CALCULUS 3

December 13, 2008

FINAL EXAM

YOUR NAME:

- ◯ **001** B. Katz-Moses .... (8am)
- ◯ **002** J. Sanders .... (9am)
- ◯ **003** J. Newhall .... (10am)
- ◯ **004** A. Spina .... (11am)
- ◯ **005** E. Angel .... (12pm)
- ◯ **006** A. Spina .... (1pm)
- ◯ **007** A. Spina .... (3pm)

SHOW ALL YOUR WORK

final answers without any supporting work
will receive no credit *even if they are right!*

No calculators allowed.
No cheat-sheets allowed.

Partial credit will be given for any **reasonable amount of work pointing in the right direction** towards the solution of your problem. You will not get any partial credit for memorizing formulas and not knowing how to use them, or for anything you write that is not directly related to the solution of your problem.

problem or part of a problem, and one of them is wrong, then it will be **the wrong one** the one that **counts** for your grading!

DO NOT WRITE INSIDE THIS BOX!

problem	points	score
1	10 pts	
2	10 pts	
3	10 pts	
4	10 pts	
5	10 pts	
6	20 pts	
7	20 pts	
8	20 pts	
9	20 pts	
10	20 pts	
11	20 pts	
12	20 pts	
13	10 pts	
TOTAL	200 pts	

1. **[10 pts]** Each of the following statements is *false*, in general. For each give a counterexample that shows it is not true.

   (a) $\mathbf{u}\times\mathbf{v} = \mathbf{v}\times\mathbf{u}$;

   (b) The cross product of two unit vectors is a unit vector;

   (c) $\mathbf{u}\cdot\mathbf{u} = \|\mathbf{u}\|$;

   (d) $\dfrac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{r}_1(t)\times\mathbf{r}_2(t)\Big) = \dfrac{\mathrm{d}\mathbf{r}_1(t)}{\mathrm{d}t}\times\dfrac{\mathrm{d}\mathbf{r}_2(t)}{\mathrm{d}t}$.

SOLUTION.

(a) Compute the cross product between any two of the unit vectors **i**, **j**, and **k**. For example

$$\begin{aligned}\mathbf{i}\times\mathbf{j} &= \mathbf{k}\,,\\ \mathbf{j}\times\mathbf{i} &= -\mathbf{k}\,.\end{aligned}$$

(b) The length of a cross product is given by

$$\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\|\,\sin\theta = \sin\theta$$

therefore any two unit vectors at an angle other than from $\pi/2$ will disprove the statement.

(c) By direct computation of the dot product we get, in three dimensions,

$$\mathbf{u}\cdot\mathbf{u} = (u_1)^2 + (u_2)^2 + (u_3)^2 = \|\mathbf{u}\|^2$$

therefore any non-unit vector will disprove the statement.

(d) By direct differentiation (and regrouping of terms) it we once obtained

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{r}_1(t)\times\mathbf{r}_1(t)\Big) = \frac{\mathrm{d}\mathbf{r}_1(t)}{\mathrm{d}t}\times\mathbf{r}_2(t) + \mathbf{r}_1(t)\times\frac{\mathrm{d}\mathbf{r}_2(t)}{\mathrm{d}t}$$

This expression is so different from the one given that almost any choice of $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ should disprove the statement. For a simple counterexample we can try

$$\mathbf{r}_1(t) =< t,0,0 >\,, \qquad \text{and} \qquad \mathbf{r}_2(t) =< 0,t,0 >\,,$$

which yield

$$\mathbf{r}_1(t)\times\mathbf{r}_2(t) =< t,0,0 > \times < 0,t,0 >= \left\langle 0,0,t^2\right\rangle\,.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{r}_1(t)\times\mathbf{r}_2(t)\Big) = \frac{\mathrm{d}}{\mathrm{d}t}\left\langle 0,0,t^2\right\rangle =< 0,0,2t >\,,$$

but

$$\frac{\mathrm{d}\mathbf{r}_1(t)}{\mathrm{d}t}\times\frac{\mathrm{d}\mathbf{r}_2(t)}{\mathrm{d}t} =< 1,0,0 > \times < 0,1,0 >=< 0,0,1 >\,.$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{r}_1(t)\times\mathbf{r}_2(t)\Big) =< 0,0,2t > \neq < 0,0,1 >= \frac{\mathrm{d}\mathbf{r}_1(t)}{\mathrm{d}t}\times\frac{\mathrm{d}\mathbf{r}_2(t)}{\mathrm{d}t}\,,$$

disproving the statement.

