CALCULUS 3

March 5, 2008

2nd TEST

YOUR NAME:

○ **001** A. Spina ....(9am)

○ **002** E. Wittenborn .... (10am)

○ **003** T. Dent ....(11am)

○ **004** J. Wiscons ....(12pm)

○ **005** A. Spina ....(1pm)

SHOW ALL YOUR WORK

No calculators allowed.
No cheat-sheets allowed.

DO NOT WRITE INSIDE THIS BOX!

problem	points	score
1	5 pts	
2	10 pts	
3	15 pts	
4	15 pts	
5	15 pts	
6	15 pts	
7	15 pts	
8	10 pts	
TOTAL	100 pts	

1. **[5 pts]** Find the natural domain of $\ln\left(4 - x^2 - y^2\right)$.

SOLUTION:

$$4 - x^2 - y^2 > 0 \qquad \Longrightarrow \qquad x^2 + y^2 < 4$$

that is, all points *inside* the circle of radius 2 with center at the origin.

2. **[10 pts]** Given the function

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0)\,, \\ 0 & (x,y) = (0,0)\,; \end{cases}$$

show that it is continuous at $(0,0)$.

SOLUTION:

To show that the function f is continuous at $(0,0)$ we must show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$. This we do by writing the function f in polar coordinates,

$$\begin{aligned} \lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{r\to 0} f\left(r\cos\theta, r\sin\theta\right) \\ &= \lim_{r\to 0} \frac{r^2\cos\theta\sin\theta}{r} \\ &= \lim_{r\to 0} r\cos\theta\sin\theta \\ &= 0\,. \end{aligned}$$

3. **[15 pts]** The function $f(x,y)$ is given by

$$f(x,y) = e^{xy}\,.$$

If $x = r\cos\theta$ and $y = r\sin\theta$ use the *chain rule* to find $\dfrac{\partial f}{\partial r}(r,\theta)$ and $\dfrac{\partial f}{\partial \theta}(r,\theta)\cdot$ Simplify your answer!

SOLUTION:

$$\begin{aligned} \frac{\partial f}{\partial r}(r,\theta) &= \left[\frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}\right]_{\substack{x=r\cos\theta \\ y=r\sin\theta}} \\ &= \left[ye^{xy}\cos\theta + xe^{xy}\sin\theta\right]_{\substack{x=r\cos\theta \\ y=r\sin\theta}} \\ &= \left[r\cos\theta\sin\theta e^{r^2\cos\theta\sin\theta} + r\cos\theta\sin\theta e^{r^2\cos\theta\sin\theta}\right]_{\substack{x=r\cos\theta \\ y=r\sin\theta}} \\ &= 2r\cos\theta\sin\theta e^{r^2\cos\theta\sin\theta} \\ &= r\,\sin(2\theta)\,e^{r^2\sin(2\theta)/2}\,; \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial \theta}(r,\theta) &= \left[\frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}\right]_{\substack{x=r\cos\theta \\ y=r\sin\theta}} \\ &= \left[ye^{xy}(-r\sin\theta) + xe^{xy}(r\cos\theta)\right]_{\substack{x=r\cos\theta \\ y=r\sin\theta}} \\ &= -r^2\sin^2\theta e^{r^2\cos\theta\sin\theta} + r^2\cos^2\theta e^{r^2\cos\theta\sin\theta} \\ &= r^2\left(\cos^2\theta - \sin^2\theta\right)e^{r^2\cos\theta\sin\theta} \\ &= r^2\,\cos(2\theta)\,e^{r^2\sin(2\theta)/2}\,. \end{aligned}$$

4. **[15 pts]** Find the directional derivative of $f(x,y) = \sin(x)\cos(y)$ at the origin in the direction $\theta = \pi/3$ measured counterclockwise with respect to the x–axis.

SOLUTION:

The directional derivative at (x_0, y_0) in the direction of $\mathbf{u}$ is given by

$$\mathrm{D}_{\mathbf{u}} = \boldsymbol{\nabla} f(x_0, y_0) \cdot \mathbf{u}\,.$$

The unit vector in the given direction is

$$\mathbf{u} = \left\langle \cos\left(\frac{\pi}{3}\right), \sin\left(\frac{\pi}{3}\right) \right\rangle = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$$

The gradient of f is

$$\boldsymbol{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle \cos(x)\cos(y)\,, -\sin(x)\sin(y) \rangle$$

Therefore

$$\mathrm{D}_{\mathbf{u}} f(0,0) = \langle \cos(0)\cos(0)\,, -\sin(0)\sin(0) \rangle \cdot \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle = \frac{1}{2}\,.$$

5. **[15 pts]** The temperature (in degrees Celsius) at a point (x, y) on a metal plate in the xy–plane is

$$T(x, y) = \frac{xy}{1 + x^2 + y^2}\,.$$

At the point $(1, 1)$ find the **unit** vector in the direction in which the temperature drops most rapidly.

SOLUTION:

The direction of maximum increase of the function T at (x_0, y_0) is given by the gradient of T at (x_0, y_0). The direction of maximum decrease at $(x_0, y_0) = (1, 1)$ is therefore given by by the direction of the vector

$$\begin{aligned} -\boldsymbol{\nabla} T(1,1) &= -\left\langle \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y} \right\rangle_{\substack{x=1\\y=1}} \\ &= -\left\langle \frac{y(1-x^2)+y^3}{(1+x^2+y^2)^2}, \frac{x(1-y^2)+x^3}{(1+x^2+y^2)^2} \right\rangle_{\substack{x=1\\y=1}} \\ &= -\left\langle \frac{1}{3}, \frac{1}{3} \right\rangle \end{aligned}$$

which is $5\pi/4$ measured counterclockwise from the x–axis.

