

CALCULUS 3
 May 7, 2008
FINAL EXAM

YOUR NAME:

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SHOW ALL YOUR WORK

final answers without any supporting work
 will receive no credit even if they are right!

**YOU DO NOT GET POINTS FOR MEMORIZING A FORMULA
 BUT FOR KNOWING HOW TO USE IT!**

No calculators allowed.
 No cheat-sheets allowed.

DO NOT WRITE INSIDE THIS BOX!

problem	points	score
1	?? pts	
2	?? pts	
3	?? pts	
4	?? pts	
5	?? pts	
6	?? pts	
7	?? pts	
8	?? pts	
9	?? pts	
TOTAL	200 pts	

1. [YES pts] Evaluate $\int_C y \sin z \, ds$, where C is the circular helix given by the equations $x = \cos t$, $y = \sin t$, $z = t$, $0 \leq t \leq 2\pi$.

SOLUTION.

If the curve C is parametrized by

$$C : \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle, \quad t \in [t_i, t_f],$$

with derivative $\mathbf{r}'(t)$, then the line integral of a scalar field $f(\mathbf{r})$ along C is given by

$$\int_C f(\mathbf{r}) \, ds = \int_{t=t_i}^{t_f} f(\mathbf{r}(t)) \|\mathbf{r}'(t)\| \, dt.$$

In our case,

$$C : \mathbf{r}(t) = \langle \cos t, \sin t, t \rangle, \quad t \in [0, 2\pi],$$

which yields

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 1 \rangle, \quad \text{and} \quad \|\mathbf{r}'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2},$$

and therefore

$$\int_C y \sin z \, ds = \int_{t=0}^{2\pi} \sin^2 t \sqrt{2} \, dt = \sqrt{2} \int_{t=0}^{2\pi} \frac{1}{2} (1 - \cos 2t) \, dt = \frac{\sqrt{2}}{2} \left[t - \frac{1}{2} \sin 2t \right]_{t=0}^{2\pi} = \sqrt{2}\pi.$$

2. [YES pts] Compute the work performed by the vector field $\mathbf{F} = x^2\mathbf{i} - xy\mathbf{j}$ on a particle moving along the curve $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$, $t \in [0, \pi/2]$.

SOLUTION.

If the curve C is parametrized by

$$C : \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle, \quad t \in [t_i, t_f],$$

with derivative $\mathbf{r}'(t)$, then the line integral of a scalar field $f(\mathbf{r})$ along C is given by

$$\int_C f(\mathbf{r}) \cdot d\mathbf{r} = \int_{t=t_i}^{t_f} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt.$$

In our case,

$$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j},$$

and

$$\mathbf{F}(\mathbf{r}(t)) = \cos^2 t\mathbf{i} - \cos t \sin t\mathbf{j}$$

therefore

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_{t=0}^{\pi/2} (-2 \cos^2 t \sin t) \, dt = \frac{2}{3} [\cos^3 t]_{t=0}^{\pi/2} = -\frac{2}{3}.$$

3. [?? pts] Determine whether or not the vector field

$$\mathbf{F}(x, y) = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$$

is conservative. If it is conservative, find a scalar function $f(x, y)$ such that $\mathbf{F} = \nabla f$.

SOLUTION.

A vector field \mathbf{F} is conservative if and only if $\nabla \times \mathbf{F} = \mathbf{0}$. Since

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ (3 + 2xy) & (x^2 - 3y^2) & 0 \end{vmatrix} = \langle 0, 0, 2x - 2x \rangle = \mathbf{0}$$

the given vector field \mathbf{F} is conservative.

One way of finding $f(x, y)$.

If $\mathbf{F} = \nabla f$, then

$$\begin{aligned} f_x(x, y) &= 3 + 2xy &\Rightarrow f(x, y) &= \int (3 + 2xy) dx = (3x + x^2y) + g(y) \\ f_y(x, y) &= x^2 - 3y^2 &\Rightarrow f(x, y) &= \int (x^2 - 3y^2) dy = (x^2y - y^3) + h(x) \end{aligned}$$

Therefore, if $h(x) = 3x$ and $g(y) = y^3$, we obtain

$$f(x, y) = 3x + x^2y - y^3 + C.$$

Another way of finding $f(x, y)$.

$$f_x(x, y) = 3 + 2xy \quad \Rightarrow \quad f(x, y) = \int (3 + 2xy) dx = (3x + x^2y) + g(y)$$

therefore, using this form of f ,

$$f_y(x, y) = x^2 - 3y^2 = x^2 + g'(y) \quad \Rightarrow \quad g'(y) = -3y^2 \quad \Rightarrow \quad g(y) = -y^3 + C$$

and so

$$f(x, y) = 3x + x^2y - y^3 + C.$$

4. [YES pts] Evaluate the integral

$$\oint_{\mathcal{C}} 2xy \, dx + (x^2 + y^2) \, dy$$

where \mathcal{C} is the ellipse $4x^2 + 9y^2 = 36$.