2. **[10 pts]** Use implicit differentiation to calculate $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

SOLUTION.

Method 1. Differentiating implicitly with respect to x while holding y constant we get

$$3x^2 + 3z^2\frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0 \qquad \Rightarrow \qquad \frac{\partial z}{\partial x} = -\frac{x^2+2yz}{z^2+2xy}\,.$$

Similarly

$$3y^2 + 3z^2\frac{\partial z}{\partial y} + 6xz + 6xy\frac{\partial z}{\partial y} = 0 \qquad \Rightarrow \qquad \frac{\partial z}{\partial y} = -\frac{y^2+2xz}{z^2+2xy}\,.$$

Method 2.

$$F(x,y,z) = x^3 + y^3 + z^3 + 6xyz - 1$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{x^2 + 2yz}{z^2 + 2xy}\,.$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{y^2 + 2xz}{z^2 + 2xy}\,.$$

3. **[10 pts]** Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$ or show that it does not exist.

SOLUTION.

1st Way.

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{u\to 0} \frac{\sin u}{u} = 1\,.$$

2nd Way.

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{r\to 0} \frac{\sin(r^2)}{r^2} = \lim_{r\to 0} \frac{2r\cos(r^2)}{2r} = \lim_{r\to 0} \cos(r^2) = 1\,.$$

by using *L'Hôpital's rule* in the one-variable limit with r.

4. **[10 pts]** Find an equation for the tangent plane to the surface $z = 2x^2 - 3y^2$ at the point $P_0(2,1,5)$.

SOLUTION.

The general equation for a plane with normal $\mathbf{n}$ and containing the point $P_0 = (2,1,5)$ with position vector $\mathbf{r}_0 =< 2, 1, 5 >$ is

$$\mathbf{n}\cdot(\mathbf{r} - \mathbf{r}_0) = 0\,.$$

The given equation is a level surface of the function

$$f(x,y,z) = 2x^2 - 3y^2 - z$$

corresponding to $f(x,y,z) = 0$. A normal vector to this surface at $\mathbf{r}_0$ is given by

$$\mathbf{n} = (\boldsymbol{\nabla} f)_0 = \langle 4x\,,\ -6y\,,\ -1\rangle_0 =< 8\,,\ -6\,,\ -1 >\ .$$

Therefore

$$\begin{aligned} \mathbf{n}\cdot(\mathbf{r} - \mathbf{r}_0) &= (\boldsymbol{\nabla} f)_0\cdot(\mathbf{r} - \mathbf{r}_0) \\ &= < 8\,, -6\,, -1 > \cdot < x-2\,,\ y-1\,,\ z-5 > \\ &= 8(x-2) - 6(y-1) - 1(z-5)\,, \end{aligned}$$

and hence, the equation is given by

$$\boxed{8(x-2) - 6(y-1) - (z-5) = 0}$$

5. **[10 pts]** Find and classify the critical points (if any) of $x^3 + 6xy - 3y^2$.

SOLUTION.

Given the two-variable function

$$f(x,y) = x^3 + 6xy - 3y^2$$

the critical points are obtained by solving

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \quad \text{or} \quad \begin{cases} 3x^2 + 6y = 0 \\ 6x - 6y = 0 \end{cases}$$

The second equation implies that

$$y = x,$$

which substituted into the first equation yields

$$3x^2 + 6x = 0 \quad \Rightarrow \quad 3x(x+2) = 0 \quad \Rightarrow \quad x = 0 \quad \text{or} \quad x = -2$$

Therefore, the critical points are $(0, 0)$ and $(-2, -2)$.

