N}^2$ such that:

(i) $\forall A \in \mathcal{Q}^2 \left[ \Pi_2(\Gamma_2(A)) = A \right]$.

(ii) $\forall A \in \mathcal{Q}^2 \left[ \Pi_1(T(\Gamma_2(A))) = T(A) \right]$.

**Proof.** Suppose that $A \in \mathcal{Q}^2$. Thus $A \in \mathcal{P}^2$. Say $T^2(A) = \Pi(X)$ with $X \in \mathcal{N}^0$. Then by Proposition 27.6 we get $\beta \in \mathcal{N}^1$ such that $\Pi_1(\beta) = T(A)$ and $T(\beta) = X$. By Proposition 28.8 let $\Gamma_2(A)$ be the set $B$ indicated there. Then $\Pi_2(\Gamma_2(A)) = A$ and $\Pi_1(T(\Gamma_2(A))) = \Pi_1(\beta)$. So $\Pi_1(T(\Gamma_2(A))) = T(A)$.

For preservation of $+$, suppose that $A_1, A_2 \in \mathcal{P}^2$. Then $T^2(A_1 + A_2) = T^2(A_1) + T^2(A_2) = \Pi(X_1) + \Pi(X_2) = \Pi(X_1 + X_2)$. Say $T^2(A_1) = \Pi(U_1)$ and $T^2(A_2) = \Pi(U_2)$.
Then there are unique $\beta_1, \beta_2$ such that $\Pi_1(\beta_1) = T(A_1), T(\beta_1) = X_1$, $\Pi_1(\beta_2) = T(A_2)$, and $T(\beta_2) = X_2$. Then we have

$$\Gamma_2(A_1) = \{ b \in n(N^1) : n \in \omega, \Pi_1 \circ b \in A_1 \text{ and } \Pi_1 \left( \sum_{i<n} b_i \right) = \Pi_1(\beta_1) \};$$

$$\Gamma_2(A_2) = \{ b \in n(N^1) : n \in \omega, \Pi_1 \circ b \in A_2 \text{ and } \Pi_1 \left( \sum_{i<n} b_i \right) = \Pi_1(\beta_2) \}.$$ 

Hence

$$\Gamma_2(A_1 + A_2) = \{ b \in n(N^1) : n \in \omega, \Pi_1 \circ b \in A_1 + A_2 \text{ and } \Pi_1 \left( \sum_{i<n} b_i \right) = \Pi_1(\beta_1 + \beta_2) \}.$$ 

Hence $\Gamma_2$ preserves +. \qed

**Proposition 28.11.** $\triangle^2(\Psi \circ \Omega) \upharpoonright N^2$ is an injective morphism from $N^2$ into $\triangle^3\Psi$.

**Proof.** By Proposition 25.15, $(\triangle(\Psi \circ \Omega)) \upharpoonright N^1$ is an injective morphism from $N^1$ into $L^1$. Hence by Proposition 25.123, $(\triangle^2(\Psi \circ \Omega)) \upharpoonright \triangle N^1$ is an injective morphism from $\triangle N^1$ into $\triangle L^1$. By Proposition 28.5, $N^2$ is a submonoid of $\triangle N^1$. So $(\triangle^2(\Psi \circ \Omega)) \upharpoonright N^2$ is an injective morphism of $N^2$ into $\triangle L^1$. By definition $\triangle L^1 \subseteq \triangle^3\Psi$. \qed

**29. First reduction step**

**Proposition 29.1.** Suppose that $N$ is a countable submonoid of $P^2$ such that:

(i) $N$ has the refinement property.

(ii) $\forall A \in N \forall (\alpha_0, \alpha_1) \in A \exists A_0, A_1 \in N [A = A_0 + A_1 \text{ and } T(A_0) = \alpha_0 \text{ and } T(A_1) = \alpha_1]$. Then $N \cap Q^2$ is isomorphic to a subsemigroup of $NBA$.

**Proof.** Let $N_1 = \{ B \in N^2 : \Pi_2(B) \in N \}$.

29.1(1) $N_1$ is a submonoid of $N^2$.

For, suppose that $B, C \in N_1$. Then $B + C \in N^2$, and $\Pi_2(B + C) = \Pi_2(B) + \Pi_2(C) \in N$. So $B + C \in N_1$, proving 29.1(1).

29.1(2) $N \cap Q^2$ is a submonoid of $P^2$, and $\Gamma_2$ is an injective homomorphism of $N \cap Q^2$ into $N_1$.

In fact, $N$ is a submonoid of $P^2$ by definition, and $Q^2$ is a submonoid of $P^2$ by Proposition 28.9. So $N \cap Q^2$ is a submonoid of $P^2$. For any $A \in N \cap Q^2$ we have $\Pi_2(\Gamma_2(A)) = A \in N$ by Proposition 28.10, so $\Gamma_2(A) \in N_1$. If $\Gamma_2(A) = \Gamma_2(A')$, then $A = \Pi_2(\Gamma_2(A)) = \Pi_2(\Gamma_2(A')) = A'$. Thus $\Gamma_2$ is injective.

29.1(3) $N_1$ is countable.

For, $N_1 = \bigcup \{ \{ B \in N^2 : \Pi_2(B) = X \} : X \in N \}$ so, since $N$ is countable, it suffices to show that for any $X \in N$ the set $\{ B \in N^2 : \Pi_2(B) = X \}$ is countable.
29.1(4) \( \forall X \in \mathcal{P}[\{\theta \in \mathcal{N}^0 : \Pi(\theta) = X\} \text{ is countable}] \).

For, suppose that \( X \in \mathcal{P} \). Say \( X = [\psi]_{ae} \). Then \( \{\theta \in \mathcal{N}^0 : \Pi(\theta) = X\} = [\psi]_{ae} \text{ is countable by Proposition 26.2.} \)

29.1(5) \( \forall X \in \mathcal{P}^1[\{b \in \mathcal{N}^1 : \Pi_1(b) = X\} \text{ is countable}] \).

For, assume that \( X \in \mathcal{P}^1 \). Let \( Y = \{b \in \mathcal{N}^1 : \Pi_1(b) = X\} \). Let \( Z = \{x \in \mathcal{N}^0 : \Pi(x) = T(X)\} \). So \( Z \text{ is countable by 29.1(4).} \) If \( b \in Y \), then \( T(X) = T(\Pi_1(b)) = \Pi(T(b)) \), so that \( T(b) \in Z \). Suppose that \( b, b' \in Y \) and \( T(b) = T(b') \). Then \( T(\Pi_1(b)) = \Pi(T(b)) \) and so by Proposition 27.3 there is a unique \( \beta \) such that \( \Pi_1(\beta) = \Pi_1(b) \) and \( T(\beta) = T(b) \). So \( \beta = b \). But also \( \Pi_1(b') = \Pi_1(b) \) and \( T(b') = T(b) \), so \( b = b' \). Hence \( T \upharpoonright Y \) is a one-one function mapping \( Y \) into \( Z \); so \( Y \) is countable.

Now we can prove 29.1(3). Suppose that \( X \in \mathcal{N} \), and let \( Y = \{B \in \mathcal{N}^2 : \Pi_2(B) = X\} \). Let \( Z = \{x \in \mathcal{N}^1 : \Pi_1(x) = T(X)\} \). So \( Z \text{ is countable by (154).} \) Suppose that \( b, b' \in Y \) and \( T(b) = T(b') \). Then \( T(\Pi_2(b)) = \Pi_1(T(b)) \) and so by Proposition 28.6 there is a unique \( \beta \) such that \( \Pi_2(\beta) = \Pi_2(b) \) and \( \Pi_1(T(\beta)) = \Pi_1(T(b)) \). So \( \beta = b \). But also \( \Pi_2(b') = \Pi_2(b) \) and \( \Pi_1(T(b')) = \Pi_1(T(b)) \), so \( b = b' \). Hence \( T \upharpoonright Y \) is a one-one function mapping \( Y \) into \( Z \); so \( Y \) is countable. This proves 29.1(3).

Now by Proposition 28.11, \( \mathcal{N}^2 \) is isomorphic to a submonoid of \( \Delta^3 \mathcal{W} \). Hence it suffices now to check conditions (i) and (ii) of Proposition 24.5 for \( N_1 \).

29.1(6) \( N_1 \) has the refinement property.

For, we use Proposition 19.7. Suppose that \( A, B \in 2N_1 \) and \( A_0 + A_1 = B_0 + B_1 \). Then \( \Pi_2(A_0) + \Pi_2(A_1) = \Pi_2(B_0) + \Pi_2(B_1) \) and \( T(A_0) + T(A_1) = T(B_0) + T(B_1) \). Now \( \Pi_2(A_0), \ldots, \Pi_2(B_1) \in N \), so by (i) there is a \( C = \sum_{j < 2} C_{ij} \) and \( \forall j < 2[\Pi_2(B_j) = \sum_{i < 2} C_{ij}] \). For all \( i, j < 2 \) let \( \gamma_{ij} = T(C_{ij}) \). Then \( \forall i < 2[\Pi_1(T(A_i)) = T(\Pi_2(A_i))] = \sum_{j < 2} \gamma_{ij} \), and similarly \( \forall j < 2[\Pi_1(T(B_j)) = \sum_{i < 2} \gamma_{ij}] \).

By Proposition 28.4 there is a \( \rho = \sum_{j < 2} \rho_{ij} \) such that \( \forall i, j < 2[\Pi_1(\rho_{ij}) = \gamma_{ij}] \), \( \forall i < 2[T(A_i) = \sum_{j < 2} \rho_{ij}] \), and \( \forall j < 2[T(B_j) = \sum_{i < 2} \rho_{ij}] \). Thus \( \forall i, j < 2[T(C_{ij}) = \gamma_{ij} = \Pi_1(\rho_{ij})] \), so by Proposition 28.8 there is a \( D = \sum_{j < 2} \rho_{ij} \) such that \( \forall i, j < 2[\Pi_2(D_{ij}) = C_{ij} \text{ and } T(D_{ij}) = \rho_{ij}] \).

It follows that \( \forall i < 2[\Pi_2(A_i) = \sum_{j < 2} \rho_{ij} = \sum_{j < 2} T(D_{ij}) = T(\sum_{j < 2} D_{ij}) \text{ and } \Pi_2(A_i) = \sum_{j < 2} C_{ij} = \sum_{j < 2} \Pi_2(D_{ij}) = \Pi_2(\sum_{i < 2} D_{ij}) \). Now \( \forall i < 2[\Pi_1(T(A_i)) = T(\Pi_2(A_i))] \), so by Proposition 28.8, for each \( i < 2 \) there is a unique \( X \) such that \( \Pi_2(X) = \Pi_2(A_i) \) and \( T(X) = T(A_i) \). It follows that \( \forall i < 2[A_i = \sum_{j < 2} D_{ij}] \). Similarly, \( \forall j < 2[B_j = \sum_{i < 2} D_{ij}] \). This proves 29.1(6).

29.1(7) \( \forall n \in \omega \setminus \{0\} \exists A \in \mathcal{N} \forall \alpha \in A[\text{dmm}(\alpha) = n \Rightarrow \exists B \in \mathcal{N}[A = \sum_{i < n} B_i \text{ and } \forall i < n[T(B_i) = \alpha_i]]] \).

We prove 29.1(7) by induction on \( n \). It is trivial for \( n = 1 \), and (ii) gives \( n = 2 \). Now assume it for \( n \), and suppose that \( \alpha \in A \) with \( \text{dmm}(\alpha) = n + 1 \). Then \( (\sum_{i < n} \alpha_i, \alpha_n) \in A \), so by (ii) there are \( C_0, C_1 \in \mathcal{N} \) such that \( A = C_0 + C_1 \), \( T(C_0) = \sum_{i < n} \alpha_i \), and \( T(C_1) = \alpha_n \).

Then by the inductive hypothesis there is a \( B \in \mathcal{N} \) such that \( C_0 = \sum_{i < n} B_i \), and \( \forall i < n[T(B_i) = \alpha_i] \). This gives 29.1(7) for \( n + 1 \).

To prove Proposition 24.5(ii), suppose that \( A \in N_1 \), \( n \in \omega \), \( \alpha \in A \), and \( \text{dmm}(\alpha) = n \). So \( T(A) = \sum_{i < n} \alpha_i \) and \( \Pi_2(A) \in \mathcal{N} \). Now \( \Pi_1 \circ \alpha \in \Pi_2(A) \). By 29.1(7) there is \( B \in \mathcal{N} \) with
domain \( n \) such that \( \Pi_2(A) = \sum_{i<n} B_i \) and \( \forall i < n[T(B_i) = \Pi_1(\alpha_i)] \). By Proposition 28.8 there is \( C \in \mathcal{N}^2 \) with domain \( n \) such that \( \forall i < n[\Pi_2(C_i) = B_i \) and \( T(C_i) = \alpha_i] \). Hence \( \forall i < n[C_i \in N_1] \). Now \( \Pi_2(\sum_{i<n} C_i) = \sum_{i<n} B_i = \Pi_2(A) \) and \( T(\sum_{i<n} C_i) = \sum_{i<n} \alpha_i = T(A) \). Hence by Proposition 28.8, \( A = \sum_{i<n} C_i \).

### 30. Distinguished elements

An element \( e \) of an \( m \)-monoid \( M \) is distinguished iff \( \forall f \in M[e+f = e \rightarrow f = 0] \). Note that 0 is distinguished. \( M_s \) is the set of all distinguished elements of \( M \). \( M \) is distinguished iff every element of \( M \) is a sum of distinguished elements.

**Proposition 30.1.** \( \forall \theta \in \mathcal{N}^0[[\theta]_{ae} \text{ is distinguished iff } \theta\ae \neq [\omega^c]_{ae}] \).

**Proof.** \( \Rightarrow: \) \([\omega^c]_{ae} + [\omega]_{ae} = [\omega]_{ae} \) and \([\omega]_{ae} \neq [0^c]_{ae} \), so \([\omega^c]_{ae} \) is not distinguished.