SOLUTION.

Green's Theorem states that

$$\oint_{\mathcal{C}} M(x, y) \, dx + N(x, y) \, dy = \iint_{\mathcal{R}} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA$$

where \mathcal{R} is the region in the xy -plane enclosed by the curve \mathcal{C} , and M and N are continuous functions that have continuous first partial derivatives throughout an open region \mathcal{D} containing \mathcal{R} .

Therefore

$$\oint_{\mathcal{C}} 2xy \, dx + (x^2 + y^2) \, dy = \iint_{\mathcal{R}} (2x - 2x) \, dA = 0.$$

5. [YES pts] Set up in spherical coordinates, **but do not evaluate** the surface integral

$$\iint_{\mathcal{S}} (x^2 + y^2) \, dS$$

where \mathcal{S} is the first-octant portion of the sphere $x^2 + y^2 + z^2 = a^2$.

SOLUTION.

- **The easy way**

Using spherical coordinates, over the sphere of radius a we have

$$f = (a \sin \phi \cos \theta)^2 + (a \sin \phi \sin \theta)^2 = a^2 \sin^2 \phi$$

and

$$dS = a^2 \sin \phi \, d\phi \, d\theta$$

we have

$$\begin{aligned} \iint_{\mathcal{S}} f(\mathbf{r}) \, dS &= \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} a^2 \sin^2 \phi \, a^2 \sin \phi \, d\phi \, d\theta \\ &= a^4 \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \sin^3 \phi \, d\phi \, d\theta. \end{aligned}$$

• **The hard way**

The surface integral of a scalar field $f(\mathbf{r})$ over a surface \mathcal{S} parametrized by $\mathbf{r} = \mathbf{r}(u, v)$, with $(u, v) \in \mathcal{R}$ is given by

$$\iint_{\mathcal{S}} f(\mathbf{r}) \, dS = \iint_{\mathcal{R}} f(\mathbf{r}(u, v)) \cdot \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \, dR$$

In this case, a parametrization of the surface \mathcal{S} is given by

$$\mathcal{S} : \quad \mathbf{r}(u, v) = \langle a \sin u \cos v, a \sin u \sin v, a \cos u \rangle \quad \text{with} \quad u \in [0, \pi/2], v \in [0, \pi/2]$$

In these new variables the scalar function takes the form

$$f = (a \sin u \cos v)^2 + (a \sin u \sin v)^2 = a^2 \sin^2 u$$

The cross product is given by

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ +a \cos u \cos v & +a \cos u \sin v & -a \sin u \\ -a \sin u \sin v & +a \sin u \cos v & 0 \end{vmatrix} \\ &= a^2 \sin^2 u \cos v \mathbf{i} + a^2 \sin^2 u \sin v \mathbf{j} + a^2 (\sin u \cos u \cos^2 v + \sin u \cos u \sin^2 v) \mathbf{k} \\ &= a^2 \sin u (\sin u \cos v \mathbf{i} + \sin u \sin v \mathbf{j} + \cos u \mathbf{k}) \end{aligned}$$

and its length is

$$\left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| = \sqrt{a^4 \sin^2 u (\sin^2 u \cos^2 v + \sin^2 u \sin^2 v + \cos^2 u)} = a^2 \sin u$$

(no need of absolute value bars here since $\sin u \geq 0$ for $u \in [0, \pi]$). We have then

$$\begin{aligned} \iint_{\mathcal{S}} f(\mathbf{r}) \, dS &= \int_{v=0}^{\pi/2} \int_{u=0}^{\pi/2} a^2 \sin^2 u \, a^2 \sin u \, du \, dv \\ &= a^4 \int_{v=0}^{\pi/2} \int_{u=0}^{\pi/2} \sin^3 u \, du \, dv. \end{aligned}$$

6. [YES pts] Find the circulation of the vector field $\mathbf{F} = \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. around the boundary \mathcal{C} of any smooth orientable surface \mathcal{S} in space is zero.

