CALCULUS 3

September 16, 2009

1st TEST

YOUR NAME:

- ◯ **001** J. SANDERS .... (8AM)
- ◯ **002** J. KISH .... (9AM)
- ◯ **003** E. WITTENBORN .... (10AM)
- ◯ **004** A. SPINA .... (11AM)
- ◯ **005** A. SPINA .... (12PM)
- ◯ **006** M. STACKPOLE .... (1PM)
- ◯ **007** C. MESA .... (3PM)

SHOW ALL YOUR WORK

final answers without any supporting work
will receive no credit *even if they are right!*

No calculators allowed.
No cheat-sheets allowed.

Partial credit will be given for any **reasonable amount of work pointing in the right direction** towards the solution of your problem. You will not get any partial credit for memorizing formulas and not knowing how to use them, or for anything you write that is not directly related to the solution of your problem.

If your tests contains **more than one solution or answer** to a problem or part of a problem, and one of them is wrong, then it will be **the wrong one** the one that **counts** for your grading!

DO NOT WRITE INSIDE THIS BOX!

problem	points	score
1	15 pts	
2	15 pts	
3	15 pts	
4	15 pts	
5	19 pts	
6	21 pts	
TOTAL	100 pts	

1. **[15 pts]** If $\mathbf{a} =< 1, 1, 0 >$, and $\mathbf{b} =< 0, 0, 1 >$, describe the set of points points $P(x, y, z)$ that satisfy

$$(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) = 0\,,$$

where $\mathbf{r} =< x, y, z >$.

SOLUTION:

Expanding the given equation we get

$$\begin{aligned}(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) = 0 \quad &\Rightarrow \quad < x-1, y-1, z-0 > \;\cdot\; < x-0, y-0, z-1 >= 0\\ &\Rightarrow \quad (x-1)(x-0) + (x-1)(x-0) + (x-0)(x-1) = 0\\ &\Rightarrow \quad x^2 + y^2 + z^2 - x - y - z = 0\\ &\Rightarrow \quad \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 + \left(z - \frac{1}{2}\right)^2\\ &\qquad = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\\ &\Rightarrow \quad \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 + \left(z - \frac{1}{2}\right)^2 = \frac{3}{4}\,.\end{aligned}$$

Therefore the equation represents a **sphere** with center and radius given by

$$C\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \qquad \text{and} \qquad R = \frac{\sqrt{3}}{2}\,.$$

NOTE: This is a generalization (in vector form) of a theorem from classical geometry that says *"the inscribed angles in a semicircle are right angles."* The semicircle in question has has its center at the midpoint between points A and B (with position vectors $\mathbf{a}$ and $\mathbf{b}$), and the radius is the semi-distance between A and B.

This problem can be worked out fully in vector form in the following way,

$$\begin{aligned}(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) = 0 \quad &\Rightarrow \quad \mathbf{r} \cdot \mathbf{r} - (\mathbf{a} + \mathbf{b}) \cdot \mathbf{r} + \mathbf{a} \cdot \mathbf{b} = 0\\ &\Rightarrow \quad \left(\mathbf{r} - \frac{\mathbf{a}+\mathbf{b}}{2}\right) \cdot \left(\mathbf{r} - \frac{\mathbf{a}+\mathbf{b}}{2}\right) = \left(\frac{\mathbf{a}+\mathbf{b}}{2}\right) \cdot \left(\frac{\mathbf{a}+\mathbf{b}}{2}\right) - \mathbf{a} \cdot \mathbf{b}\\ &\Rightarrow \quad \left(\mathbf{r} - \frac{\mathbf{a}+\mathbf{b}}{2}\right) \cdot \left(\mathbf{r} - \frac{\mathbf{a}+\mathbf{b}}{2}\right) = \left(\frac{\mathbf{a}-\mathbf{b}}{2}\right) \cdot \left(\frac{\mathbf{a}-\mathbf{b}}{2}\right)\\ &\Rightarrow \quad \left\|\mathbf{r} - \frac{\mathbf{a}+\mathbf{b}}{2}\right\|^2 = \left\|\frac{\mathbf{a}-\mathbf{b}}{2}\right\|^2.\end{aligned}$$

This is the equation of a **sphere** with center at

$$\mathbf{r}_0 = \frac{1}{2}(\mathbf{a} + \mathbf{b}) = \frac{1}{2}\big(< 1, 1, 0 > + < 0, 0, 1 > \big) = \left\langle \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\rangle$$

and radius

$$R = \frac{1}{2}\|\mathbf{a} - \mathbf{b}\| = \frac{1}{2}\| < 1, 1, 0 > - < 0, 0, 1 > \| = \frac{1}{2}\| < 1, 1, -1 > \| = \frac{\sqrt{3}}{2}\,.$$

2. **[15 pts]** Find an equation of the plane through the points $Q_1(q, 0, 0)$, $Q_2(q, 2q, 0)$, and $Q_3(q, 2q, 3q)$, where q is a given non-zero number.