\( \Leftarrow: \) Obviously \([0^c] \) is distinguished. Now suppose that \( \theta \in \mathcal{N}^0 \) satisfies \((135), \psi \in \mathcal{N}^0 \), and \([\theta]_{ae} + [\psi]_{ae} = [\theta]_{ae} \). Then \( F \overset{\text{def}}{=} \{ k \in \omega + 1 : \theta(k) + \psi(k) \neq \theta(k) \} \) is finite. Clearly \( \forall k \in (\omega + 1)\setminus F[\psi(k) = 0] \), so \([\psi]_{ae} = [0^c]_{ae} \). \( \square \)

**Proposition 30.2.** In \( \mathcal{P} \), \([\omega^c]_{ae} \) is not a sum of distinguished elements. Hence \( \mathcal{P} \) is not distinguished.

**Proof.** Suppose that \([\omega^c]_{ae} = \sum_{i<m} a_i \), each \( a_i \in \mathcal{P} \). Then some \( a_i \) is equal to \([\omega^c]_{ae} \), and so by Proposition 30.1 is not distinguished. \( \square \)

**Proposition 30.3.** If \( e \) is a distinguished element of \( M \), then \( e \) has the Schröder-Bernstein property.

**Proof.** Assume that \( e \) is a distinguished element of \( M \) and \( e \leq b \leq e \). Thus there are \( c,d \in M \) such that \( b = e + c \) and \( e = b + d \). Thus \( e = e + c + d \), so \( c + d = 0 \), hence \( c = d = 0 \), hence \( b = e \). \( \square \)

**Proposition 30.4.** In \( \mathcal{P} \), \([\omega^c]_{ae} \) has the Schröder-Bernstein property, but \([\omega^c]_{ae} \) is not distinguished.

**Proof.** By Propositions 30.1, \([\omega^c]_{ae} \) is not distinguished. Now suppose that \([\omega^c]_{ae} \leq a \leq [\omega^c]_{ae} \). Choose \( b,c \in \mathcal{P} \) such that \( a = [\omega^c]_{ae} + b \) and \([\omega^c]_{ae} = a + c \). Clearly then \( a = [\omega^c]_{ae} \). \( \square \)

For \( m \)-monoids \( M,N \), we say that \( N \) is a distinguished extension of \( M \) if \( N \) is an extension of \( M \) and \( M_s \subseteq N_s \).

**Proposition 30.5.** If \( P \) is a distinguished extension of \( N \) and \( N \) is a distinguished extension of \( M \), then \( P \) is a distinguished extension of \( M \). \( \square \)

**Proposition 30.6.** If \( \langle M_k : k \in \omega \rangle \) is a system of \( m \)-monoids such that \( \forall k \in \omega[M_{k+1} \) is a distinguished extension of \( M_k \)], then \( \forall k \in \omega[\bigcup_{i \in \omega} M_i \) is a distinguished extension of \( M_k \)]. \( \square \)
Proposition 30.7. If \( e \) is a nonzero element of an \( m \)-monoid \( M \), then there exist a distinguished extension \( N \) of \( M \) and \( e_0, e_1 \in N \) such that

(i) \( e = e_0 + e_1 \) and \( e_0 \) and \( e_1 \) are distinguished and nonzero.

(ii) \( N \) is generated as a monoid by \( M \cup \{e_0, e_1\} \).

(iii) \( \forall m, n \in \omega \cup f, f' \in M[f' = f + me_0 + ne_1 \rightarrow m = n \text{ and } f' = f + me] \).

(iv) If \( \Lambda : M \rightarrow M' \) is a morphism of \( m \)-monoids, then \( \forall f_0, f_1 \in M'[\Lambda(e) = f_0 + f_1 \rightarrow \exists \Lambda'[\Lambda \subseteq \Lambda' \text{ and } \Lambda' : N \rightarrow M' \text{ is a morphism and } \forall i < 2[\Lambda'(e_i) = f_i]] \).

\textbf{Proof.} Let \((\omega, +)\) be the natural monoid. Define \( P = M \times \omega \times \omega \). Define

\[
(f, m, n) \sim (f', m', n') \iff (f, m, n), (f', m', n') \in P, m - n = m' - n',
\]

\[f + me = f' + m'e, \text{ and } f + ne = f' + n'e.\]

30.7(1) \( \sim \) is a congruence relation on \( P \).

In fact, it is clearly an equivalence relation on \( P \). Now suppose that \((f, m, n) \sim (f', m', n')\) and \((f'', m'', n'') \sim (f''', m''', n''')\). Then

\[
(f, m, n) + (f'', m'', n'') = (f + f'', m + m'', n + n'');
(f', m', n') + (f''', m''', n''') = (f' + f''', m' + m''', n' + n''');
\]

\[
m + m'' - (n + n'') = m - n + m'' - n' ' = m' - n' + m''' - n'''
= m' + m''' - (n' + n''');
\]

\[
f + f'' + (m + m'')e = f + me + f'' + m''e
= f' + m'e + f'' + m''e
= f' + f''' + (m' + m'''e)
\]

\[
f + f'' + (n + n'')e = f + ne + f'' + n''e
= f' + n'e + f'' + n'''e
= f' + f''' + (n' + n''')e.
\]

This proves 30.7(1).

Let \( N = P/\sim \) and let \( \Pi \) be the natural mapping from \( P \) to \( N \). For any \( f \in M \) let \( \Psi(f) = [(f, 0, 0)]_\sim \).

30.7(2) \( \Psi \) is an injective morphism from \( M \) to \( N \).

In fact, clearly \( \Psi \) is a morphism from \( M \) to \( N \). Now suppose that \( \Psi(f) = \Psi(f') \). Thus \((f, 0, 0) \sim (f', 0, 0)\). Clearly \( f = f' \). This proves 30.7(2).

30.7(3) If \( f \) is distinguished in \( M \), then \( \Psi(f) \) is distinguished in \( N \).

For, suppose that \( f \) is distinguished in \( M \), and \( \Psi(f) + [(f', m, n)]_\sim = \Psi(f) \). Thus \((f, 0, 0) + (f', m, n) \sim (f, 0, 0)\), i.e., \((f + f', m, n) \sim (f, 0, 0)\). Hence \( f + f' + me = f \); since \( f \) is distinguished, this implies that \( f' + me = 0 \), hence \( f' = 0 = m \). Also, \( m - n = 0 \), so \( m = n \). Hence \([(f', m, n)] = 0 \), proving 30.7(3).
In fact, \( \Psi(e) = 30.7(5) \) proves \( 30.7(4) \). Similarly, \( 30.7(4) \) PROVES \( 30.7(5) \). For, suppose that \( 30.7(7) \) PROVES \( 30.7(10) \).

In fact, let \( f = (m, n, \omega) \), so \( f = (m, n, \omega) \). Then \( \Psi(e) = [(e, 0, 0)] = [(0, 1, 1)] \). We have \( 0 - 0 = 1 - 1, e + 0e = 0 + 1e \), and \( e + 0e = 0 + 1e \). So \( \Psi(e) = [(e, 0, 0)] = [(0, 1, 1)] = e_0 + e_1 \).

30.7(7) \( e_0 \neq 0 \).

In fact, suppose that \( [(0, 1, 0)] = [(0, 0, 0)] \). then \( 1 - 0 = 0 - 0 \), contradiction. Similarly, 30.7(8) \( e_1 \neq 0 \).

30.7(9) \( N \) is generated by \( \Psi[M] \cup \{e_0, e_1\} \).

In fact, let \( [(f, m, n)] \in N \). Then \( \Psi(f) + me_0 + ne_1 = [(f, m, n)] \), as desired.

30.7(10) \( \forall m, n \in \omega \forall f, f' \in M, \Psi(f') = \Psi(f) + me_0 + ne_1 \rightarrow m = n \) and \( f' = f + me \).

For, suppose that \( m, n \in \omega, f, f' \in M, \) and \( \Psi(f') = \Psi(f) + me_0 + ne_1 \). Thus \( (f', 0, 0) \sim (f, m, n) \), so \( 0 - 0 = m - n \) and \( f' = f + me \). So \( m = n \), proving 30.7(4).

Now suppose that \( \Lambda : M \rightarrow M' \) is a morphism of \( m \)-monoids, \( f_0, f_1 \in M' \), and \( \Lambda(e) = f_0 + f_1 \). Define \( \Lambda' : N \rightarrow M' \) by \( \Lambda'([(f, m, n)]) = \Lambda(f) + mf_0 + nf_1 \). Then \( \Lambda' \) is well-defined: Suppose that \( [(f, m, n)] = [(f', m', n')] \). Thus \( (f, m, n) \sim (f', m', n') \), so \( m - n = m' - n', f + me = f' + m'e \), and \( f + ne = f' + n'e \). Say \( m \geq n \). Then also \( m' \geq n' \), and

\[
\begin{align*}
\Lambda(f) + mf_0 + nf_1 &= \Lambda(f) + nf_0 + f_1 + (m - n)f_0 = \Lambda(f) + n\Lambda(e) + (m - n)f_0 \\
&= \Lambda(f + ne) + (m' - n')f_0 = \Lambda(f' + n'e) + (m' - n')f_0 \\
&= \Lambda(f') + n'\Lambda(e) + (m' - n')f_0 = \Lambda(f') + n'(f_0 + f_1) + (m' - n')f_0 \\
&= \Lambda(f') + m'f_0 + n'f_1.
\end{align*}
\]

So \( \Lambda' \) is well-defined.

\( \Lambda' \) preserves the operation:

\[
\begin{align*}
\Lambda'([(f, m, n)]) + [(f', m', n')] &= \Lambda'([(f + f', m + m', n + n')]) \\
&= \Lambda(f + f') + (m + m')f_0 + (n + n')f_1 \\
&= \Lambda(f) + \Lambda(f') + mf_0 + m'f_0 + nf_1 + n'f_1 \\
&= \Lambda'([(f, m, n)]) + \Lambda'([(f', m', n')]).
\end{align*}
\]

Clearly \( \Lambda'([(f, 0, 0)]) = \Lambda(f) \), and \( \Lambda(e_i) = f_i \) for \( i < 2 \).
Proposition 30.8. If $M$ is distinguished and $N$ is as in Proposition 30.7, then $N$ is distinguished.

Proposition 30.9. (2.13.4) If $M$ is a countable $m$-monoid, then there is a distinguished extension $N$ of $M$ such that $N$ is distinguished, countable and atomless.

Proof. Assume that $M$ is a countable $m$-monoid. Let $\langle e_k; k < \omega \rangle$ enumerate $M \setminus \{0\}$. We define $N_0 = M$, and $N_{k+1}$ is obtained from $N_k$ by applying Proposition 30.7 to $N_k$ and $e_k$. Thus we have $e_k = e_{k0} + e_{k1}$ for some distinguished nonzero elements of $N_{k+1}$. Let $P = \bigcup_{k \in \omega} N_k$. Thus

$30.9(1) \quad \text{P is a countable distinguished extension of } M \text{ such that for each nonzero } x \in M \text{ there are nonzero distinguished } e, e \' \in P \text{ such that } x = e + e \'.$

Now let $Q^0 = M$ and let $Q^{k+1}$ be obtained from $Q^k$ by the proof of 30.9(1). Let $N = \bigcup_{k \in \omega} Q^k$. Then $N$ is a distinguished extension of $M$ such that for every nonzero $x \in N$ there are nonzero distinguished elements $e, e \' \in N$ such that $x = e + e \'$. So $N$ is atomless.

31. Constructing refinements

$M_2(\omega)$ is the monoid of $2 \times 2$ matrices with entries in $\omega$ under matrix addition. For $i, j < 2$, $\varepsilon_{ij}$ is the $2 \times 2$ matrix with 1 in the $(i, j)$-position and 0 elsewhere. Define $\alpha_i = \sum_{j < 2} \varepsilon_{ij}$ for $i < 2$ and $\beta_j = \sum_{i < 2} \varepsilon_{ij}$ for $j < 2$. Define $\varphi : M_2(\omega) \rightarrow \mathbb{Z}$ by $\varphi(\sum_{i,j < 2} r_{ij} \varepsilon_{ij}) = r_{00} + r_{11} - r_{01} - r_{10}$.

Proposition 31.1. $\varphi$ is a semigroup homomorphism from $M_2(\omega)$ to $\mathbb{Z}$.

Proof.

$$
\varphi \left( \sum_{i,j < 2} r_{ij} \varepsilon_{ij} + \sum_{i,j < 2} r_{ij}' \varepsilon_{ij} \right) = \varphi \left( \sum_{i,j < 2} (r_{ij} + r_{ij}') \varepsilon_{ij} \right) = r_{00} + r_{00}' + r_{11} + r_{11}' - r_{01} - r_{01}' - r_{10} - r_{10}'
$$

$$
= r_{00} + r_{11} - r_{01} - r_{10} + r_{00}' + r_{11}' - r_{01}' - r_{10}'
$$

$$
= \varphi \left( \sum_{i,j < 2} r_{ij} \varepsilon_{ij} \right) + \varphi \left( \sum_{i,j < 2} r_{ij}' \varepsilon_{ij} \right).
$$

\hfill \square

Proposition 31.2. $\forall i < 2 [\varphi(\alpha_i) = \varphi(\beta_i) = 0]$.

\hfill \square

Proposition 31.3.

$$
\alpha_0 + \alpha_1 = \beta_0 + \beta_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}
$$

A $2 \times 2$ matrix $\gamma \in M_2(\omega)$ is counter-diagonal iff $\gamma_{00} = \gamma_{11} = 0$.

137
Proposition 31.4. Every $\gamma \in M_2(\omega)$ can be represented in the form

$$31.4(1) \quad \gamma = \delta + \sum_{i<2} m_i \alpha_i + \sum_{j<2} n_j \beta_j$$

where $\delta$ is diagonal or counter-diagonal, $\forall i < 2 [m_i, n_i \in \omega]$, and $n_0 n_1 = 0$.