To classify the critical points we need the second derivatives,

$$f_{xx} = 6x\,, \qquad f_{yy} = -6\,, \qquad f_{xy} = 6\,,$$

and the discriminant

$$D = f_{xx}(x_0, y_0)\, f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0) = -36x - 36\,.$$

- at $(0, 0)$ a *saddle* because

$$D(0, 0) = -36 < 0\,;$$

- at $(-2, -2)$ a *relative minimum* because

$$D(-2, -2) = +36 > 0 \quad \text{and} \quad f_{xx}(-2, -2) = -12 < 0\,.$$

6. **[20 pts]** Evaluate the line integral $\int_{\mathcal{C}} xy^4 \, \mathrm{d}s$ where $\mathcal{C}$ is the right half of the circle $x^2 + y^2 = 4$, from $(0, -2)$ to $(0, +2)$.

SOLUTION.

Firstly, we parametrize the curve $\mathcal{C}$ with

$$\begin{cases} x(t) &= 2\cos t \\ y(t) &= 2\sin t \end{cases} \qquad \text{with} \qquad t \in \left[-\frac{\pi}{2}\,, +\frac{\pi}{2}\right]$$

Secondly,

$$\mathrm{d}s = \|\,\mathrm{d}\mathbf{r}\,\| = \left\|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right\| \,\mathrm{d}t = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = 2\,\mathrm{d}t$$

Finally,

$$\begin{aligned} \int_{\mathcal{C}} xy^4 \, \mathrm{d}s &= \int_{t=-\pi/2}^{+\pi/2} (2\cos t)(2\sin t)^4 2\,\mathrm{d}t \\ &= 2^6 \left[\frac{\sin^5 t}{5}\right]_{t=-\pi/2}^{+\pi/2} \\ &= \frac{2 \times 2^6}{5}\,. \end{aligned}$$

Therefore

$$\boxed{\int_{\mathcal{C}} xy^4 \, \mathrm{d}s = \frac{2^7}{5}}$$

7. **[20 pts]** Evaluate the integral

$$\oint_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

where the vector field $\mathbf{F}$ is given by

$$\mathbf{F} = \langle e^y + ye^x\,,\; xe^y + e^x \rangle\,,$$

and the closed path $\mathcal{C}$ by the polygonal line joining the points

$$(0, 0) \longrightarrow (1, 1) \longrightarrow (0, 2) \longrightarrow (-3, 2) \longrightarrow (-1, 1) \longrightarrow (-3, 0) \longrightarrow (0, 0)$$

SOLUTION.

1st Way. Since the curve $\mathcal{C}$ is closed, let us find first if the vector field is conservative. We know that *the circulation* (the work-type integral over a closed curve) *of a conservative field is zero.*

If

$$\mathbf{F}(x,y) =< F_1(x,y)\,,\, F_2(x,y) >= \left\langle \frac{\partial\phi(x,y)}{\partial x}\,,\, \frac{\partial\phi(x,y)}{\partial y} \right\rangle = \boldsymbol{\nabla}\phi(x,y)$$

then

$$\frac{\partial^2\phi}{\partial y\partial x} = \frac{\partial^2\phi}{\partial x\partial y} \qquad \Rightarrow \qquad \frac{\partial F_1(x,y)}{\partial y} = \frac{\partial F_2(x,y)}{\partial x}\,.$$

In our case

$$\frac{\partial F_1(x,y)}{\partial y} = (e^y + e^x) = \frac{\partial F_2(x,y)}{\partial x}$$

therefore the given $\mathbf{F}$ is a conservative field, and

$$\boxed{\oint_{\mathcal{C}} \mathbf{F}\boldsymbol{\cdot}\,\mathrm{d}\mathbf{r} = 0}$$

2nd Way. Same thing can be done with the curl, by appending a null z-component to the given vector field $\mathbf{F}$ and getting

$$\boldsymbol{\nabla}\times\mathbf{F} = \mathbf{0}\,.$$

We know that if the curl of $\mathbf{F} = \mathbf{0}$ then $\mathbf{F}$ is conservative, or we may use *Stokes' theorem.*

3rd Way. Since this is a two-dimensional problem and the curve is closed, we can first look at Green's theorem, just in case! Green's theorem is a two-dimensional version of Stokes' theorem that Green discovered roughly 50 years before Stokes. It wasn't Stokes' fault, he was only a kid when Green published his results.

