CALCULUS 3

February 6, 2008

1st TEST

YOUR NAME:

○ **001** A. Spina .... (9am)
○ **002** E. Wittenborn .... (10am)
○ **003** T. Dent .... (11am)
○ **004** J. Wiscons .... (12pm)
○ **005** A. Spina .... (1pm)

SHOW ALL YOUR WORK

No calculators allowed.
No cheat-sheets allowed.

Make sure you write an arrow on top of vector quantities to differentiate them from scalar quantities (numbers). A word-processor and boldface fonts were used in writing test, but you are writing by hand! Remember that, within the same context, $\vec{v}$ (with the arrow) is a *vector* ($\vec{v} \equiv \mathbf{v}$) and v (without the arrow) is the *norm* of the previous vector ($v \equiv ||\vec{v}|| \equiv ||\mathbf{v}||$). If a vector is the null vector, write an arrow on top of the zero!

DO NOT WRITE INSIDE THIS BOX!

problem	points	score
1	10 pts	
2	10 pts	
3	10 pts	
4	10 pts	
5	10 pts	
6	10 pts	
7	10 pts	
8	10 pts	
9	10 pts	
10	10 pts	
TOTAL	100 pts	

1. **[10 pts]** Find a parametric equation of the line L through the points $P_0(1,3,2)$ and $P_1(-4,3,0)$.

SOLUTION:

A parametric equation of a line through $P_0(x_0,y_0,z_0)$ and parallel to $\mathbf{v} =< v_1, v_2, v_3 >$ is given by

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} =< x_0, y_0, z_0 > +t < v_1, v_2, v_3 >$$

or

$$\begin{cases} x(t) &= x_0 + t\,v_1 \\ y(t) &= y_0 + t\,v_2 \\ z(t) &= z_0 + t\,v_3 \end{cases}$$

A vector $\mathbf{v}$ parallel to the line L is given by

$$\mathbf{v} = \vec{OP_1} - \vec{OP_0} =< -4,3,0 > - < 1,3,2 >=< -5,0,-2 >$$

therefore the equation we are looking for is

$$\boxed{\mathbf{r}(t) =< 1,3,2 > +t < -5,0,-2 >}$$

or

$$\boxed{\begin{cases} x(t) &= 1-5t \\ y(t) &= 3 \\ z(t) &= 2-2t \end{cases}}$$

2. **[10 pts]** Find the equation of the plane through the points $P_0(0,1,1)$, $P_1(1,0,1)$, and $P_2(1,1,0)$.

SOLUTION:

1st Way. The equation of the plane through $P_0(x_0,y_0,z_0)$ and with normal $\mathbf{n} =< n_1, n_2, n_3 >$ is given by

$$\mathbf{n}\cdot(\mathbf{r}-\mathbf{r}_0) = 0\,,$$

or

$$n_1(x-x_0) + n_2(y-y_0) + n_3(z-z_0) = 0\,.$$

Two vectors, $\mathbf{v}$ and $\mathbf{w}$ parallel to the plane are

$$\begin{aligned} \mathbf{v} &= \vec{P_0P_1} = < 1,0,1 > - < 0,1,1 > = < 1,-1,0 > \\ \mathbf{w} &= \vec{P_0P_2} = < 1,1,0 > - < 0,1,1 > = < 1,0,-1 > \end{aligned}$$

A third vector perpendicular to these two vectors (and therefore perpendicular to the plane) is given by

$$\mathbf{n} = \mathbf{v}\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} =< 1,1,1 >$$

The equation of the plane is therefore

$$\boxed{x + (y-1) + (z-1) = 0 \qquad \text{or} \qquad x+y+z=2}$$

2nd Way. Replacing the coordinates of P_0, P_1, and P_2 in the general equation of a plane

$$ax + by + cz = d$$

we obtain the following system of equations

$$\begin{aligned} b + c &= d \\ a \quad + c &= d \\ a + b \quad &= d \end{aligned}$$

Solving this system we get

$$a = b = c = \frac{d}{2}\cdot$$

Therefore

$$\frac{d}{2}x + \frac{d}{2}y + \frac{d}{2}z = d$$

Since $d \neq 0$ in order for a solution to exist, we divide all the equation by d and multiply by 2, thus obtaining

$$\boxed{x + y + z = 2}$$

3. **[10 pts]** Let $\mathbf{v} =< v_1, v_2 >= v_1\mathbf{i} + v_2\mathbf{j}$ be a vector in 2D with $||\mathbf{v}|| \neq 0$. Find $\mathbf{v}^\perp$, a vector perpendicular to $\mathbf{v}$. Write it in terms of the components v_1 and v_2 of the original vector.