The unit vector in the direction of maximum temperature decrease is then given by

$$\begin{aligned} \mathbf{u} &= \frac{-\left\langle \frac{1}{3}, \frac{1}{3} \right\rangle}{\sqrt{(1/3)^2 + (1/3)^2}} \\ &= \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle\,. \end{aligned}$$

6. **[15 pts]** Find the direction of the tangent line to the curve of intersection of the cone $z = \sqrt{x^2 + y^2}$ and the plane $x + 2y + 2z = 20$ at the point $(4, 3, 5)$. Just any vector along the tangent line will do, you do not need to normalize it!

SOLUTION:

A vector $\mathbf{v}$ having the same direction as that of the tangent line to the curve of intersection of two surfaces, $f(x, y, z) = 0$, and $g(x, y, z) = 0$ at a point (x_0, y_0, z_0) must be perpendicular to the normal vectors to each of the surfaces at that point. Therefore, we can take

$$\mathbf{v} = \boldsymbol{\nabla} f(x_0, y_0, z_0) \times \boldsymbol{\nabla} g(x_0, y_0, z_0)\,.$$

Four our problem,

$$\begin{aligned} z = \sqrt{x^2+y^2} &\longrightarrow f(x,y,z) = x^2 + y^2 - z^2 = 0\,, \quad z \geq 0\,, \\ x + 2y + 2z = 20 &\longrightarrow h(x,y,z) = x + 2y + 2z - 20 = 0\,, \end{aligned}$$

therefore,

$$\begin{aligned}\nabla f(x,y,z) &= \langle 2x\,,\, 2y\,,\, -2z\rangle\,,\\ \nabla h(x,y,z) &= \langle 1\,,\, 2\,,\, 2\rangle\,,\end{aligned}$$

$$\begin{aligned}\nabla f(4,3,5) &= \langle 8\,,\, 6\,,\, -10\rangle\,,\\ \nabla h(4,3,5) &= \langle 1\,,\, 2\,,\, 2\rangle\,,\end{aligned}$$

and

$$\mathbf{v} = \nabla f(4,3,5)\times\nabla g(3,3,5) = \langle 8\,,\, 6\,,\, -10\rangle \times \langle 1\,,\, 2\,,\, 2\rangle = \langle 32, -26, 10\rangle\,.$$

7. **[15 pts]** Find the equation of the plane that is tangent to the paraboloid

$$z = \frac{x^2}{2^2} + \frac{y^2}{3^2}$$

at the point $(2,3,2)$.

SOLUTION:

The equation of a plane is given by

$$\mathbf{n}\cdot(\mathbf{r}-\mathbf{r}_0) = 0\,.$$

The equation of the paraboloid can be written as

$$f(x,y,z) = \frac{x^2}{2^2} + \frac{y^2}{3^2} - z = 0\,.$$

A vector normal to the paraboloid at $(2,3,2)$ is given by the gradient of f at $(2,3,2)$,

$$\nabla f(2,3,2) = \left\langle \frac{\partial f}{\partial x}\,,\, \frac{\partial f}{\partial y}\,,\, \frac{\partial f}{\partial z}\right\rangle_{\substack{x=2\\y=3\\z=2}} = \left\langle \frac{2x}{2^2}\,,\, \frac{2y}{3^2}\,,\, -1\right\rangle_{\substack{x=2\\y=3\\z=2}} = \left\langle 1\,,\, \frac{2}{3}\,,\, -1\right\rangle$$

The equation of the tangent plane is then

$$\begin{aligned}0 &= \nabla f(2,3,2)\cdot\langle x-2, y-3, z-2\rangle\\ &= \left\langle 1\,,\, \frac{2}{3}\,,\, -1\right\rangle\cdot\langle x-2, y-3, z-2\rangle\\ &= (x-2) + \frac{2}{3}(y-3) - (z-2)\end{aligned}$$

or,

$$x + \frac{2}{3}y - z = 1\,.$$

8. **[10 pts]** The following statements are either **true** or **false**. If true, then say so and explain why. If false, then say so and give a simple counter-example to show why the statement is false.

   (a) If $f(x,y)$ is differentiable at (x_0,y_0) and $\nabla f(x_0,y_0) =< 0,0 >$ then the tangent plane to $z = f(x,y)$ at (x_0,y_0) is parallel to the xy–plane.

   (b) Let $\mathbf{r}(t)$ be a vector valued function. Then $\frac{\mathrm{d}}{\mathrm{d}t}||\mathbf{r}(t)|| = ||\mathbf{r}'(t)||$.

SOLUTION:

(a) **True.** If f is differentiable at (x_0,y_0) then the equation of the tangent plane to $z = f(x,y)$ at (x_0,y_0) is

$$< 0,0,-1 > \cdot < x-x_0, y-y_0, z-z_0 >= 0 \qquad \Longrightarrow \qquad z = z_0 = f(x_0,y_0)$$

which is the equation of a plane parallel to the xy–plane (the variables x and y do not appear in the equation).

(b) **False.**

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}||\mathbf{r}(t)|| &= \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\mathbf{r}(t)\cdot\mathbf{r}(t)}\\ &= \frac{\mathbf{r}'(t)\cdot\mathbf{r}(t) + \mathbf{r}(t)\cdot\mathbf{r}'(t)}{2\sqrt{\mathbf{r}(t)\cdot\mathbf{r}(t)}}\\ &= \frac{\mathbf{r}(t)\cdot\mathbf{r}'(t)}{||\mathbf{r}(t)||}\,.\end{aligned}$$