Proof. Suppose that $\gamma \in M_2(\omega)$. We prove the existence of a decomposition 31.4(1) by induction on $p = \sum_{i,j<2} \gamma_{ij}$. Note that if $\gamma$ is diagonal or counter-diagonal then we can take each $m_i$ and $n_i$ equal to 0. If $p = 0$, then $\gamma$ is diagonal. Assume the result for values $\leq p$, and now assume that $\sum_{i,j<2} \gamma_{ij} = p + 1$. We may assume that $\gamma$ is not diagonal or counter-diagonal. Thus

$$(\gamma_{01} \neq 0 \text{ or } \gamma_{10} \neq 0) \text{ and } (\gamma_{00} \neq 0 \text{ or } \gamma_{11} \neq 0).$$

This gives four cases.

Case 1. $\gamma_{01} \neq 0 \neq \gamma_{00}$.

Subcase 1.1. $\gamma_{01} \leq \gamma_{00}$. Let $\gamma' = \gamma - \gamma_{01} \alpha_0$. Then $\sum_{i,j<2} \gamma'_{ij} \leq p - 1$, so $\gamma'$ can be written in the form 31.4(1), and hence so can $\gamma = \gamma' + \gamma_{01} \alpha_0$.

Subcase 1.2. $\gamma_{00} < \gamma_{01}$. Let $\gamma' = \gamma - \gamma_{00} \alpha_0$; 31.4(1) follows.

Case 2. $\gamma_{01} \neq 0 \neq \gamma_{11}$.

Subcase 2.1. $\gamma_{01} \leq \gamma_{11}$. Let $\gamma' = \gamma - \gamma_{01} \beta_1$; 31.4(1) follows.

Subcase 2.2. $\gamma_{11} < \gamma_{01}$. Similarly.

Case 3. $\gamma_{10} \neq 0 \neq \gamma_{00}$.

Subcase 3.1. $\gamma_{10} \leq \gamma_{00}$. Let $\gamma' = \gamma - \gamma_{10} \beta_0$; 31.4(1) follows.

Subcase 3.2. $\gamma_{00} < \gamma_{10}$. Similarly.

Case 4. $\gamma_{10} \neq 0 \neq \gamma_{11}$.

Subcase 4.1. $\gamma_{10} \leq \gamma_{11}$. Let $\gamma' = \gamma - \gamma_{10} \alpha_1$; 31.4(1) follows.

Subcase 4.2. $\gamma_{11} < \gamma_{10}$. Similarly.

Thus we have the representation 31.4(1). To see that we can add the condition $n_0 n_1 = 0$, we take two cases. Recall that $\alpha_0 + \alpha_1 = \beta_0 + \beta_1$.

Case 1. $n_0 \leq n_1$. Then

$$\delta + \sum_{i<2} m_i \alpha_i + \sum_{j<2} n_j \beta_j = \delta + \sum_{i<2} m_i \alpha_i + n_0 (\beta_0 + \beta_1) + (n_1 - n_0) \beta_1$$

$$= \delta + \sum_{i<2} m_i \alpha_i + n_0 (\alpha_0 + \alpha_1) + (n_1 - n_0) \beta_1$$

$$= \delta + (m_0 + n_0) \alpha_0 + (m_1 + n_0) \alpha_1 + (n_1 - n_0) \beta_1.$$ 

Case 2. $n_1 < n_0$. Similarly.

A representation of $\gamma$ as in Proposition 31.4 is called a reduced representation of $\gamma$.

Proposition 31.5. In a reduced representation of a matrix $\gamma \in M_2(\omega)$, $\varphi(\gamma) = \varphi(\delta)$.
Proof. In fact, for each $i < 2$ we have

$$m_i \alpha_i = \sum_{j < 2} m_i \varepsilon_{ij} = \begin{pmatrix} m_i & m_i \\ 0 & 0 \end{pmatrix},$$

so $\varphi(m_i \alpha_i) = 0$. Similarly $\varphi(m_i \beta_i) = 0$. Hence the proposition holds. \qed

Proposition 31.6. In a reduced representation of a matrix $\gamma \in M_2(\omega)$, $\delta = 0$ iff $\varphi(\gamma) = 0$.

Proof. Let $\gamma \in M_2(\omega)$, and consider a reduced representation of $\gamma$ as in Proposition 31.3. If $\delta = 0$, then $\varphi(\gamma) = \varphi(\delta) = 0$ by Proposition 31.4. Assume that $\varphi(\gamma) = 0$. Then $\varphi(\delta) = \varphi(\delta) = 0$ by Proposition 31.4. If $\delta$ is diagonal, then $\delta_{00} + \delta_{11} = 0$, hence $\delta = 0$. If $\delta$ is counter-diagonal, then $\delta_{01} + \delta_{10} = 0$, hence $\delta = 0$. \qed

Proposition 31.7. If $\gamma$ has the reduced representation 34.1(1) with $\delta = 0$, then in any reduced representation

$$\gamma = \delta' + \sum_{i < 2} m_i' \alpha_i + \sum_{j < 2} n_j' \beta_j$$

we must have $\delta' = 0$, $\forall i < 2[m_i = m_i']$, and $\forall j < 2[n_j = n_j']$.

Proof. By Proposition 31.5 we have $\varphi(\gamma) = \varphi(\delta') = \varphi(\delta) = 0$. Clearly $\varphi(A) \neq 0$ if $A \neq 0$ is diagonal or counter-diagonal. So $\delta' = 0$. So assume that we have reduced representations

$$\gamma = \sum_{i < 2} m_i \alpha_i + \sum_{j < 2} n_j \beta_j = \sum_{i < 2} m_i' \alpha_i + \sum_{j < 2} n_j' \beta_j.$$ 

Then

$$m_0 \alpha_0 = \begin{pmatrix} m_0 & m_0 \\ 0 & 0 \end{pmatrix}; \quad m_1 \alpha_1 = \begin{pmatrix} 0 & 0 \\ m_1 & m_1 \end{pmatrix};$$

$$n_0 \beta_0 = \begin{pmatrix} n_0 & 0 \\ n_0 & 0 \end{pmatrix}; \quad n_1 \beta_1 = \begin{pmatrix} 0 & n_1 \\ 0 & n_1 \end{pmatrix}.$$ 

Say wlog $n_0 = 0$. Then

$$\sum_{i < 2} m_i \alpha_i + \sum_{j < 2} n_j \beta_j = \begin{pmatrix} m_0 & m_0 + n_1 \\ m_1 & m_1 + n_1 \end{pmatrix}.$$ 

Case 1. $n_0' = 0$. Then also

$$\sum_{i < 2} m_i' \alpha_i + \sum_{j < 2} n_j' \beta_j = \begin{pmatrix} m_0' & m_0' + n_1' \\ m_1' & m_1' + n_1' \end{pmatrix}.$$ 

It follows that $m_0 = m_0'$, $m_1 = m_1'$, $n_0 = n_0'$, and $n_1 = n_1'$. 139
Case 2. \(n'_1 = 0\). Then
\[
\sum_{i < 2} m'_i \alpha_i + \sum_{j < 2} n'_j \beta_j = \begin{pmatrix} m'_0 + n'_0 & m'_0 \\ m'_1 + n'_0 & m'_1 \end{pmatrix}
\]
This gives the equations
\[
\begin{align*}
m_0 &= m'_0 + n'_0 \\
m_1 &= m'_1 + n'_0 \\
m_0 + n_1 &= m'_0 \\
m_1 + n_1 &= m'_1
\end{align*}
\]
It follows that \(n'_0 = n_1 = 0\), \(m_0 = m'_0\), and \(m_1 = m'_1\). So \(n_0 = n'_0\) and \(n_1 = n'_1\).

\[\square\]

**Proposition 31.8.** Suppose that \(M\) is an \(m\)-monoid, \(e_0, e_1, f_0, f_1 \in M \setminus \{0\}\), and \(e_0 + e_1 = f_0 + f_1\). Then there is a distinguished extension \(N\) of \(M\) such that \(N\) contains distinguished elements \(g_{ij}\) for \(i, j < 2\) such that:

(i) \(\forall i < 2[e_i = \sum_{j < 2} g_{ij}\).

(ii) \(\forall j < 2[f_j = \sum_{i < 2} g_{ij}\).

(iii) \(N\) is generated by \(M \cup \{g_{ij} : i, j < 2\}\).

(iv) \(\forall g, g' \in M \forall r \in 2 \times 2 \omega[g' = g + \sum_{i,j < 2} r_{ij} g_{ij} \rightarrow r_{00} + r_{11} - r_{01} - r_{10} = 0 \text{ and } \exists m_i, n_i \in \omega \text{ for } i < 2[\forall i, j < 2[r_{ij} = m_i + n_j] \text{ and } g' = g + \sum_{i,j < 2} m_i e_i + \sum_{j < 2} n_j f_j]]\).

(v) If \(\Lambda : M \rightarrow M'\) is a morphism, \(\forall i, j < 2[h_{ij} \in M']\), \(\forall i < 2[\Lambda(e_i) = \sum_{j < 2} h_{ij}]\), and \(\forall j < 2[\Lambda(f_j) = \sum_{i < 2} h_{ij}]\), then there is an extension \(\Lambda'\) of \(\Lambda\) to a morphism of \(N\) to \(M'\) such that \(\forall i, j < 2[\Lambda'(g_{ij}) = h_{ij}]\).

**Proof.** Let \(P = M \times M_2(\omega)\). We define \((g, \gamma) \sim (g', \gamma')\) iff there exist \(\delta \in M_2(\omega)\) and \(m_i, n_i, m'_i, n'_i \in \omega\) for \(i < 2\) such that the following conditions hold:

\[
\gamma = \delta + \sum_{i < 2} m_i \alpha_i + \sum_{j < 2} n_j \beta_j, \quad \gamma' = \delta + \sum_{i < 2} m'_i \alpha_i + \sum_{j < 2} n'_j \beta_j, \quad \text{and}
\]

31.8(1) \(g + \sum_{i < 2} m_i e_i + \sum_{j < 2} n_i f_i = g' + \sum_{i < 2} m'_i e_i + \sum_{j < 2} n'_j f_i\)

We may assume that the representations in 31.8(1) are reduced. In fact, let
\[
\delta = \delta' + \sum_{i < 2} m''_i \alpha''_i + \sum_{j < 2} n''_j \beta''_j
\]
be a reduced representation of \(\delta\). Then the first equation of 31.8(1) becomes
\[
\gamma = \delta' + \sum_{i < 2} (m_i + m''_i) \alpha_i + \sum_{j < 2} (n_j + n''_j) \beta_j,
\]
140
and the argument at the end of the proof of Proposition 31.3 can be used to put this in reduced form, with $\delta'$ retained. Namely, if $n_0 + n_0'' \leq n_1 + n_1''$, then

$$\sum_{i<2} (m_i + m_i'')\alpha_i + \sum_{j<2} (n_j + n_j'')\beta_j$$

$$= \sum_{i<2} (m_i + m_i'')\alpha_i + \sum_{j<2} (n_j + n_j'')\beta_j + (n_0 + n_0'')(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}) - (n_0 + n_0'')(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix})$$

$$= \sum_{i<2} (m_i + m_i'')\alpha_i + \sum_{j<2} (n_j + n_j'')\beta_j + (n_0 + n_0'')(\alpha_0 + \alpha_1) - (n_0 + n_0'')(\beta_0 + \beta_1)$$

$$= (m_0 + m_0'' + n_0 + n_0'')\alpha_0 + (m_1 + m_1'' + n_1 + n_1'')\alpha_1 + (n_0 + n_0'' - n_1 - n_1'')\beta_0,$$

while if $n_1 + n_1' < n_0 + n_0''$ then

$$\sum_{i<2} (m_i + m_i'')\alpha_i + \sum_{j<2} (n_j + n_j'')\beta_j$$

$$= (m_0 + m_0'' + n_0 + n_0'')\alpha_0 + (m_1 + m_1'' + n_1 + n_1'')\alpha_1 + (n_0 + n_0'' - n_1 - n_1'')\beta_0.$$

The second equation in 31.8(1) is treated similarly.

The condition 31.8(2) can be treated similarly. Namely, if $n_0 + n_0'' \leq n_1 + n_1''$, then

$$\sum_{i<2} (m_i + m_i'')e_i + \sum_{j<2} (n_j + n_j'')f_j$$

$$= \sum_{i<2} (m_i + m_i'')e_i + (n_0 + n_0'')(f_0 + f_1) + (n_1 + n_1'' - n_0 - n_0'')f_1$$

$$= \sum_{i<2} (m_i + m_i'')e_i + (n_0 + n_0'')(e_0 + e_1) + (n_1 + n_1'' - n_0 - n_0'')f_1$$

$$= (m_0 + m_0'' + n_0 + n_0'')e_0 + (m_1 + m_1'' + n_1 + n_1'')e_1 + (n_0 + n_0'' - n_1 - n_1'')f_1,$$

while if $n_1 + n_1'' < n_0 + n_0''$ then

$$\sum_{i<2} (m_i + m_i'')e_i + \sum_{j<2} (n_j + n_j'')f_j$$

$$= (m_0 + m_0'' + n_0 + n_0'')e_0 + (m_1 + m_1'' + n_1 + n_1'')e_1 + (n_0 + n_0'' - n_1 - n_1'')f_0.$$

Now

$$g + \sum_{i<2} (m_i + m_i'')e_i + \sum_{j<2} (n_j + n_j'')f_j$$

$$= g' + \sum_{i<2} (m_i' + m_i'')e_i + \sum_{j<2} (n_j' + n_j'')f_j.$$

So 38.1(2) can be changed corresponding to the change of 38.1(1).
31.8(3) If \((g, \gamma) \sim (g', \gamma')\), then \(\varphi(\gamma) = \varphi(\gamma')\).

This follows from Proposition 31.5.

Now for any \((g, \gamma) \in M \times M_2(\omega)\) it is clear that \((g, \gamma) \sim (g, \gamma)\). So \(\sim\) is reflexive on \(M \times M_2(\omega)\). Clearly \(\sim\) is symmetric. Next, let \((g, \gamma) \sim (g', \gamma')\) and \((g'', \gamma'') \sim (g'''', \gamma'''')\).