   NOTE: Even if you already know the answer to this problem, **show** how you obtain the components of your $\mathbf{v}^\perp$ from the given condition it must satisfy.

   SOLUTION:

   Let $\mathbf{v}^\perp$ be the vector perpendicular to $\mathbf{v}$. We have,

$$\mathbf{v} \cdot \mathbf{v}^\perp = 0$$

   or, in terms of their components,

$$\mathbf{v} \cdot \mathbf{v}^\perp =< v_1, v_2 > \cdot < v_1^\perp, v_2^\perp >= v_1 v_1^\perp + v_2 v_2^\perp = 0$$

   the easy solutions to this equation are given by

$$v_1^\perp = -v_2\,, \quad v_2^\perp = +v_1\,,$$

   which represents a *counter-clockwise* rotation by $\pi/2$ of the vector $\mathbf{v}$, or

$$v_1^\perp = +v_2\,, \quad v_2^\perp = -v_1\,,$$

   which represents a *clockwise* rotation by $\pi/2$ of the vector $\mathbf{v}$. Therefore

$$\boxed{\mathbf{v}^\perp = \mp v_2\mathbf{i} \pm v_1\mathbf{j} =< \mp v_2, \pm v_1 >}$$

4. **[10 pts]** The following statements are either **true** or **false**. If true, then say so and explain why. If false, then say so and give a simple counter-example to show why the statement is false.

   (a) $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = ||\mathbf{a}||^2 - ||\mathbf{b}||^2$;

   (b) $(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b}) = -2\mathbf{a}\cdot\mathbf{b}$;

   (c) If $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$, then $(\mathbf{a}\times\mathbf{b})\times\mathbf{a}$ is parallel to $\mathbf{b}$;

   (d) The plane given by $x - y + 2z = 2$ intersects the plane given by $3x - y - z = 1$.

   NOTE: Just *one lucky* example doesn't prove a statement right, but it can prove it false!

   SOLUTION:

   (a) **True**. The expression on the left can be easily expanded, yielding

$$(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = \underbrace{\mathbf{a}\cdot\mathbf{a}}_{||\mathbf{a}||^2} - \mathbf{a}\cdot\mathbf{b} + \underbrace{\mathbf{b}\cdot\mathbf{a}}_{\mathbf{a}\cdot\mathbf{b}} - \underbrace{\mathbf{b}\cdot\mathbf{b}}_{||\mathbf{b}||^2}\,,$$

   therefore

$$\boxed{(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = ||\mathbf{a}||^2 - ||\mathbf{b}||^2}$$

   (b) **False**. The left-hand side is a vector, the right hand side is a scalar. The correct answer to the expansion of the expression on the left is

$$(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b}) = \underbrace{\mathbf{a}\times\mathbf{a}}_{\mathbf{0}} - \mathbf{a}\times\mathbf{b} + \underbrace{\mathbf{b}\times\mathbf{a}}_{-\mathbf{a}\times\mathbf{b}} - \underbrace{\mathbf{b}\times\mathbf{b}}_{\mathbf{0}}$$

   therefore

$$\boxed{(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b}) = -2\,\mathbf{a}\times\mathbf{b}}$$

(c) **False**. As a counter-example take $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = \mathbf{j}$, then

$$\begin{aligned}\Big((\mathbf{i}+\mathbf{j})\times\mathbf{j}\Big)\times(\mathbf{i}+\mathbf{j}) &= (\mathbf{i}\times\mathbf{j}+\mathbf{j}\times\mathbf{j})\times(\mathbf{i}+\mathbf{j}) \\ &= (\mathbf{k}+\mathbf{0})\times(\mathbf{i}+\mathbf{j}) \\ &= \mathbf{k}\times(\mathbf{i}+\mathbf{j}) \\ &= \mathbf{k}\times\mathbf{i}+\mathbf{k}\times\mathbf{j} \\ &= \mathbf{j}-\mathbf{i}\end{aligned}$$

which is not parallel to $\mathbf{j}$.