SOLUTION:

1st Way. The vector equation of a plane through a point P_0 and with normal $\mathbf{n}$ is given by

$$\mathbf{n} \cdot \left(\mathbf{r} - \mathbf{r}_0\right) = 0\,,$$

where $\mathbf{r} = \vec{OP} =< x, y, z >$ is the position vector of a generic point P on the plane, and $\mathbf{r}_0 = \vec{OP_0} =< x_0, y_0, z_0 >$ is the position vector of P_0, a reference point in the plane.

As a reference point in the plane we can take Q_1, that is

$$\mathbf{r}_0 = \vec{OP_0} =< q, 0, 0 > \ .$$

A vector perpendicular to the plane can be obtained by computing

$$\vec{Q_1Q_2} \times \vec{Q_1Q_3} =< 0, 2q, 0 > \times < 0, 2q, 3q >= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2q & 0 \\ 0 & 2q & 3q \end{vmatrix} =< 6q^2, 0, 0 > \ .$$

The normal vector $\mathbf{n}$ only needs to be parallel to the vector $\vec{Q_1Q_2} \times \vec{Q_1Q_3}$ so we take, for simplicity,

$$\mathbf{n} =< 1, 0, 0 > \ .$$

With $\mathbf{r}_0$ and $\mathbf{n}$ we write

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) =< 1, 0, 0 > \ \cdot \ < x - q, y - 0, z - 0 >= x - q\,.$$

Therefore, an equation of the plane is

$$\boxed{x = q}$$

2nd Way. Replacing the coordinates of the points Q_1, Q_2, and Q_3 into the general plane equation in rectangular form,

$$ax + by + cz = d\,,$$

we obtain the system of linear equations

$$\begin{cases} q\,a & + & 0\,b & + & 0\,c & = & d\,, \\ q\,a & + & 2q\,b & + & 0\,c & = & d\,, \\ q\,a & + & 2q\,b & + & 3q\,c & = & d\,, \end{cases}$$

whose solutions are

$$a = d/q \neq 0\,, \qquad b = c = 0\,.$$

If we take, for simplicity, $d = q$, then $a = 1$, and we obtain

$$\boxed{x = q}$$

3. **[15 pts]** Find an equation of the line L along which the two planes

$$x + 2y + 3z = 1 \qquad \text{and} \qquad 3x + 2y + z = 3$$

intersect.

SOLUTION:

The vector equation of a line L is given by,

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}\,,$$

where $\mathbf{r}_0$ is the position vector of a reference point P_0 on the line L, and the vector $\mathbf{v}$ is a vector parallel to the line L.

A vector $\mathbf{v}$ parallel to the line L is going to be perpendicular to the two vectors $\mathbf{n}^{(1)}$ and $\mathbf{n}^{(2)}$ normal to the given planes or, what is the same, parallel to

$$\mathbf{n}^{(1)} \times \mathbf{n}^{(2)} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix} = < -4, 8, -4 >$$

We can take, for simplicity,

$$\mathbf{v} = < 1, -2, 1 > \ .$$

A point P_0 on the line is obtained by choosing any solution (of the infinitely many) to the system of equations

$$\begin{array}{rcrcrcl} x & + & 2y & + & 3z & = & 1\,, \\ 3x & + & 2y & + & z & = & 3\,. \end{array}$$

One such solution is obtained by setting arbitrarily $x = 0$ and solving for y and z,

$$\mathbf{r}_0 = < 0, 2, -1 > \ .$$

Therefore,

$$\boxed{\mathbf{r}(t) = < 0, 2, -1 > + \, t < 1, -2, 1 >}$$

or

$$\boxed{\left\{ \begin{array}{rcl} x(t) & = & t \\ y(t) & = & 2 - 2t \\ z(t) & = & -1 + t \end{array} \right.}$$

4. **[15 pts]** Find an equation of the plane that passes through the point $(6, 0, -2)$ and contains the line $x = 4 - 2t,\ y = 3 + 5t,\ z = 7 + 4t$.

SOLUTION:

If the problem is going to have a unique solution, then the given point should not be on the given line, as can be verified by trying to solve t the equations $6 = 4 - 2t,\ 0 = 3 + 5t,\ -2 = 7 + 4t$ and finding that no solution can exist.