SOLUTION.

We have to find

$$\oint_{\mathcal{C}} \mathbf{r} \cdot d\mathbf{r}.$$

Stokes' theorem states that

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathcal{S}} \nabla \times \mathbf{F} \cdot \mathbf{n} \, dS$$

where \mathcal{C} is the boundary of the oriented surface \mathcal{S} in the direction counterclockwise with respect to the surface's unit normal vector \mathbf{n} . But

$$\nabla \times \mathbf{F} = \nabla \times \mathbf{r} = \mathbf{0},$$

Therefore

$$\oint_{\mathcal{C}} \mathbf{r} \cdot d\mathbf{r} = \iint_{\mathcal{S}} \nabla \times \mathbf{r} \cdot \mathbf{n} \, dS = \mathbf{0}.$$

7. [YES pts]

(a) Compute the circulation of the vector field

$$\mathbf{F} = \frac{-y}{x^2 + y^2} \mathbf{i} + \frac{x}{x^2 + y^2} \mathbf{j} + z \mathbf{k}$$

around the curve \mathbf{C} given by the circle $x^2 + y^2 = 1$ in the xy -plane in the counterclockwise direction when viewed from the positive z -axis.

- (b) In this problem, can you use *Stokes' Theorem* to transform the line integral for the circulation into a surface integral? **Justify your answer!**

SOLUTION.

- (a) We have to compute the circulation by direct evaluation of the integral $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$. Parametrizing the curve as

$$\mathcal{C} : \mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle, \quad t \in [0, 2\pi]$$

we have

$$\begin{aligned} \oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} &= \int_{t=0}^{2\pi} \langle -\sin t, \cos t, 0 \rangle \cdot \langle -\sin t, \cos t, 0 \rangle dt \\ &= \int_{t=0}^{2\pi} (\sin^2 t + \cos^2 t) dt \\ &= 2\pi. \end{aligned}$$

- (b) Stokes' formula cannot be used to transform the circulation integral into a surface integral because the domain of \mathbf{F} is not *simply connected*. The vector field \mathbf{F} is not defined along the z -axis so there is no way to contract the curve \mathcal{C} to a point without leaving the domain of \mathbf{F} .

8. [YES pts] Find a function $h(x, y, z)$ such that the outward flux of the vector field $\mathbf{F}(x, y, z) = \langle \sin x, e^z, h(x, y, z) \rangle$ across any sphere is zero. **Justify your answer.**

SOLUTION.

Gauss' theorem states that,

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_{\mathcal{V}} \nabla \cdot \mathbf{F} \, dV$$

when \mathcal{S} is a closed surface with interior \mathcal{V} , and $\nabla \cdot \mathbf{F}$ is defined within \mathcal{V} .

If $\nabla \cdot \mathbf{F} = 0$ over all \mathbf{R}^3 , then the flux integral over any sphere (or any closed surface) of such vector field will be zero. Since

$$\begin{aligned} \nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x} \sin x + \frac{\partial}{\partial y} e^z + \frac{\partial}{\partial z} h(x, y, z) \\ &= \cos x + \frac{\partial}{\partial z} h(x, y, z) \end{aligned}$$

we can obtain a divergenceless field by choosing a function $h(x, y, z)$ such that

$$\frac{\partial}{\partial z} h(x, y, z) = -\cos x \quad \Rightarrow \quad h(x, y, z) = -z \cos x + k,$$

where k is a constant of integration. Therefore,

$$\mathbf{F}(x, y, z) = \langle \sin x, e^z, -z \cos x + k \rangle.$$

9. [YES pts] Find the flux of the vector field $\mathbf{F} = (x - z)\mathbf{i} + (y - x)\mathbf{j} + (z - y)\mathbf{k}$ across the surface \mathcal{S} given by $x^2 + y^2 = a^2$, $z = 0$, and $z = 1$, with outward orientation.

SOLUTION.

The given surface is a closed surface, therefore we can use *Gauss' theorem* that states

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_{\mathcal{V}} \nabla \cdot \mathbf{F} \, dV$$

For our problem we have

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x - z) + \frac{\partial}{\partial y}(y - x) + \frac{\partial}{\partial z}(z - y) = 3,$$

therefore

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_{\mathcal{V}} \nabla \cdot \mathbf{F} \, dV = \iiint_{\mathcal{V}} 3 \, dV = 3V(\mathcal{V}) = 3\pi a^2.$$