(d) **True**. The normals to the planes, $\mathbf{n}_1 =< 1, -1, 2 >$ and $\mathbf{n}_2 =< 3, -1, -1 >$ are not parallel, therefore the planes intersect.

5. **[10 pts]** Find the angle of intersection of the planes given by the equations

$$\Pi_1 : \quad x - 2y + z = 2 , \qquad \text{and} \qquad \Pi_2 : \quad 2x + y + z = 1 .$$

SOLUTION:

The angle of intersection θ of two planes is the angle between their normals. A vector perpendicular to the first plane Π_1 is given by

$$\mathbf{n}_1 =< 1, -2, 1 > ,$$

and a vector perpendicular to the second plane Π_2 by

$$\mathbf{n}_2 =< 2, 1, 1 > .$$

We know that

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = ||\mathbf{n}_1||\,||\mathbf{n}_2||\cos\theta \qquad \Longrightarrow \qquad \cos\theta = \frac{|\mathbf{n}_1\cdot\mathbf{n}_2|}{||\mathbf{n}_1||\,||\mathbf{n}_2||} = \frac{1}{\sqrt{6}\sqrt{6}}$$

where the absolute value bars around the dot product are to ensure that we obtain the smallest of the two angles. Therefore

$$\boxed{\theta = \arccos\left(\frac{1}{6}\right)}$$

6. **[10 pts]** Find a parametric equation of the line of intersection of the the planes

$$\Pi_1 : \quad x - 2y + z = 2 , \qquad \text{and} \qquad \Pi_2 : \quad 2x + y + z = 1 .$$

SOLUTION:

The parametric vector equation of a line L, that goes through P_0 and is parallel to $\mathbf{v}$ is given by

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} .$$

A vector perpendicular to the first plane Π_1 is given by

$$\mathbf{n}_1 =< 1, -2, 1 > ,$$

and a vector perpendicular to the second plane Π_2 by

$$\mathbf{n}_2 =< 2, 1, 1 > .$$

Since the line L is going to be perpendicular to $\mathbf{n}_1$ (because it lies on Π_1), and also perpendicular to $\mathbf{n}_2$ (because it lies on Π_2), it will be parallel to the cross product of $\mathbf{n}_1$ with $\mathbf{n}_2$. Therefore we can take

$$\mathbf{v} = \mathbf{n}_1\times\mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 2 & 1 & 1 \end{vmatrix} =< -3, 1, 5 >$$

For $\mathbf{r}_0$ we need to find one point of intersection of the planes, that is we look for a solution of the system of equations,

$$\begin{array}{rcrcrcl} x & - & 2y & + & z & = & 2 \\ 2x & + & y & + & z & = & 1 \end{array}$$

any solution will do, so we set $y = 0$ and solve the resulting 2×2 system

$$\begin{array}{rcrcl} x & + & z & = & 2 \\ 2x & + & z & = & 1 \end{array} \qquad \Longrightarrow \qquad x = -1\,, z = 3\,.$$

Therefore

$$\mathbf{r}_0 = \langle -1\,, 0\,, 3 \rangle$$

will be the position vector of a point on the line (the one of its intersection with the xz–plane).

A parametric equation of the line can therefore be written as

$$\boxed{\mathbf{r} = \langle -1\,, 0\,, 3 \rangle + t < -3, 1, 5 > \qquad \text{or} \qquad \left\{ \begin{array}{rcl} x & = & -1 - 3t \\ y & = & t \\ z & = & 3 + 5t \end{array} \right.}$$

7. **[10 pts]** Given the equation $z - x^2 - y^2 = 0$ that represents a quadric surface, in the grid below
   - draw the traces on the xz–plane, the yz–plane and on the planes $z = 9$, $z = 4$ and $z = 0$ (beware, these planes are not drawn);
   - with the above curves as a guide complete a sketch of the quadric making sure that you do not spoil the traces you already draw.

   and sketch it.