To compute a (non-parametric) equation of the plane, which is of the form

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0\,,$$

we need two pieces of information, a reference point $\mathbf{r}_0$ on the plane and a vector $\mathbf{n}$ perpendicular to the plane.

We can take the given point on the plane as reference point, that is,

$$\mathbf{r}_0 = < 6, 0, -2 > \ .$$

If we knew two vectors $\mathbf{v}$ and $\mathbf{w}$ parallel to the plane, then we can obtain a vector perpendicular to the plane by taking their cross product, that is,

$$\mathbf{n} = \mathbf{v} \times \mathbf{w}$$

One such vector we know, is the direction vector of the line on the plane, that is,

$$\mathbf{v} = < -2, 5, 4 > \ .$$

The other vector we take as the displacement vector between any point on the line and the given point on the plane (but not on the line). Any vector of the form

$$< (4 - 2t) - 6\,,\, (3 + 5t) - 0\,,\, (7 + 4t) - (-2) >$$

will do. For simplicity we evaluate the above expression at $t = 0$ and obtain

$$\mathbf{w} = < -2, 3, 9 > \ .$$

Therefore

$$\mathbf{n} = \mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 5 & 4 \\ -2 & 3 & 9 \end{vmatrix} = < 33\,,\, 10\,,\, 4 >$$

and an equation for the plane is obtained in the form

$$\boxed{33(x-6) + 10y + 4(z+2) = 0}$$

or

$$\boxed{33x + 10y + 4z = 190}$$

5. **[19 pts]** By converting coordinates, decide which of the following equations have the same graph. The first column contains equations in rectangular form, the second contains equations in cylindrical form, and the third in spherical form.

(1)	$(x-1)^2 + y^2 = 1$	(a)	$r = 2\cos\theta$	(i)	$\rho^2(1 - \sin^2\phi\sin^2\theta) = 1$
(2)	$x^2 + y^2 = 4$	(b)	$r^2\cos^2\theta = 1 - z^2$	(ii)	$\rho = 2\csc\phi$
(3)	$x^2 + z^2 = 1$	(c)	$r = 2$	(iii)	$\rho\sin\phi = 2\cos\theta$

NOTE ON THE POINTS FOR THE PROBLEM: no work $\Longrightarrow$ no credit. . . NO EXCUSES!

SOLUTION:

The change of coordinates from rectangular to cylindrical and to spherical is accomplished by means of the following change of variables

rectangular → cylindrical	rectangular → spherical
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$
$z = z$	$z = \rho\cos\phi$

For equation (1) we have

$$\begin{aligned}(x-1)^2 + y^2 = 1 \quad &\to \quad (r\cos\theta - 1)^2 + (r\sin\theta)^2 = 1 \\ &\to \quad r^2\cos^2\theta - 2r\cos\theta + 1 + r^2\sin^2\theta = 1 \\ &\to \quad r^2 - 2r\cos\theta = 0 \\ &\to \quad r = 2\cos\theta\,,\end{aligned}$$

$$\begin{aligned}(x-1)^2 + y^2 = 1 \quad &\to \quad (\rho\sin\phi\cos\theta - 1)^2 + (\rho\sin\phi\sin\theta)^2 = 1 \\ &\to \quad \rho^2\sin^2\phi\cos^2\theta - 2\rho\sin\phi\cos\theta + 1 + \rho^2\sin^2\phi\sin^2\theta = 1 \\ &\to \quad \rho^2\sin^2\phi - 2\rho\sin\phi\cos\theta = 0 \\ &\to \quad \rho\sin\phi = 2\cos\theta\,.\end{aligned}$$

Therefore

$$\boxed{(1) \equiv (\text{a}) \equiv (\text{iii})}$$

For equation (2) we have

$$\begin{aligned}x^2 + y^2 = 4 \quad &\to \quad r^2\cos^2\theta + r^2\sin^2\theta = 4 \\ &\to \quad r^2 = 4 \\ &\to \quad r = 2\,,\end{aligned}$$

$$\begin{aligned}x^2 + y^2 = 4 \quad &\to \quad \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta = 4 \\ &\to \quad \rho^2\sin^2\phi = 4 \\ &\to \quad \rho\sin\phi = 2 \\ &\to \quad \rho = 2\csc\phi\,.\end{aligned}$$