> *Green's Theorem* states that if $\mathbf{C}$ is positively oriented, piecewise-smooth, simple closed curve in the plane, and $\mathcal{D}$ is the region bounded by $\mathcal{C}$, then
>
> $$\oint_{\mathcal{C}} P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y\,.$$

In our case,

$$P(x,y) = e^y + ye^x\,, \qquad \text{and} \qquad Q(x,y) = xe^y + e^x\,,$$

which gives

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial}{\partial x}(xe^y + e^x) - \frac{\partial}{\partial y}(e^y + ye^x) = (e^y + e^x) - (e^y + e^x) = 0\,.$$

Therefore

$$\boxed{\oint_{\mathcal{C}} \mathbf{F}\boldsymbol{\cdot}\,\mathrm{d}\mathbf{r} = \oint_{\mathcal{C}} P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y = 0}$$

8. **[20 pts]** Evaluate $\displaystyle\oint_{\mathcal{C}} y^3\,\mathrm{d}x - x^3\,\mathrm{d}y$ where $\mathcal{C}$ is the *counterclockwise (positively-oriented)* circle $x^2 + y^2 = 4$.

SOLUTION.

> *Green's Theorem* states that if $\mathbf{C}$ is positively oriented, piecewise-smooth, simple closed curve in the plane, and $\mathcal{D}$ is the region bounded by $\mathcal{C}$, then
>
> $$\oint_{\mathcal{C}} P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y\,.$$

In our case

$$P(x,y) = y^3\,, \qquad \text{and} \qquad Q(x,y) = -x^3\,.$$

Hence,

$$\begin{aligned}
\oint_{\mathcal{C}} y^3\,\mathrm{d}x - x^3\,\mathrm{d}y &= \iint_{x^2+y^2\le 4} \left(-3x^2 - 3y^2\right)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\theta=0}^{2\pi}\int_{r=0}^{2} \left(-3r^2\right) r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_{\theta=0}^{2\pi} \left[-\frac{3}{4}r^4\right]_{r=0}^{2}\,\mathrm{d}\theta \\
&= -12\int_{\theta=0}^{2\pi} \mathrm{d}\theta \\
&= -24\pi\,.
\end{aligned}$$

Therefore,

$$\boxed{\oint_{\mathcal{C}} y^3\,\mathrm{d}x - x^3\,\mathrm{d}y = -24\pi}$$

9. **[20 pts]** Evaluate the surface integral $\iint_{\mathcal{S}} \left(x^2z + y^2z\right)\,\mathrm{d}S$ where $\mathcal{S}$ is the hemisphere $x^2+y^2+z^2 = 4$, $z \ge 0$.

SOLUTION.

Method 1. Since the equation of the surface can be parametrized in Cartesian coordinates,

$$z = +\sqrt{4 - x^2 - y^2}\,, \qquad x^2 + y^2 \le 4\,.$$

we begin working the integral in Cartesian coordinates and later, if necessary, transform to a more suitable coordinate system.

The surface differential takes the form

$$\begin{aligned}
\mathrm{d}S &= \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{2}{\sqrt{4 - x^2 - y^2}}\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

Hence,

$$\begin{aligned}
\iint_{\mathcal{S}} \left(x^2z + y^2z\right)\,\mathrm{d}S &= \iint_{\mathcal{S}} \left(x^2 + y^2\right) z\,\mathrm{d}S \\
&= \iint_{x^2+y^2\le 4} \left(x^2+y^2\right)\sqrt{4 - x^2 - y^2}\,\frac{2}{\sqrt{4 - x^2 - y^2}}\,\mathrm{d}x\,\mathrm{d}y \\
&= 2\iint_{x^2+y^2\le 4} \left(x^2+y^2\right)\,\mathrm{d}x\,\mathrm{d}y \\
&= 2\int_{\theta=0}^{2\pi}\int_{r=0}^{2} r^2\,r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= 4\pi\int_{r=0}^{2} r^3\,\mathrm{d}r\,\mathrm{d}\theta \\
&= 4\pi\left[\frac{r^4}{4}\right]_{r=0}^{2} \\
&= 16\pi\,.
\end{aligned}$$