Say
\[
\gamma = \delta + \sum_{i < 2} m_i \alpha_i + \sum_{j < 2} n_j \beta_j, \quad \gamma' = \delta + \sum_{i < 2} m'_i \alpha_i + \sum_{j < 2} n'_j \beta_j,
\]
\[
g + \sum_{i < 2} m_i e_i + \sum_{j < 2} n_i f_i = g' + \sum_{i < 2} m'_i e_i + \sum_{j < 2} n'_i f_i,
\]
\[
\gamma'' = \delta'' + \sum_{i < 2} m''_i \alpha_i + \sum_{j < 2} n''_j \beta_j, \quad \gamma''' = \delta''' + \sum_{i < 2} m'''_i \alpha_i + \sum_{j < 2} n'''_j \beta_j,
\]
\[
g'' + \sum_{i < 2} m''_i e_i + \sum_{j < 2} n''_i f_i = g''' + \sum_{i < 2} m'''_i e_i + \sum_{j < 2} n'''_i f_i.
\]

Then
\[
\gamma + \gamma'' = \delta + \delta'' + \sum_{i < 2} (m_i + m''_i) \alpha_i + \sum_{j < 2} (n_j + n''_j) \beta_j;
\]
\[
\gamma' + \gamma''' = \delta + \delta''' + \sum_{i < 2} (m'_i + m'''_i) \alpha_i + \sum_{j < 2} (n'_j + n'''_j) \beta_j;
\]
\[
g + g'' + \sum_{i < 2} (m_i + m''_i) e_i + \sum_{j < 2} (n_i + n''_i) f_i
\]
\[
= g' + g''' + \sum_{i < 2} (m'_i + m'''_i) e_i + \sum_{j < 2} (n'_i + n'''_i) f_i
\]

Thus \((g + g'', \gamma + \gamma'') \sim (g' + g''', \gamma' + \gamma''')\).

It follows that the transitive closure \(\approx\) of \(\sim\) is a congruence relation on \(P\).

Next we claim 31.8(4) If \((g, 0) \sim (g', \theta)\), then \(\theta\) has the form \(\sum_{i < 2} p_i \alpha_i + \sum_{j < 2} q_i \beta_i\), and \(g = g' + \sum_{i < 2} p_i e_i + \sum_{j < 2} q_i f_i\).

For, say \((g, 0) = (g_0, \gamma_0) \sim (g_1, \gamma_1) \sim \cdots \sim (g_s, \gamma_s) = (g', \theta)\). By (171) we have \(\varphi(\gamma_i) = 0\) for all \(i\). Hence in the reduced representations of the \(\gamma_i\), all the \(\delta\)-s are 0 by Proposition 31.6. So we get
\[
\gamma_k = \sum_{i < 2} m^k_i \alpha_i + \sum_{j < 2} n^k_j \beta_j\]
\[
g_k + \sum_{i < 2} m^k_i e_i + \sum_{j < 2} n^k_j f_j = g_{k+1} + \sum_{i < 2} m^{k+1}_i e_i + \sum_{j < 2} n^{k+1}_j f_j.
\]
By uniqueness (Proposition 31.7), \( m_i^0 = 0 = n_i^0 \) for all \( i < 2 \). Hence the desired conclusion holds.

We let \( N = P / \sim \), and let \( \Omega : P \to N \) be the natural map. Define \( \Psi(g) = \Omega(g, 0) \) for any \( g \in M \). Clearly \( \Psi \) is a morphism from \( M \) to \( N \). We claim that \( \Psi \) is one-one. For, suppose that \( \Psi(g) = \Psi(g') \). Thus \((g, 0) \approx (g', 0)\), so by 31.8(4) we get \( g = g' \). Thus \( \Psi \) is one-one.

31.8(5) If \( g \) is distinguished in \( M \), then \( \Psi(g) \) is distinguished in \( N \).

For suppose that \( g \) is distinguished in \( M \), and \( \Psi(g) + [(h, \gamma)] \approx = \Psi(g) \). Then \((g, 0) \approx (g + h, \gamma)\). Hence by 31.8(4), \( \gamma \) has the form \( \sum_{i < 2} m_i \alpha_i + \sum_{j < 2} n_i \beta_i \), and \( g = g + h + \sum_{i < 2} m_i e_i + \sum_{j < 2} n_i f_i \). Since \( g \) is distinguished, it follows that \( h = 0 \) and \( \sum_{i < 2} m_i e_i + \sum_{j < 2} n_i f_i = 0 \). So \((h, \gamma) \sim (0, 0)\), as desired.

31.8(6) \( \forall i < 2, (e_i, 0) \sim (0, \alpha_i) \) and \((f_i, 0) \sim (0, \beta_i)\).

In fact, for \((e_0, 0) \sim (0, \alpha_0)\) we have
\[
0 = 0 + \sum_{i < 2} 0 \alpha_i + \sum_{i < 2} 0 \beta_i, \quad \alpha_0 = 0 + \alpha_0 + 0 \alpha_1 + \sum_{j < 2} 0 \beta_j \quad \text{and}
\]
\[
e_0 + \sum_{i < 2} 0 \alpha_i + \sum_{j < 2} 0 \beta_j = 0 + e_0 + 0 e_1 + \sum_{j < 2} 0 \beta_j.
\]

Similarly for the other three cases.

Now we define \((i, j) < 2, g_{ij} = [(0, \varepsilon_{ij})] \approx = \Psi(e_i), \) using 31.8(6). Similarly, \( \forall j < 2, g_{ij} = \Psi(f_j) \). This gives (i) and (ii) of the proposition.

31.8(7) \( N \) is generated by \( \Psi[M] \cup \{g_{ij} : i, j < 2\} \).

In fact, if \((g, \gamma) \in P\), then \([\gamma] \approx = \Psi(g) + \sum_{i, j < 2} \gamma_{ij} g_{ij}\).

31.8(8) \( \forall i, j < 2, g_{ij} \text{ is distinguished} \).

For, suppose that \((i, j) < 2, g_{ij} + [(h, \gamma)] \approx = g_{ij}\). Thus \((h, \gamma + \varepsilon_{ij}) \approx (0, \varepsilon_{ij})\); say
\[
(0, \varepsilon_{ij}) = (g_0, \gamma_0) \sim \cdots \sim (g_s, \gamma_s) = (h, \gamma + \varepsilon_{ij}).
\]

Thus
\[
\varepsilon_{ij} = \varepsilon_{ij} + \sum_{i < 2} 0 \alpha_i + \sum_{j < 2} 0 \beta_j, \quad \gamma_1 = \varepsilon_{ij} + \sum_{i < 2} m_i^1 \alpha_i + \sum_{j < 2} n_i^1 \beta_j,
\]
\[
0 = g^1 + \sum_{i < 2} m_i^1 e_i + \sum_{j < 2} n_i^1 f_j.
\]

It follows that \( g^1 = 0 \) and \( m_i^1 = n_i^1 \) for all \( i < s \), and \( h = \gamma = 0 \). So 31.8(8) holds.

Now we prove (iv). Suppose that \( g, g' \in M, r \in 2^{\times 2} \omega \), and \( \Psi(g') = \Psi(g) + \sum_{i, j < 2} r_{ij} g_{ij} \). Thus \((g', 0) \sim (g_1, \gamma_1) \sim \cdots \sim (g, \sum_{i, j < 2} r_{ij} \varepsilon_{ij})\). By 31.8(4) we have
\[
0 = \varphi(\gamma_1) = \cdots = \varphi(r). \quad \text{So } r_{00} + r_{11} - r_{01} - r_{10} = 0.
\]
Case 1. $r_{00} \leq r_{01}$. Let $n_0 = 0$, $n_1 = r_{01} - r_{00}$, $m_0 = r_{00}$, $m_1 = r_{10}$
Case 2. $r_{01} < r_{00}$. Let $n_0 = r_{00} - r_{01}$, $n_1 = 0$, $m_0 = r_{01}$, $m_1 = r_{11}$.

Then in either case we have $\forall i, j < 2[r_{ij} = m_i + n_j]$. Now

$$
\sum_{i<2} m_i \Psi(e_i) + \sum_{j<2} n_j \Psi(f_j) = m_0 \Psi(e_0) + m_1 \Psi(e_1) + n_0 \Psi(f_0) + n_1 \Psi(f_1)
$$

$$
= m_0 g_{00} + m_0 g_{01} + m_1 g_{10} + m_1 g_{11}
$$

$$
+ n_0 g_{00} + n_0 g_{11} + n_1 g_{01} + n_1 g_{11}
$$

$$
= r_{00} g_{00} + r_{10} g_{01} + r_{10} g_{10} + r_{11} g_{11}
$$

This proves (iv).

Now for (v), suppose that $\Lambda : M \rightarrow M'$ is a morphism, $\forall i, j < 2[h_{ij} \in M']$, $\forall i < 2[\Lambda(e_i) = \sum_{j<2} h_{ij}]$, and $\forall j < 2[\Lambda(f_j) = \sum_{i<2} h_{ij}]$. Define $\Gamma : P \rightarrow M'$ by

$$
\Gamma(g, r) = \Lambda(g) + \sum_{i,j<2} r_{ij} h_{ij}.
$$

Clearly $\Gamma$ preserves $\cdot$. Now

$$
\Gamma(0, \alpha_0) = \Gamma\left(0, \sum_{j<2} \varepsilon_{0j}\right) = \Gamma\left(0, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\right) = h_{00} + h_{01} = \Lambda(e_0);
$$

$$
\Gamma(0, \alpha_1) = \Gamma\left(0, \sum_{j<2} \varepsilon_{1j}\right) = \Gamma\left(0, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\right) = h_{10} + h_{11} = \Lambda(e_1);
$$

$$
\Gamma(0, \beta_0) = \Gamma\left(0, \sum_{i<2} \varepsilon_{i0}\right) = \Gamma\left(0, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\right) = h_{00} + h_{10} = \Lambda(f_0);
$$

$$
\Gamma(0, \beta_1) = \Gamma\left(0, \sum_{i<2} \varepsilon_{i1}\right) = \Gamma\left(0, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\right) = h_{01} + h_{11} = \Lambda(f_1).
$$

Suppose now that $(g, \gamma) \sim (g', \gamma')$ as in (169), (170). Then

$$
\Gamma(g, \gamma) = \Gamma(g, 0) + \gamma(0, \gamma)
$$

$$
= \Lambda(g) + \Gamma(0, \delta) + \sum_{i<2} m_i \Gamma(0, \alpha_i) + \sum_{j<2} n_j \Gamma(0, \beta_j)
$$

$$
= \Lambda(g) + \Gamma(0, \delta) + \sum_{i<2} m_i \Lambda(e_i) + \sum_{j<2} n_j \Lambda(f_j)
$$

$$
= \Gamma(0, \delta) + \Lambda \left( g + \sum_{i<2} m_i e_i + \sum_{j<2} n_j f_j \right)
$$
Clearly R is the set of all countable nonempty subsets α of \(<<\omega \mathcal{P}\) such that α has C.P. and \(\forall m, n \in \omega \forall a, b \in \alpha [\text{dmn}(a) = m \text{ and dmn}(b) = n \rightarrow \sum_{i<m} a_i = \sum_{j<n} b_j].\)

**Proposition 32.1.** R is an m-monoid which contains \(\Delta \mathcal{P}\) as a submonoid.

**Proof.** The sum of elements of R is defined as in the definition of \(\Delta M\). If \(\alpha, \beta \in R\), \(a, b \in \alpha + \beta\), \(\text{dmn}(a) = m\), \(\text{dmn}(b) = n\), then there exist \(c, d \in \alpha\) and \(e, f \in \beta\) such that \(\text{dmn}(c) = \text{dmn}(e) = m\) and \(a = c + e\), and \(\text{dmn}(d) = \text{dmn}(f) = n\) and \(b = d + f\). Then \(\sum_{i<m} c_i = \sum_{i<n} d_i\) and \(\sum_{i<m} e_i = \sum_{i<n} f_i\), so \(\sum_{i<m} a_i = \sum_{i<n} (c_i + e_i) = \sum_{i<n} (d_i + f_i) = \sum_{i<n} b_i\). By the proof of Proposition 8.7, \(\alpha + \beta\) has C.P. Hence \(\alpha + \beta \in R\). Clearly R contains \(\Delta \mathcal{P}\) as a submonoid.

**Proposition 32.2.** If \(\alpha, \beta \in R\) and

\[
\forall m, n \in \omega \setminus \{0\} \forall a \in \alpha \forall b \in \beta [\text{dmn}(a) = m \text{ and dmn}(b) = n \rightarrow \sum_{i<m} a_i = \sum_{j<n} b_j],
\]

then \(\alpha \cup \beta \in R\).

For any \(a \in \mathcal{P}\) let

\[
\hat{a} = \{b \in \omega \mathcal{P} : \exists l \in \text{dmn}(b)[b_l = a \text{ and } \forall j \neq l[b_l = 0]\}.
\]

**Proposition 32.3.** If \(a \in \mathcal{P}\), then \(\hat{a} \in R\).

**Proof.** For the collection property, suppose that \(b \in \hat{a}\) and \(b \prec c\). Say \(b_l = a\) and \(\forall s \neq l[b_s = 0]\). Also say \(b\) has domain \(m\), \(c\) has domain \(n\). \(\lambda : m \to n\), and \(\forall j < n[c_j = \sum\{b_i : \lambda(i) = j\}\). Say \(\lambda(l) = k\). Then \(c_k = a\) and \(c_s = 0\) for all \(s \neq k\). So \(c \in \hat{a}\). Thus \(\hat{a}\) has C.P. Clearly the other condition for \(\hat{a} \in R\) holds.

**Proposition 32.4.** \(T(\hat{a}) = a\).

**Proposition 32.5.** If \(\alpha \in R\), \(b \in \alpha\), and \(\text{dmn}(b) = m\), then \(\sum_{i<m} b_i \subseteq \alpha\).
Proposition 32.6. If \( \Psi \) is a model of \( M \), then \( \forall e \in M [\Psi(e) = 0 \iff e = 0] \).