Therefore

$$\boxed{(2) \equiv (c) \equiv (ii)}$$

Finally, for equation (3) we have

$$\begin{aligned} x^2 + z^2 = 1 \quad &\rightarrow \quad r^2\cos^2\theta + z^2 = 1 \\ &\rightarrow \quad r^2\cos^2\theta = 1 - z^2\,, \end{aligned}$$

$$\begin{aligned} x^2 + z^2 = 1 \quad &\rightarrow \quad \rho^2\sin^2\phi\cos^2\theta + \rho^2\cos^2\phi = 1 \\ &\rightarrow \quad \rho^2\sin^2\phi\left(1 - \sin^2\theta\right) + \rho^2\cos^2\phi = 1 \\ &\rightarrow \quad \rho^2 - \rho^2\sin^2\theta = 1 \\ &\rightarrow \quad \rho^2\left(1 - \sin^2\phi\sin^2\theta\right) = 1\,. \end{aligned}$$

Therefore

$$\boxed{(3) \equiv (b) \equiv (i)}$$

6. **[21 pts]** The following statements are either **true** or **false**. If true, then say so and explain why. If false, then say so and explain why or give a *counter-example* to show why the statement is false.

NOTE ON THE POINTS FOR THE PROBLEM: 3 points each part, 1 point for correct *true* or *false* answer, 2 points for justification or counter-example. Also, a decent and commented sketch can be taken as a "proof" or "counterexample."

(a) $\|k\mathbf{v}\| = k\,\|\mathbf{v}\|$, for any vector $\mathbf{v}$ and any scalar k.

(b) $\mathbf{u}\times\mathbf{v} = \mathbf{v}\times\mathbf{u}$.

(c) If $\mathbf{u}$ and $\mathbf{v}$ are any two vectors, then $\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$.

(d) If $\mathbf{u}$ and $\mathbf{v}$ are any two vectors, then $|\mathbf{u}\cdot\mathbf{v}| > \|\mathbf{u}\|\,\|\mathbf{v}\|$.

(e) If $\mathbf{u}$ and $\mathbf{v}$ are any two vectors, then $(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v}) = 2\mathbf{u}\times\mathbf{v}$.

(f) If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are any three vectors, then $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\times\mathbf{w}$.

(g) If $\mathbf{u}$ and $\mathbf{v}$ are any two non-zero and non-parallel vectors vectors of equal length, then $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are perpendicular.

SOLUTION:

(a) **False.**
If $k < 0$ this would mean that the length of a (scaled) vector is negative. The correct answer is

$$\|k\mathbf{v}\| = k\,\|\mathbf{v}\|$$

for any vector $\mathbf{v}$ and any scalar k.

(b) **False.**
Cross product is anti-commutative. The correct expression is

$$\mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u}\,.$$

(c) **True.**

$$\begin{aligned} \|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) + (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) \\ &= \big(\mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u}\big) + \big(\mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u}\big) \\ &= 2\mathbf{u}\cdot\mathbf{u} + 2\mathbf{v}\cdot\mathbf{v} \\ &= 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2\,. \end{aligned}$$

(d) **False.**
If $\angle(\mathbf{u},\mathbf{v}) = \theta$, then

$$\begin{aligned} \mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\,\cos\theta \quad &\Rightarrow \quad -\|\mathbf{u}\|\,\|\mathbf{v}\| \le \mathbf{u}\cdot\mathbf{v} \le \|\mathbf{u}\|\,\|\mathbf{v}\| \\ &\Rightarrow \quad |\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\,\|\mathbf{v}\|\,. \end{aligned}$$

(e) **False.**

$$\begin{aligned}(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v}) &= (\mathbf{u}+\mathbf{v})\times\mathbf{u}-(\mathbf{u}+\mathbf{v})\times\mathbf{v}\\ &= \mathbf{u}\times\mathbf{u}+\mathbf{v}\times\mathbf{u}-\mathbf{u}\times\mathbf{v}-\mathbf{v}\times\mathbf{v}\\ &= \mathbf{0}+\mathbf{v}\times\mathbf{u}-\mathbf{u}\times\mathbf{v}+\mathbf{0}\\ &= -\mathbf{u}\times\mathbf{v}-\mathbf{u}\times\mathbf{v}\\ &= -2\mathbf{u}\times\mathbf{v}\,.\end{aligned}$$

(f) **False.**
Cross product is not associative. As a counterexample take

$$\mathbf{0}=\mathbf{i}\times(\mathbf{j}\times\mathbf{j})\neq(\mathbf{i}\times\mathbf{j})\times\mathbf{j}=-\mathbf{i}\,.$$

(g) **True.**

$$(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})=\mathbf{u}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{u}-\mathbf{u}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{v}=\|\mathbf{u}\|^2-\|\mathbf{v}\|^2=0\,.$$

This is a restatement, in vector language, of a an old-known theorem of classical geometry,

"If the four sides of a parallelogram are equal, then its diagonals are perpendicular."