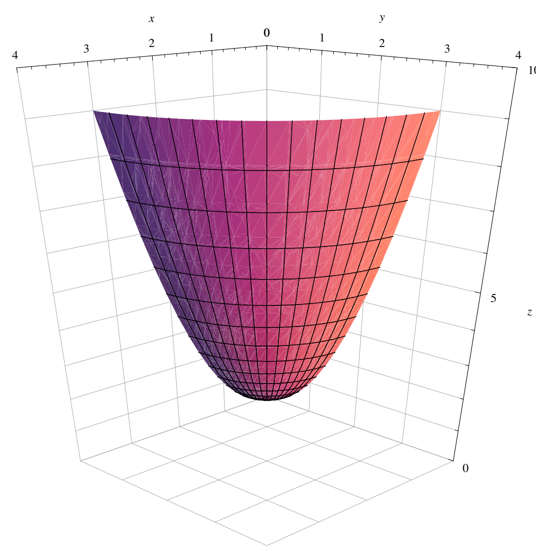
SOLUTION:

The traces yield

$$x = 0 \quad \longrightarrow \quad \left\{ \begin{array}{l} z = y^2 \\ x = 0 \end{array} \right. \qquad \text{parabola}$$

$$y = 0 \quad \longrightarrow \quad \left\{ \begin{array}{l} z = x^2 \\ y = 0 \end{array} \right. \qquad \text{parabola}$$

$$z = k^2 \quad \longrightarrow \quad \left\{ \begin{array}{l} x^2 + y^2 = k^2 \\ z = k^2 \end{array} \right. \qquad \text{circles}$$



8. **[10 pts]** Write an equation for the ellipsoid $4x^2 + 4y^2 + z^2 = 1$ in
   (a) cylindrical coordinates;
   (b) in spherical coordinates.

SOLUTION:

(a)

$$4x^2 + 4y^2 + z^2 = 1 \qquad \Longrightarrow \qquad z^2 = 1 - 4\left(x^2 + y^2\right)$$

therefore

$$\boxed{z^2 = 1 - 4r^2}$$

(b)

$$\begin{aligned} 4x^2 + 4y^2 + z^2 = 1 &\implies 4\left(x^2+y^2+z^2\right) = 1 + 3z^2 \\ &\implies 4\rho^2 = 1 + 3\rho^2\cos^2\phi \end{aligned}$$

therefore

$$\boxed{\rho^2\left(4 - 3\cos^2\phi\right) = 1}$$

9. **[10 pts]** Find a rectangular equation for the surface whose cylindrical equation is $r = 2\cos\theta$ and identify the surface it describes.

SOLUTION:

$$\begin{aligned} r = 2\cos\theta &\implies r^2 = 2r\cos\theta \\ &\implies x^2 + y^2 = 2x \\ &\implies (x-1)^2 + y^2 = 1 \end{aligned}$$

This is a cylinder with axis parallel the z–axis that intersects the xy–plane on a circle of radius 1 with center $(1, 0, 0)$. Therefore

$$\boxed{(x-1)^2 + y^2 = 1}$$

10. **[10 pts]** Find a rectangular equation for the surface whose spherical equation is $\rho = \sin\theta\,\sin\phi$ and identify the surface it describes.

SOLUTION:

$$\begin{aligned} \rho = \sin\theta\,\sin\phi &\implies \rho^2 = \rho\,\sin\theta\,\sin\phi \\ &\implies x^2 + y^2 + z^2 = y \end{aligned}$$

Therefore

$$\boxed{x^2 + \left(y - \frac{1}{2}\right)^2 + z^2 = \frac{1}{4}}$$

which is the equation of a sphere with center $(0, 1/2, 0)$ and radius $1/2$.