Proof. Suppose that \( \Psi \) is a model of \( M \) and \( e \in M \). Clearly \( e = 0 \) implies that \( \Psi(e) = 0 \). If \( e \neq 0 \) then, since 0 is distinguished, \( \forall f \in M [0 \neq f \rightarrow 0 = \Psi(0) \neq \Psi(f)] \).

Proposition 32.7. Let \( \Psi : M \rightarrow \mathcal{R} \) be a submodel of an \( m \)-monoid \( M \). For any \( e \in M \) let \( \Psi'(e) = T(\Psi(e)) \). Then \( \Psi' \) is a morphism from \( M \) to \( \mathcal{R} \).

Proof. If \( e = f = 0 \), clearly \( \Psi'(e + f) = \Psi'(e) + \Psi'(f) \). If \( e \neq 0 \) or \( f \neq 0 \), then by (M1) and (M2), \( \Psi(e) + \Psi(f) \) is a nonempty subset of \( \Psi(e + f) \). Hence \( T(\Psi(e + f)) = T(\Psi(e)) + T(\Psi(f)) \).

Proposition 32.8. Let \( E \) be a generating set for a countable \( m \)-monoid \( M \). Suppose that \( \Psi : E \rightarrow \mathcal{R} \) is a mapping such that

(i) \( \forall e \in E [\Psi(e) = 0 \iff e = 0] \).

(ii) \( \forall m, n \in \omega \) \( \forall e \in mE \forall f \in mE [\sum_{i < m} e_i = \sum_{j < n} f_j \rightarrow \sum_{i < m} T(\Psi(e_i)) = \sum_{j < n} T(\Psi(f_j))] \).

For each \( g \in M \) let

\[
\Psi^+(g) = \bigcup \left\{ \sum_{i < m} \Psi(e_i) : m \in \omega, e \in mE, \sum_{i < m} e_i = g \right\}.
\]

Then \( \Psi^+ \) is a submodel of \( M \), and \( \forall e \in M [\Psi(e) \subseteq \Psi^+(e)] \).

Proof. Assume the hypotheses. Suppose that \( g \in M \); we want to show that \( \Psi^+(g) \in \mathcal{R} \). Clearly \( \Psi^+(g) \) is countable and nonempty. Clearly also \( \sum_{i < m} \Psi(e_i) \in \mathcal{R} \) for any \( m \in \omega \) and \( e \in mE \), so \( \Psi^+(g) \subseteq \langle\langle \omega, \mathcal{R} \rangle \). C.P. is closed under unions, so \( \Psi^+(g) \) has C.P. Now suppose that \( m, n \in \omega \), \( a, b \in \Psi^+(g) \), \( \mathrm{dmm}(a) = m \), and \( \mathrm{dmm}(b) = n \). Say \( a \in \sum_{i < p} \Psi(e_i) \) with \( e \in \langle\langle \omega, \mathcal{R} \rangle \) and \( \sum_{i < p} e_i = g \); and \( b \in \sum_{i < q} \Psi(e'_i) \) with \( e' \in \langle\langle \omega, \mathcal{R} \rangle \) and \( \sum_{i < q} e'_i = g \).

By (ii), \( \sum_{i < p} T(\Psi(e_i)) = \sum_{i < q} T(\Psi(e'_i)) \). By Proposition 32.1, \( \sum_{i < p} \Psi(e_i) \cup \sum_{i < q} \Psi(e'_i) \in \mathcal{R} \), and by Proposition 32.2, \( \sum_{i < p} \Psi(e_i) \subseteq \sum_{i < q} \Psi(e'_i) \subseteq \mathcal{R} \). Hence \( \sum_{i < m} a_i = \sum_{i < n} b_i \). So \( \Psi^+(g) \in \mathcal{R} \).

Clearly \( \Psi^+(0) = 0 \). If \( g \neq 0 \), choose \( m \in \omega \) and \( e \in mE \) so that \( \sum_{i < m} e_i = g \). Say \( i < m \) and \( e_i \neq 0 \). Then by (i), \( \Psi(e_i) \neq 0 \). So \( \sum_{j < m} \Psi(e_j) \neq 0 \). Thus \( \Psi^+(g) \neq 0 \). Thus (M1) holds for \( \Psi^+ \).
Now suppose that $g, h \in M$ and $x \in \Psi^+(g) + \Psi^+(h)$. Then there exist $y, z$ with equal domains such that $x = y + z$, $y \in \Psi^+(g)$, and $z \in \Psi^+(h)$. Say $y \in \sum_{i<m} \Psi(e_i)$ with $m \in \omega$, $e \in \mathcal{V}E$, and $\sum_{i<n} e_i = g$; and $z \in \sum_{i<n} \Psi(e'_i)$ with $n \in \omega$, $e' \in \mathcal{V}E$, and $\sum_{i<n} e'_i = h$. Clearly then $x \in \Psi^+(g + h)$.

Clearly then $x \in \Psi^+(g + h)$. \hfill \Box

The mapping $\Psi^+$ of Proposition 32.8 is called the closure of $\Psi$.

**Proposition 32.9.** Suppose that $\Psi$ is a submodel of $M$, and $\Theta : M \to \mathcal{R}$ satisfies $\forall e \in M[\Psi(e) \subseteq \Theta(e)]$. Then $\Theta$ satisfies (i) and (ii) of Proposition 32.8.

**Proof.** (i) is clear. (ii) holds since for any $e \in M$ $\Psi(e) \subseteq \Theta(e)$ and so $T(\Psi(e)) = T(\Theta(e))$.

### 33. Expansions of submodels

For any $\alpha \subseteq \mathcal{V} \mathcal{P}$ let

$$\text{Dom}(\alpha) = \{ a \in \mathcal{P} : a \text{ is an entry in some } x \in \alpha \}.$$

If $M$ is a collection of subsets of $\mathcal{V} \mathcal{P}$, then

$$\text{Dom}(M) = \bigcup_{\alpha \in M} \text{Dom}(\alpha).$$

**Proposition 33.1.** If $M$ is a submonoid of $\mathcal{R}$, then $\text{Dom}(M)$ is a submonoid of $\mathcal{P}$.

**Proof.** Suppose that $M$ is a submonoid of $\mathcal{R}$ and $a, b \in \text{Dom}(M)$. Say $a \in \text{Dom}(\alpha)$ and $b \in \text{Dom}(\beta)$ with $\alpha, \beta \in M$. Say $a = x_i$ with $x \in \alpha$ and $b = y_j$ with $y \in \beta$. Say $x$ has domain $m$. Define $\lambda : m \to 2$ by $\lambda(k) = 0$ if $k \neq i$ and $\lambda(i) = 1$. Let $c_0 = \sum \{x_k : k < m, k \neq i\}$ and $c_1 = i$. Then $\forall j < 2|c_j = \sum \{x_k : \lambda(k) = j\}$. So $\langle \sum \{x_k : k < m, k \neq i\}, x_i \rangle \in \alpha$. Similarly, $\langle \sum \{y_k : k < m, k \neq j\}, y_j \rangle \in \beta$. It follows that $x_i + y_j$ is an entry in a member of $\alpha + \beta$. \hfill \Box

Let $M$ and $N$ be $m$-monoids, with $M$ a submonoid of $N$. If $\Psi$ is a submodel of $M$ and $\Theta$ is a submodel of $N$, then $\Theta$ is an expansion of $\Psi$ iff

(E) $\forall e \in M[\Theta(e) \cap \mathcal{V}\mathcal{P}(\text{Dom}(\Psi[M])) = \Psi(e)]$.

**Proposition 33.2.** If $\Theta$ is an expansion of $\Psi$, then $\forall e \in M[\Psi(e) \subseteq \Theta(e) \text{ and } T(\Psi(e)) = T(\Theta(e))]$.

**Proof.** Assume that $\Theta$ is an expansion of $\Psi$. Then $\forall e \in M[\Psi(e) \subseteq \Theta(e) \subseteq \mathcal{R}]$, so the proposition follows from the definition of $\mathcal{R}$. \hfill \Box

**Proposition 33.3.** If $\Theta$ is an expansion of $\Psi$, then $\text{Dom}(\Psi[M]) \subseteq \text{Dom}(\Theta[M])$.
Proof.

\[ \text{Dom}(\Psi[M]) = \bigcup_{\alpha \in \Psi[M]} \text{Dom}(\alpha) = \bigcup_{e \in M} \text{Dom}(\Psi(e)) \]
\[ \subseteq \bigcup_{e \in M} \text{Dom}(\Theta(e)) = \bigcup_{\alpha \in \Theta[M]} \text{Dom}(\alpha) = \text{Dom}(\Theta[M]). \]

\[ \square \]

**Proposition 33.4.** If \( \Theta \) is an expansion of \( \Psi \), then \( \forall e, f \in M[\Psi(e) \neq \Psi(f) \rightarrow \Theta(e) \neq \Theta(f)] \).

**Proof.** Suppose that \( \Theta \) is an expansion of \( \Psi \), \( e, f \in M \), and \( \Theta(e) = \Theta(f) \). Then \( \Psi(e) = \Theta(e) \cap <\omega \Psi[M] = \Theta(f) \cap <\omega \Psi[M] = \Psi(f) \).

\[ \square \]

**Proposition 33.5.** Suppose that:
(i) \( M_0, M_1, M_2 \) are \( m \)-monoids.
(ii) \( M_0 \) is a submonoid of \( M_1 \) and \( M_1 \) is a submonoid of \( M_2 \).
(iii) \( \Theta_0, \Theta_1, \Theta_2 \) are submodels of \( M_0, M_1, M_2 \) respectively.
(iv) \( \Theta_1 \) is an expansion of \( \Theta_0 \) and \( \Theta_2 \) is an expansion of \( \Theta_1 \). Then \( M_0 \) is a submonoid of \( M_2 \), and \( \Theta_2 \) is an expansion of \( \Theta_0 \).

**Proof.** Clearly \( M_0 \) is a submonoid of \( M_2 \). For any \( e \in M_0 \) we have
\[ \Theta_2(e) \cap <\omega (\text{Dom}(\Theta_0[M_0])) \subseteq \Theta_2(e) \cap <\omega (\text{Dom}(\Theta_1[M_1])) = \Theta_1(e). \]
Hence
\[ \Theta_2(e) \cap <\omega (\text{Dom}(\Theta_0[M_0])) \subseteq \Theta_1(e) \cap <\omega (\text{Dom}(\Theta_0[M_0])) = \Theta_0(e). \]
On the other hand, clearly \( \Theta_0(e) \subseteq \Theta_2(e) \cap <\omega (\text{Dom}(\Theta_0[M_0])) \). So \( \Theta_2 \) is an expansion of \( \Theta_0 \).

\[ \square \]

**Proposition 33.6.** Let \( M_0 \subseteq M_1 \subseteq \cdots \) be an \( \omega \)-chain of submonoids of an \( m \)-monoid \( M \), with \( \bigcup_{k \in \omega} M_k = M \). Suppose that each \( M_{k+1} \) is a distinguished extension of \( M_k \). For each \( k \in \omega \) let \( \Psi_k \) be a submodel of \( M_k \), and suppose that \( \forall k \in \omega[\Psi_{k+1} \text{ is an expansion of } \Psi_k] \). For each \( k \in \omega \) and \( e \in M_k \) let \( \Theta_k(e) = \bigcup_{n \geq k} \Psi_n(e) \). Then the following hold:
(i) \( \forall k \in \omega[\Theta_k \text{ is a submodel of } M_k] \).
(ii) \( \forall k \in \omega[\Theta_k \text{ is an expansion of } \Psi_k] \).
(iii) \( \forall k, n \in \omega [k \leq n \rightarrow \Theta_k = \Theta_n \upharpoonright M_k] \).
(iv) \( \forall k \in \omega[\forall n \geq k[\Psi_n \text{ is a model} \rightarrow \Theta_k \text{ is a model}]] \).

**Proof.** Assume the hypotheses. (i): (M1) is clear. For (M2), suppose that \( e, f \in M_k \) and \( x \in \Theta_k(e) + \Theta_k(f) \). So there exist \( y \in \Theta_k(e) \) and \( z \in \Theta_k(f) \) such that \( x = y + z \). Say \( k \leq n \) and \( y \in \Psi_n(e) \) and \( z \in \Psi_n(f) \). So \( x \in \Psi_n(e) + \Psi_n(f) \subseteq \Psi_n(e + f) \subseteq \Theta_k(e + f) \). This proves (M2).

For (ii), assume that \( k \in \omega \). Suppose that \( e \in M_k \). Then
\[ \Theta_k(e) \cap <\omega (\text{Dom}(\Psi_k[M_k])) = \bigcup_{n \geq k} (\Psi_n(e) \cap <\omega (\text{Dom}(\Psi_k[M_k]))) = \Psi_k(e). \]

148
For (iii), suppose that \( n, k \in \omega \) and \( k \leq n \). Then for any \( e \in M_k \), if \( k \leq m \leq n \) then \( \Psi_m(e) \subseteq \Psi_n(e) \), and so \( \Theta_n(e) = \bigcup_{m \geq n} \Psi_m(e) = \bigcup_{m \geq k} \Psi_m(e) = \Theta_k(e) \). For (iv), assume that \( k \in \omega \) and \( \forall n \geq k [\Psi_n \text{ is a model}] \). Suppose that \( e \in M_{k_0} \) and \( f \in M_k \), with \( e \neq f \). Say \( \Theta_k(e) = \Psi_m(e) \) and \( \Theta_k(f) = \Psi_m(f) \). Then \( e \in M_{m_0} \), so \( \Psi_m(e) \neq \Psi_m(f) \).

### Proposition 33.7

Suppose that the hypotheses of Proposition 33.6 hold. For each \( e \in M \) let \( \Theta_\omega(e) = \Theta_k(e) \), where \( k \) is minimum such that \( e \in M_k \). Then

(i) \( \Theta_\omega \) is a submodel of \( M \).

(ii) \( \forall k < \omega [\Theta_\omega \text{ is an expansion of } \Psi_k] \).

**Proof.** Assume the hypotheses. For (i), clearly \( \Theta_\omega(e) = 0 \) iff \( e = 0 \). Now take \( e, f \in M \). Say \( k \) minimum such that \( e \in M_k \), and \( l \) is minimum such that \( f \in M_l \). Say \( k \leq l \). Take any \( u \in \Theta_\omega(e) + \Theta_\omega(f) \). Then \( \Theta_\omega(e) = \Theta_k(e) \) and \( \Theta_\omega(f) = \Theta_l(f) \). Say \( u = x + y \) with \( x \in \Theta_k(e) \) and \( y \in \Theta_l(f) \). Then \( x \in \Theta_l(e) \), so \( u \in \Theta_l(e) + \Theta_l(f) \subseteq \Theta_l(e + f) \).

Say \( s \) is minimum such that \( e + f \in M_s \). So \( s \leq l \). \( \Theta_s(e + f) = \bigcup_{m \geq s} \Psi_m(e + f) \); if \( s \leq m \leq l \) then \( \Psi_m(e + f) \leq \Psi_l(e + f) \). So \( \Theta_s(e + f) = \Theta_l(e + f) \). Hence (i) holds.

For (ii), suppose that \( k \in \omega \) and \( e \in M_k \). Let \( s \) be minimum such that \( e \in M_s \). Then

\[
\Theta_\omega(e) \cap <\omega(\text{Dom}(\Psi_k[M_k])) = \Theta_s(e) \cap <\omega(\text{Dom}(\Psi_k[M_k])) = \Theta_k(e) \cap <\omega(\text{Dom}(\Psi_k[M_k])) = \Psi_k(e).
\]

The submodel \( \Theta_\omega \) is called the limit of \( \langle \Psi_k : k \in \omega \rangle \).

### Proposition 33.8

Under the notation of Proposition 33.7, if all \( \Psi_k \) are models, then \( \Theta_\omega \) is a model.

**Proof.** Suppose that \( e \in M_s \) and \( f \in M \). Say \( e \in M_k \) and \( f \in M_k \). Clearly \( e \in M_{k_0} \). Hence \( \Psi_k(e) \neq \Psi_k(f) \). By Proposition 33.7(ii), clearly \( \Theta_\omega(e) \neq \Theta_\omega(f) \).

### 34. Second reduction step

#### Proposition 34.1

Let \( M_0 \subseteq M_1 \subseteq \cdots \) be a chain of countable submonoids of an \( m \)-monoid \( M \), and suppose that for each \( k \in \omega \), \( \Psi_k \) is a model of \( M_k \), with \( \Psi_{k+1} \) an expansion of \( \Psi_k \). Also assume:

(i) Each \( M_k \) is distinguished, and \( M_{k+1} \) is a distinguished extension of \( M_k \).

(ii) \( \forall k \in \omega \forall e_0, e_1, f_0, f_1 \in M_k \) \( e_0 + e_1 = f_0 + f_1 \rightarrow \exists g \in 2^{\times 2} M_{k+1} \forall i < 2 \) then \( e_i = \sum_{j < 2} g_{ij} \) and \( \forall j < 2 \sum_{i < 2} g_{ij} \).

(iii) \( \forall k \in \omega \forall e \in M_k \forall n \in \omega \forall e_0, \ldots, e_{n-1} \in \Psi_k(e) \exists e'_0, \ldots, e'_{n-1} \in M_{k+1} \) \( e = \sum_{i < n} e_i \) and \( \forall i < n [T(\Psi_{k+1}(e'_i)) = a_i] \).

(iv) \( \Psi_0(M_0^*) \subseteq 2^1 \).

Then \( M_0^* \) is isomorphic to a subsemigroup of \( \text{NBA} \).

**Proof.** We may assume that \( M = \bigcup_{k \in \omega} M_k \). Hence \( M \) is a countable \( m \)-monoid. By Proposition 19.7 and (ii), \( M \) is an \( r \)-monoid.

149
We claim that $M$ is atomless. For, suppose that $e \in M^*$. Say $e \in M_k$. Then $\Psi_k(e) \neq 0$. Now $\Psi_k(e) \in \mathcal{P}^1 = \Delta \mathcal{P}$, so by the splitting property there is an $(a_0, a_1) \in \Psi_k(e)$ with $a_0 \neq 0 \neq a_1$. Then (iii) shows that $e$ is not an atom.

By Propositions 33.7 and 33.8, $\Theta_\omega$ is a model of $M$ and is an expansion of each $\Psi_k$. For each $e \in M$ let

$$\Lambda(e) = \left\{ \Theta_\omega \circ f : \sum_{i<n} f_i = e \right\}.$$ 

We are going to apply Proposition 29.1 to $\Lambda[M^*_0]$.

Let $N = \{ \Lambda(e) : e \in M \}$. Note that $\Lambda = (\Delta \Theta_\omega) \circ \delta_M$. In fact, if $e, f \in N$, then

$$(\Delta \Theta_\omega) (\delta_M(e)) = (\Delta \Theta_\omega) \left( \left\{ b : \sum_{i<n} b_j = e \right\} \right) = \left\{ \Theta_\omega \circ f : \sum_{i<n} f_i = e \right\}.$$ 

Now by Proposition 19.12, $\delta_M$ is a morphism from $M$ to $\Delta M$. $\Theta_\omega$ is a morphism from $M$ to $\mathcal{P}^1$. Since $M$ is distinguished, the proof of Proposition 25.9 gives: $\Delta \Theta_\omega$ is a morphism from $\Delta M$ to $\Delta \mathcal{P}^1$. Hence $\Lambda$ is a morphism from $M$ to $N$, and $N$ is a submonoid of $\Delta \mathcal{P}^1$. Since $\Lambda$ maps $M$ onto $N$, it follows that $N$ has the refinement property. Now suppose that $A \in N$ and $(\alpha_0, \alpha_1) \in A$. Then there exist $g \in M$ such that $A = \Lambda(g)$, and $f_0, f_1 \in M$ with $f_0 + f_1 = g$, $\alpha_0 = \Omega_\omega(f_0)$, and $\alpha_1 = \Theta_\omega(f_1)$. Say $A_0 = \Lambda(f_0)$ and $A_1 = \Lambda(f_1)$. So $A = A_0 + A_1$. Moreover,

$$T(A_0) = T(\Lambda(f_0)) = \sum_{i<m} \Omega_\omega(g_i) = \Omega_\omega \left( \sum_{i<m} g_i \right) = \Omega_\omega(f_0) = \alpha_0,$$

and similarly $T(A_1) = \alpha_1$. This checks (ii) of Proposition 29.1.

34.1(1) $N \subseteq \mathcal{P}^2$.

In fact, if $e \in M$, then by the above, $\Lambda(e) \in \Delta \mathcal{P}^1$, so we just need to show that $\Lambda(e)$ has the new heirarchy property. Suppose that $m, n \in \omega \setminus \{0\}$, $\langle \alpha, \beta_1, \ldots, \beta_{m-1} \rangle \in \Lambda(e)$, and $\langle a_0, \ldots, a_{n-1} \rangle \in \alpha$. Then there are $f, g_1, \ldots, g_{m-1}$ such that $f + \sum_{i<m} g_i = e$, $\alpha = \Theta_\omega(f)$, and $\forall i < m[\beta_i = \Theta_\omega(g_i) = \hat{\beta}_i]$. Since $\langle a_0, \ldots, a_{n-1} \rangle \in \alpha$, there is a $k < \omega$ such that $f \in M_k$ and $\langle a_0, \ldots, a_{n-1} \rangle \in \Psi_k(f)$. By (iii), there are $e_0, \ldots, e_{n-1} \in M_{k+1}$ such that $\sum_{i<n} e_i = f$ and $\forall i < n[T(\Psi_{k+1}(e_i)) = a_i]$. Hence $\forall i < n[T(\Omega_\omega(e_i)) = a_i]$. Now $\sum_{i<n} e_i + \sum_{i<m} g_i = e$, so $\langle \Theta_\omega(e_0), \ldots, \Theta_\omega(e_{n-1}), \Theta_\omega(g_0), \ldots, \Theta_\omega(g_{m-1}) \rangle \in \Lambda(e)$. This checks the new heirarchy property; so 34.1(1) holds.

34.1(2) $\Lambda[M^*_0] \subseteq \mathcal{P}^2$.

In fact, let $e \in M^*_0$. By (iv), $\Psi_0(e) \in \mathcal{P}^1$, so $T(\Psi_0(e)) = [\omega^c]_{ae}$. Also, $T(\Lambda(e)) = \Omega_\omega(e) \supseteq \Psi_0(e)$. Hence $T^2(\Lambda(e)) = [\omega^c]_{ae}$. This proves (182).

Now by Proposition 29.1, $\Lambda[M^*_0]$ is isomorphic to a subsemigroup of $\mathbf{NBA}$. Hence it suffices now to show that $\Lambda$ is injective. Suppose that $e, f \in M$ and $\Lambda(e) = \Lambda(f)$. Since $M$ is distinguished, we can write $e = \sum_{i<m} e_i$ with each $e_i \in M_s$. Then $\Theta_\omega \circ e \in \Lambda(e) = \Lambda(f)$, so there exist $g_i$ such that $\sum_{i<m} g_i = f$ and $\Theta_\omega \circ e = \Theta_\omega \circ g$. Since $\Omega_\omega$ is a model, $\forall i < m[g_i = e_i]$, so $e = f$. 

\[\square\]
35. Two expansions

Proposition 35.1. Suppose that $M$ is a countable $m$-monoid, $\Theta$ is a submodel of $M$, $e \in M^*$, $\langle a, b_0, \ldots, b_{m-1} \rangle \in \Theta(e)$, $a \neq 0$, and $\exists i < m[b_i \neq 0]$. Then there exist a distinguished extension $N$ of $M$ and an expansion $\Theta'$ of $\Theta$ to a submodel of $N$ such that:

(i) $\exists e_0, e_1[e_0, e_1$ are distinguished in $N$, $e = e_0 + e_1$, and $N$ is generated by $M \cup \{e_0, e_1\}$.

(ii) $\langle a \rangle \in \Theta'(e_0)$ and $\langle b_0, \ldots, b_{m-1} \rangle \in \Theta'(e_1)$.

(iii) $\Theta' | M = \Theta$.

Proof. By Proposition 30.7 let $N$ be a distinguished extension of $M$ with $e_0, e_1 \in N$ such that $e_0, e_1$ are distinguished and nonzero, $e = e_0 + e_1$, $N$ is generated by $M \cup \{e_0, e_1\}$, and Proposition 30.7(iii),(iv) hold. Define $\Psi : M \cup \{e_0, e_1\} \rightarrow \mathcal{P}$ by

$$
\Psi(f) = \Theta(f) \text{ if } f \in M;
$$
$$
\Psi(e_0) = \hat{a};
$$
$$
\Psi(e_1) = \sum_{i < m} \hat{b}_i.
$$

35.1(1) The hypotheses of Proposition 32.8 hold for $\Psi$.

Proposition 32.8(i) is clear. Now by Proposition 32.7, $T \circ \Theta$ is a morphism from $M$ to $\mathcal{P}$. Moreover,

35.1(2) There is a morphism $\Lambda' : N \rightarrow \mathcal{P}$ such that $T \circ \Theta \subseteq \Lambda'$, $\Lambda'(e_0) = T(\hat{a}) = a$, and $\Lambda'(e_1) = T(\sum_{i < m} \hat{b}_i) = \sum_{i < m} b_i$.

This holds by Proposition 30.7(iv), since $T(\Theta(e)) = a + \sum_{i < p} b_i$.

For Proposition 32.8(ii), suppose $p, n \in \omega$, $u \in p(M \cup \{e_0, e_1\})$, $v \in n(M \cup \{e_0, e_1\})$, and $\sum_{i < p} u_i = \sum_{j < n} v_j$. Say $\sum_{i < p} u_i = g + pe_0 + qe_1$ and $\sum_{j < n} v_j = g' + p'e_0 + q'e_1$ with $g, g' \in M$. Then

$$
\sum_{i < p} T(\Psi(u_i)) = T(\Psi(g')) + pT(\Psi(e_0)) + qT(\Psi(e_1))
$$

$$
= T(\Theta(g)) + pa + q \sum_{i < m} b_i
$$

$$
= \Lambda'(g) + p \Lambda'(e_0) + q \Lambda'(e_1)
$$

$$
= \Lambda'(g + pe_0 + qe_1) = \Lambda'\left(\sum_{i < p} u_i\right) = \Lambda'\left(\sum_{j < n} v_j\right)
$$

$$
= \Lambda'(g') + p' \Lambda'(e_0) + q' \Lambda'(e_1)
$$

$$
= T(\Psi(g')) + p'T(\Psi(e_0)) + q'T(\Psi(e_1))
$$

$$
= \sum_{j < n} T(\Psi(v_j)).
$$
Thus Proposition 32.8(ii) holds. So (183) holds. We thus get a submodel $\Psi^+$ of $N$.

To see that $\Psi^+ \upharpoonright M = \Theta$, suppose that $g \in M$. Take $e'$ such that $\sum_{i<m} e'_i = g$. Then we can write $\sum_{i<m} e'_i = f + me_0 + ne_1$ with $f \in M$. By Proposition 30.7(iii) we have $m = n$ and $\sum_{i<m} e'_i = f + me$. Hence $\sum_{i<m} \Psi(e'_i) = \sum_{i<m} e'_i = \Psi f + \Psi(me) = \Psi(g) = \Theta(g)$. So $\Psi^+ \upharpoonright M = \Theta$. In particular, $\Psi^+$ is an expansion of $\Theta$.

Now $\langle a \rangle \in \hat{a} = \Psi(e_0) \subseteq \Psi^+(e_0)$. Also, $\langle b_0, \ldots, b_{m-1} \rangle \in \sum_{i<m} \hat{b}_i = \Psi(e_1) \subseteq \Psi^+(e_1)$.

**Proposition 35.2.** Suppose that $M$ is a countable $m$-monoid, $\Theta$ is a submodel of $M$, and $e_0 + e_1 = f_0 + f_1$ in $M^*$.

Then there exist a distinguished extension $N$ of $M$ and elements $g_{00}, g_{01}, g_{10}, g_{11}$ of $N$ such that $N$ is generated by $M \cup \{g_{00}, g_{01}, g_{10}, g_{11}\}$, the $g_{ij}$ are distinguished elements, $\forall i < 2[e_i = \sum_{j<2} g_{ij}]$, and $\forall j < 2[f_j = \sum_{i<2} g_{ij}]$. Also, there is a submodel $\Lambda$ of $N$ such that $\forall e \in M[\Theta(e) \subseteq \Lambda(e)]$.

**Proof.** We take $N$ and $g_{ij}$ as in Proposition 31.8. So it remains only to find $\Lambda$. For all $i < 2$ let $a_i = T(\Theta(e_i))$ and $b_i = T(\Theta(f_i))$. Then $a_i, b_i \in \mathcal{P}$ for each $i < 2$. By Proposition 26.6 there is a $c : 2 \times 2 \rightarrow \mathcal{P}$ such that $\forall i < 2[a_i = \sum_{j<2} c_{ij}]$ and $\forall j < 2[b_j = \sum_{i<2} c_{ij}]$.

Now define $\Psi : M \cup \{g_{00}, g_{01}, g_{10}, g_{11}\} \rightarrow \mathcal{P}$ by:

$$
\Psi(x) = \begin{cases} 
\Theta(x) & \text{if } x \in M, \\
\hat{c}_{ij} & \text{if } x = g_{ij}.
\end{cases}
$$

35.2(1) The hypotheses of Proposition 32.8 hold for $\Psi$.

In fact, Proposition 32.8(i) is clear. For Proposition 32.8(ii), assume that $m, n \in \omega$, $h \in {}^m E$, $f \in {}^n E$, and $\sum_{i<m} h_i = \sum_{j<n} f_j$, where $E = M \cup \{g_{00}, g_{01}, g_{10}, g_{11}\}$. Now $T \circ \Theta$ is a morphism from $M$ into $\mathcal{P}$, $\forall i < 2[T(\Theta(e_i)) = a_i = \sum_{j<2} c_{ij}]$, and $\forall j < 2[T(\Theta(f_j)) = b_j = \sum_{i<2} c_{ij}]$. Hence by Proposition 31.8(v) there is an extension $\Lambda$ of $T \circ \Theta$ to a morphism of $N$ into $\mathcal{P}$ such that $\forall i, j < 2[\Lambda(g_{ij}) = c_{ij}]$. Note that for any $i, j < 2$, $T(\Psi(g_{ij})) = c_{ij} = \Lambda(g_{ij})$. So $\Lambda$ is an extension of $T \circ \Psi$. Hence

$$
\sum_{i<m} T(\Psi(h_i)) = \sum_{i<m} \Lambda(h_i) = \Lambda \left( \sum_{i<m} h_i \right) = \Lambda \left( \sum_{j<n} f_j \right) = \sum_{j<n} \Lambda(f_j) = \sum_{j<n} T(\Psi(f_j)).
$$

Thus 35.2(1) holds.

By Proposition 32.8, $\Psi^+$ is a submodel of $N$. Clearly $\forall e \in M[\Theta(e) \subseteq \Psi^+(e)]$. 

\[\Box\]
36. The basic lemma

Proposition 36.1. Suppose that $M$ is a countable $m$-monoid, $\Psi$ is a submodel of $M$, and $e, f \in M$. Suppose that $e \not\leq f$ and $a = T(\Psi(e))$.

Then there exist $a_0, a_1 \in \mathcal{P}$ such that $a = a_0 + a_1$ and there is an expansion $\Theta$ of $\Psi$ such that $(a_0, a_1) \in \Theta(e) \setminus \Theta(f)$.

Proof. $\text{Dom}(\Psi[M])$ is countable, so by Proposition 26.7 there exist $a_0, a_1$ such that $a = a_0 + a_1$ and $a_0$ is linearly independent from $\text{Dom}(\Psi[M])$, with $a_0 \not\in [\omega^c]_e$. Define $\Lambda : M \rightarrow \mathcal{R}$ by:

$$\Lambda(g) = \begin{cases} 
\Psi(g) & \text{if } g \neq e, \\
\Psi(e) \cup (\hat{a}_0 + \hat{a}_1) & \text{if } g = e.
\end{cases}$$

Clearly $\Lambda : M \rightarrow \mathcal{R}$. Suppose that $m, n \in \omega, h \in mM, f' \in nM$, and $\sum_{i<n} h_i = \sum_{j<n} f'_j$. Say $\sum_{i<m} h_i = x + sa$ with $x \in M \setminus \{a\}$, and $\sum_{j<n} f'_j = y + ta$ with $y \in M \setminus \{a\}$. Then

$$\sum_{i<m} T(\Lambda(h_i)) = T(\Lambda(x)) + sT(\Lambda(e)) = T(\Psi(x)) + sT(\Psi(e) \cup (\hat{a}_0 + \hat{a}_1))$$

$$= T(\Lambda(x)) + sa = T(\Lambda(x)) + sT(\lambda(e))$$

$$= T(\Lambda(x + se)) = T(\Lambda(y + te))$$

$$= \sum_{j<n} T(\Lambda(f'_j)).$$

By Proposition 32.8, let $\Theta$ be the closure of $\Lambda$; so $\Theta$ is a submodel of $M$. Moreover, $(a_0, a_1) \in \hat{a}_0 + \hat{a}_1 \subseteq \Lambda(e) \subseteq \Theta(e)$. If $\sum_{i<m} u_i = f$, then each $u_i \neq e$, and so $\Psi(f) = \sum_{i<m} \Psi(u_i) = \sum_{i<m} \Lambda(u_i)$; hence $\Psi(f) = \Theta(f)$. Since $a_0 \not\in \text{Dom}(\Psi[M])$, it follows that $(a_0, a_1) \not\in \Theta(f)$.

It remains only to show that $\Theta$ is an expansion of $\Psi$. Suppose that $v \in M$. Obviously $\Psi(v) \subseteq \Theta(v) \cap \omega^c(\text{Dom}(\Psi[M]))$. Now suppose that $b \in \Theta(v)$, with $\text{dmm}(b) = m$ and $\forall i < m[b_i \in \text{Dom}(\Psi[M])]$. By the definition of closure, there is an $m \in \omega$ and $w \in mM$ such that $\sum_{i<m} w_i = v$ and $b \in \sum_{i<m} \Lambda(w_i)$.

Case 1. $\forall i < m[w_i \neq e]$. Then $\sum_{i<m} \Lambda(w_i) = \sum_{i<m} \Psi(w_i) = \Psi(\sum_{i<m} w_i) = \Psi(v)$.

So $b \in \Psi(v)$.

Case 2. $\exists i < m[w_i = e]$. Then there exist $y \in M \setminus \{e\}$ and $p \in \omega$ such that $\sum_{i<m} w_i = y + pe$. Then $b \in \Lambda(y) + p\Lambda(e) = \Psi(y) + p(\Psi(e) \cup (\hat{a}_0 + \hat{a}_1))$. Thus we may assume that $b = b' + b_1 + \ldots + b_q + b + b_q + \ldots + b_p$, with $b' \in \Psi(y), b_1, \ldots, b_q \in \Psi(e)$, and $b_q + b_{q+1} + \ldots + b_p \in \hat{a}_0 + \hat{a}_1$. For each $r \in \{q, q+1, \ldots, p\}$ choose $c^e_r, c^f_r$ such that $b_r = c^e_r + c^f_r$, $c^e_r \in \hat{a}_0, c^f_r \in \hat{a}_1$. Moreover, there are $d', a_1, \ldots, a_p \in M$ and $\gamma' \in \Psi(d'), \gamma_1 \in \Psi(d_1), \ldots, \gamma_p \in \Psi(d_p)$ such that $b'$ is an entry in $\gamma'$ and $\forall r \in \{1, \ldots, p\}[b_r$ is an entry in $\gamma_r]$. Thus for all $r = q + 1, \ldots, p$, $c^e_r + c^f_r$ is an entry in $\gamma_r \subseteq \text{Dom}(\Psi[M])$. It follows that $a_0$ and $a_1$ appear at the same place in $c^e_0$ and $c^f_1$. Thus for all $r = q + 1, \ldots, p$, $b_r \in \hat{a}$. Now clearly $\hat{a} \subseteq \Psi(e)$, so for all $r = q + 1, \ldots, p$, $\Psi(e)$. Hence $b \in \Psi(y) + p\Psi(e) = \Psi(y + pe) = \Psi(\sum_{i<m} w_i) = \Psi(v)$. \qed

Proposition 36.2. Suppose that $M$ is a countable $m$-monoid, $\Psi$ is a submodel of $M$, $e_0, e_1 \in M$, and $a \in \Psi(e_0 + e_1)$. Then there is a submodel $\Theta$ of $M$ such that $\forall e \in M[\Psi(e) \subseteq \Theta(e)]$ and $a \in \Theta(e_0) + \Theta(e_1)$. 153
Proof. We proceed by induction on the length $n$ of $a$. For $n = 1$, say $a = \langle x \rangle$. Then $x = T(\Psi(e_0 + e_1)) = T(\Psi(e_0)) + T(\Psi(e_1)) = T(\Psi(e_0) + \Psi(e_1))$, so $a \in \Psi(e_0) + \Psi(e_1)$.

Now suppose inductively that $n > 1$. Write $a = \langle a_i : i < n \rangle$. We may assume that $a_0 \neq 0$ and $\exists i[0 < i < n \text{ and } a_i \neq 0]$. Then by Proposition 35.1 there exist a distinguished extension $N$ of $M$ and an expansion $\Theta'$ of $\Theta$ to a submodel of $N$ such that:

36.2(1) $\exists f_0, f_1, f_0, f_1$ are distinguished in $N$, $e_0 + e_1 = f_0 + f_1$, and $N$ is generated by $M \cup \{f_0, f_1\}$.

36.2(2) $\langle a_0 \rangle \in \Theta'(f_0)$ and $\langle a_1, \ldots, a_{n-1} \rangle \in \Theta'(f_1)$.

36.2(3) $\Theta' \upharpoonright M = \Theta$.

Now by Proposition 35.2 there exist a distinguished extension $N'$ of $N$ and elements $g_{00}, g_{01}, g_{10}, g_{11}$ of $N'$ such that $N'$ is generated by $N \cup \{g_{00}, g_{01}, g_{10}, g_{11}\}$, the $g_{ij}$ are distinguished elements, $\forall i < 2[e_i = \sum_{j < 2} g_{ij}]$, and $\forall j < 2[f_j = \sum_{i < 2} g_{ij}]$. Also, there is a submodel $\Lambda$ of $N'$ such that $\forall e \in N[\Theta'(e) \subseteq \Lambda(e)]$. Hence $\langle a_1, \ldots, a_{n-1} \rangle \in \Theta'(f_1) \subseteq \Lambda(f_1) = \Lambda(g_{01} + g_{11})$. By the inductive hypothesis there is a submodel $\Theta''$ of $N'$ such that $\forall e \in N'[\Lambda(e) \subseteq \Theta''(e)]$ and $\langle a_1, \ldots, a_{n-1} \rangle \in \Theta'(g_{01}) + \Theta''(g_{11})$. Say $\langle a_1, \ldots, a_{n-1} \rangle = b_0 + b_1$ with $b_0 \in \Theta''(g_{01})$ and $b_1 = \Theta''(g_{11})$. Also, $a_0 = T(\Theta'(f_0)) = T(\Lambda(f_0)) = T(\Theta''(f_0)) = T(\Theta''(g_{00})) + T(\Theta''(g_{10}))$. Let $a_0'' = T(\Theta''(g_{00}))$ and $a_1'' = T(\Theta''(g_{10}))$. Then for any $i < 2$, $\langle a_i'' \rangle - b_i = (\langle a_i'' \rangle - 0) + (0 + b_i) \in \Theta''(g_{01}) + \Theta''(g_{11}) \subseteq \Theta''(g_{01} + g_{11}) = \Theta''(e_i)$. Now let $\Theta''' = \Theta'' \upharpoonright M$. Then $\Theta'''$ is a submodel of $M$, $\forall e \in M[\Theta(e) \subseteq \Theta''(e)]$, and $a \in \Theta'''(e_0) + \Theta'''(e_1)$.

\[\Box\]

Proposition 36.3. Suppose that $M$ is a countable $m$-monoid, $\Psi$ is a submodel of $M$, $e \in M$, $m, n \in \omega$, $a \in m(\Psi(e))$, and $b \in n(\Psi(e))$. Then there is a submodel $\Theta$ of $M$ such that $\forall x \in M[\Psi(x) \subseteq \Theta(x)]$ and there is a $c \in m \times n(\Theta(e))$ such that $\forall i < m[a_i = \sum_{j < n} c_{ij}]$ and $\forall j < n[b_j = \sum_{i < m} c_{ij}]$.

Proof. By Proposition 35.1 and induction, there exist an extension $N$ of $M$, a submodel $\Theta$ of $M$ and elements $e' \in mN$ such that $\forall i < m[T(\Theta(e'_i)) = a_i]$, $\sum_{i < m} e'_i = e$, and $\Theta \upharpoonright M = \Psi$. Now $b \in n(\Psi(e)) \subseteq n(\Theta(e))$, so by Proposition 36.2 there is a submodel $\Lambda$ of $M$ such that $\forall x \in M[\Theta(x) \subseteq \Lambda(x)]$ and $b \in \sum_{i < m} \Lambda(e'_i)$. Say $b = \sum_{i < m} c_i$ with each $c_i \in n(\Lambda(e_i))$. Thus $\forall j < n[b_j = \sum_{i < m} c_{ij}]$. Then $\forall i < m[\sum_{j < n} c_{ij} = T(\Lambda(e'_i)) = T(\Theta(e'_i)) = a_i]$. Now $c \in m \times n(\Psi)$, and

\[c = c_0 \cdots c_{m-1} = (c_0 \cdots 0 + 0 \cdots c_0 \cdots 0 + \cdots + (0 \cdots 0 \cdots c_{m-1}) \in \sum_{i < m} \Lambda(e_i) \subseteq \Lambda \left( \sum_{i < m} e_i \right) = \Lambda(e).\]

Thus $\Lambda$ is as desired.

\[\Box\]

Proposition 36.4. If $\Psi$ is a submodel of $M$, $(a, b) \in \Psi(e)$ with $a \neq 0$, $\exists i < \text{dmm}(b)[b_i \neq 0]$, then there exist $a_0, a_1$ and $\Theta$ such that $\forall x \in M[\Psi(x) \subseteq \Theta(x)]$, $a_0 \neq 0 \neq a_1$, $a = a_0 + a_1$, and $(a_0, a_1, b) \in \Theta(e)$.
Proof. By Proposition 35.1 there exist \( e_0, e_1 \), an distinguished extension \( N \) of \( M \), and an expansion \( \Psi' \) of \( \Psi \) to a submodel of \( N \), such that \( e = e_0 + e_1 \), \( a = T(\Psi'(e_0)) \), \( b \in \Psi'(e_1) \), and \( \Psi' \upharpoonright M = \Psi \). Then by Proposition 36.1, taking \( f = 0 \), there exist \( a_0, a_1 \in \mathcal{P} \) and an expansion \( \Theta \) of \( \Psi' \) such that \( a = a_0 + a_1 \) and \( (a_0, a_1) \in \Theta(e_0) \). Hence \( (a_0, a_1, b) \in \Theta(e_0) + \Theta(e_1) \subseteq \Theta(e) \).

\[ \Box \]

Proposition 36.5. If \( M \) is a countable \( m \)-monoid and \( \Psi \) is a submodel of \( M \), then there is a submodel \( \Theta \) of \( M \) such that \( \forall x \in M[\Psi(x) \subseteq \Theta(x)] \) for all \( e, f \in M \) with \( e \) distinguished and \( e \neq f \) we have \( \Theta(e) \neq \Theta(f) \).

Proof. Let \( \langle (e_i, f_i) : i \in \omega \rangle \) list all pairs \( (e, f) \) such that \( e, f \in M \), \( e \) is distinguished, and \( e \neq f \). Define \( \Theta^0 = \Psi \). Suppose that \( \Theta^i \) has been defined so it is a submodel of \( M \) and \( \forall j < i \forall x \in M[\Theta^j(x) \subseteq \Theta^i(x)] \). By Proposition 30.3 we have \( e_i \not\subseteq f_i \) or \( f_i \not\subseteq e_i \). Hence by Proposition 36.1 there is a submodel \( \Theta^{i+1} \) of \( M \) such that \( \forall x \in M[\Theta^i(x) \subseteq \Theta^{i+1}(x)] \) and \( \Theta^{i+1}(e_i) \neq \Theta^{i+1}(f_i) \).

Now define \( \Lambda(x) = \bigcup_{i \in \omega} \Theta^i(x) \) for all \( x \in M \). Clearly \( \Lambda \) is as desired.

\[ \Box \]

Proposition 36.6. If \( M \) is a countable distinguished \( m \)-monoid and \( \Psi \) is a submodel of \( M \), then there is a submodel \( \Theta \) of \( M \) such that \( \forall x \in M[\Psi(x) \subseteq \Theta(x)] \) and \( \forall e_0, e_1 \in M[\Psi(e_0 + e_1) \subseteq \Theta(e_0 + e_1)] \).

Proof. This follows by an easy construction using Proposition 36.2.

\[ \Box \]

Proposition 36.7. If \( M \) is a countable distinguished \( m \)-monoid and \( \Psi \) is a submodel of \( M \), then there is a model \( \Theta \) of \( M \) such that \( \forall x \in M[\Psi(x) \subseteq \Theta(x)] \).

Proof. This follows by an inductive construction using Propositions 36.5 and 36.6.

\[ \Box \]

37. Proof of Ketonen’s theorem

Proposition 37.1. Let \( M \) be a countable distinguished \( m \)-monoid, and suppose that \( \Psi \) is a model of \( M \). Suppose that \( e \in M \) and \( (a_0, a_1) \in \Psi(e) \). Then there exist a distinguished extension of \( M \) to a countable distinguished monoid \( N \), a model \( \Theta \) of \( N \) such that \( \forall x \in M[\Psi(x) \subseteq \Theta(x)] \), and elements \( e_0, e_1 \) of \( N \), such that \( e = e_0 + e_1 \) and \( \forall i < 2[T(\Theta(e_i)) = a_i] \).

Proof. By Propositions 35.1 and 36.7.

\[ \Box \]

Proposition 37.2. Let \( M \) be a countable distinguished \( m \)-monoid, and suppose that \( \Psi \) is a model of \( M \). If \( e_0 + e_1 = f_0 + f_1 \) in \( M^* \), then there exist a distinguished extension \( N \) of \( M \) and elements \( g_{ij} \) of \( N \) for \( i, j < 2 \) such that \( N \) is countable and distinguished, \( \forall i < 2[e_i = \sum_{j < 2} g_{ij}] \), and \( \forall j < 2[f_j = \sum_{i < 2} g_{ij}] \).

Proof. By Propositions 36.3 and 36.7.

\[ \Box \]

Theorem 37.3. (Ketonen) Every countable monoid can be isomorphically embedded into NBA.
Proof. Let \( M \) be a countable monoid. Let \( 0 \) be a new element, define \( M' = M \cup \{0\} \), and define for any \( x, y \in M' \),
\[
x + M' y = \begin{cases} 
  x + M y & \text{if } x, y \in M, \\
  x & \text{if } y = 0, \\
  y & \text{if } x = 0 \neq y.
\end{cases}
\]
Thus \( M' \) is a countable \( m \)-monoid. By Proposition 30.9 it can be embedded in a countable atomless distinguished \( m \)-monoid \( N \). For each \( x \in N \) let \( \Psi(x) = ([\omega^e]_{ae}) \). Obviously \( \Psi \) is a submodel of \( N \). By Proposition 36.7 there is a model \( \Psi_0 \) of \( N \) such that \( \forall x \in N[\Psi(x) \subseteq \Psi_0(x)] \). Note that \( \forall x \in N[T(\Psi(x)) = T(\Psi_0(x)) = [\omega^e]_{ae}] \), so \( \Psi_0[P] \subseteq \Psi_0[N^*] \subseteq \mathcal{P}^1 \).

Let \( M_0 = N \) and \( \Psi_0 = \Psi \). Suppose that we have defined countable distinguished \( m \)-monoids \( M_0, \ldots, M_m \) and models \( \Psi_k \) of \( M_k \) for \( k \leq m \) such that if \( k < m \) then \( M_{k+1} \) is a distinguished extension of \( M_k \), \( \forall x \in M_k[\Psi_k(x) \subseteq \Psi_{k+1}(x)] \), and the following conditions hold:

37.3(1) \( \forall e_0, e_1, f_0, f_1 \in M_k[e_0 + e_1 = f_0 + f_1 \rightarrow \exists g \in 2^{x+2}M_{k+1}[\forall i < 2[e_i = \sum_{j<2} g_{ij}] \text{ and } \forall j < 2[f_j = \sum_{i<2} g_{ij}]]] \).

37.3(2) \( \forall e \in M_k[\forall (a_0, a_1) \in \Psi_k(e) \exists e_0, e_1 \in M_{k+1}[e = e_0 + e_1 \text{ and } \forall i < 2[T(\Psi_{k+1}(e_i)) = a_i]] \).

Using Propositions 37.1 and 37.2 we define \( M_{k+1} \) and \( \Psi_{k+1} \) satisfying the above conditions.

Now define \( R = \bigcup_{k \in \omega} M_k \) and \( \Theta(x) = \bigcup\{\Psi_k(x) : x \in M_k\} \). By Proposition 34.1, \( M \) is a submonoid of \( \text{NBA} \).

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INDEX

relatively complemented ................................................................. 1
ABID ..................................................................................................... 1
\( a \backslash b \) .................................................................................. 1
\( A' \) .............................................................................................. 1
ideal ................................................................................................. 2
prime ideal ....................................................................................... 2
\( a \triangle b \) ................................................................................... 4
filter ................................................................................................. 5
\( a \uparrow \) ......................................................................................... 5
\( F \downarrow \) ...................................................................................... 5
\( [a]_I \) ............................................................................................. 6
Φ(α) ................................................... .............................. 78
Φ_a(α) ........................................................................ 78
fragments ................................................................. 78
α-fragments ............................................................. 80
local refinement property ........................................... 80
L.P. ........................................................................... 80
(L1)–(L2) ................................................................. 80
uniformly dense .......................................................... 81
△ζM .......................................................... 84
T^ζ .......................................................... 84
L^ζ .......................................................... 87
σ .......................................................... 87
(ζ_1)-(ζ_2) ............................................................. 87
uniformly dense .......................................................... 81
△ζM .......................................................... 84
T^ζ .......................................................... 84
L^ζ .......................................................... 87
σ .......................................................... 87
Boolean hierarchy ........................................................ 87
Σ^ζ .......................................................... 87
Σ^η .......................................................... 87
Σ^η .......................................................... 87
Σ^η .......................................................... 87
Σ^η .......................................................... 87
L^ζ .......................................................... 88
{(Z1)-(Z3)} ............................................................. 88
ρ^ζ .......................................................... 88
heirarchy property .................................................. 90
H.P. ........................................................................... 90
NBA ........................................................................... 92
MBA ........................................................................... 92
a ≤ b .......................................................... 92
Schröder-Bernstein property ...................................... 93
S.-B. ........................................................................... 93
M ↾ a .......................................................... 93
locally countable ..................................................... 93
hereditary submonoid ................................................... 93
refinement monoid ................................................... 93
r-monoid ............................................................. 93
(2,2)-refinement property ......................................... 94
atomless monoid ....................................................... 95
δ(a) = δ_M(a) .......................................................... 95
δ_M(a) ............................................................. 95
V-m-relation .......................................................... 96
weak V-m-relation ................................................... 96
V-m-morphism ........................................................ 96
V-m-congruence ..................................................... 96
(Vm1)-(Vm2) ........................................................... 96
Y(M) ........................................................................... 98
V-simple .............................................................. 98
<table>
<thead>
<tr>
<th>V-radical</th>
<th>99</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma \oplus_k \tau$</td>
<td>103</td>
</tr>
<tr>
<td>strict $\xi$-hierarchy property</td>
<td>107</td>
</tr>
<tr>
<td>$\mathcal{L}^c$</td>
<td>107</td>
</tr>
<tr>
<td>$\text{supp}(\theta)$</td>
<td>110</td>
</tr>
<tr>
<td>$\varphi_a$</td>
<td>110</td>
</tr>
<tr>
<td>$\Psi$</td>
<td>110</td>
</tr>
<tr>
<td>equal almost everywhere</td>
<td>110</td>
</tr>
<tr>
<td>$\theta \equiv_{ae} \chi$</td>
<td>110</td>
</tr>
<tr>
<td>$0^c$</td>
<td>110</td>
</tr>
<tr>
<td>$\omega^c$</td>
<td>110</td>
</tr>
<tr>
<td>$\mathcal{N}_0^*$</td>
<td>110</td>
</tr>
<tr>
<td>$\text{(AE1)-(AE7)}$</td>
<td>110,111</td>
</tr>
<tr>
<td>$\mathcal{N}_1^*$</td>
<td>111</td>
</tr>
<tr>
<td>$\equiv'$</td>
<td>111,116</td>
</tr>
<tr>
<td>$\text{(AE8),(AE9)}$</td>
<td>116</td>
</tr>
<tr>
<td>$\Omega \theta$</td>
<td>115</td>
</tr>
<tr>
<td>$\mathcal{N}_1^*$</td>
<td>116</td>
</tr>
<tr>
<td>$\mathcal{N}_1^1$</td>
<td>116</td>
</tr>
<tr>
<td>$\Delta \Theta$</td>
<td>117</td>
</tr>
<tr>
<td>$\mathcal{P}$</td>
<td>120</td>
</tr>
<tr>
<td>$\Pi_1$</td>
<td>120</td>
</tr>
<tr>
<td>$\mathcal{P}_1^1$</td>
<td>122</td>
</tr>
<tr>
<td>$\Pi_1(\alpha)$</td>
<td>122</td>
</tr>
<tr>
<td>$\mathcal{O}_1^1$</td>
<td>124</td>
</tr>
<tr>
<td>$\Gamma_1$</td>
<td>124</td>
</tr>
<tr>
<td>new hierarchy property</td>
<td>125</td>
</tr>
<tr>
<td>$\mathcal{N}_2^*$</td>
<td>125</td>
</tr>
<tr>
<td>$\mathcal{O}_2^2$</td>
<td>125</td>
</tr>
<tr>
<td>$\text{(N1)-(N4)}$</td>
<td>125</td>
</tr>
<tr>
<td>$\Pi_2$</td>
<td>127</td>
</tr>
<tr>
<td>$\mathcal{O}_2^2$</td>
<td>131</td>
</tr>
<tr>
<td>distinguished</td>
<td>134</td>
</tr>
<tr>
<td>$M_s$</td>
<td>134</td>
</tr>
<tr>
<td>distinguished extension</td>
<td>134</td>
</tr>
<tr>
<td>$\varphi$</td>
<td>137</td>
</tr>
<tr>
<td>$M_2(\omega)$</td>
<td>137</td>
</tr>
<tr>
<td>$\varepsilon_{ij}$</td>
<td>137</td>
</tr>
<tr>
<td>$\alpha_i$</td>
<td>37</td>
</tr>
<tr>
<td>$\beta_j$</td>
<td>137</td>
</tr>
<tr>
<td>reduced representation</td>
<td>138</td>
</tr>
<tr>
<td>$\mathcal{R}$</td>
<td>145</td>
</tr>
<tr>
<td>$\hat{a}$</td>
<td>145</td>
</tr>
<tr>
<td>submodel</td>
<td>146</td>
</tr>
</tbody>
</table>
model ................................................... .......................... 146
(M1)–(M3) .......................................................................... 146
closure of Ψ ...................................................................... 147
Dom(α) .................................................................................. 147
Dom(M) ................................................................................ 147
expansion ............................................................................. 147
limit ...................................................................................... 149

161