Therefore,

$$\boxed{\iint_{\mathcal{S}} \left(x^2 z + y^2 z\right)\,\mathrm{d}S = 16\pi}$$

Method 2. Since the surface $\mathcal{S}$ is the upper hemisphere of radius $\rho_0 = 2$, we parametrize the problem in spherical coordinates,

$$\begin{aligned} x(\phi,\theta) &= 2\sin\phi\cos\theta\,, \\ y(\phi,\theta) &= 2\sin\phi\sin\theta\,, \\ z(\phi,\theta) &= 2\cos\phi\,, \end{aligned} \qquad \text{with} \qquad \begin{aligned} \phi &\in [0,\pi/2]\,, \\ \theta &\in [0,2\pi]\,. \end{aligned}$$

It is not necessary to compute $\|\mathbf{r}_\phi \times \mathbf{r}_\theta\|$ since we already know that on the surface of a sphere of radius ρ_0 centered at the origin the element of surface $\mathrm{d}S$ takes the form

$$\mathrm{d}S = \rho_0^2 \sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta = 4\sin\phi\;\mathrm{d}\phi\,\mathrm{d}\theta\,.$$

Hence,

$$\begin{aligned} \iint_{\mathcal{S}} \left(x^2 z + y^2 z\right)\,\mathrm{d}S &= \iint_{\mathcal{S}} \left(x^2 + y^2\right) z\;\mathrm{d}S \\ &= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2} \left(4\sin^2\phi\cos^2\theta + 4\sin^2\phi\sin^2\theta\right)(2\cos\phi)\;4\sin\phi\;\mathrm{d}\phi\,\mathrm{d}\theta \\ &= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2} \left(4\sin^2\phi\right)(2\cos\phi)\;4\sin\phi\;\mathrm{d}\phi\,\mathrm{d}\theta \\ &= 32\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2} \sin^3\phi\;\cos\phi\;\mathrm{d}\phi\,\mathrm{d}\theta \\ &= 32(2\pi)\int_{\phi=0}^{\pi/2} \sin^3\phi\;\cos\phi\;\mathrm{d}\phi \\ &= 64\pi\left[\frac{\sin^4\phi}{4}\right]_{\phi=0}^{\pi/2} \\ &= 16\pi\,. \end{aligned}$$

Therefore,

$$\boxed{\iint_{\mathcal{S}} \left(x^2 z + y^2 z\right)\,\mathrm{d}S = 16\pi}$$

10. **[20 pts]** Evaluate the surface integral $\iint_{\mathcal{S}} \mathbf{F}\cdot\mathrm{d}\mathbf{S}$, where the vector field $\mathbf{F}$ is given by $\mathbf{F}(\mathbf{r}) = xze^y\mathbf{i} - xze^y\mathbf{j} + z\mathbf{k}$ and $\mathcal{S}$ is the part of the plane $x+y+z=1$ in the first octant and has downward orientation.

SOLUTION

This is a problem that can be solved in Cartesian coordinates, with x and y as parameters and the equation of the surface (the plane) being given by

$$z(x,y) = 1 - x - y\,, \qquad \text{with} \qquad x \geq 0\,, \quad y \geq 0\,, \quad 0 \leq x + y \leq 1\,.$$

The element of surface is given by

$$\mathrm{d}S = \left(\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\right)\,\mathrm{d}x\,\mathrm{d}y = \sqrt{3}\;\mathrm{d}x\,\mathrm{d}y\,,$$

and the unit normal $\mathbf{n}$ with downward orientation is

$$= \frac{1}{\sqrt{3}}\left(-\mathbf{i} - \mathbf{j} - \mathbf{k}\right)\,.$$

Putting all this information together we have

$$\begin{aligned}\mathbf{F}\cdot \mathrm{d}\mathbf{S} &= \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S \\ &= \big(xze^y\mathbf{i} - xze^y\mathbf{j} + z\mathbf{k}\big)\cdot\frac{1}{\sqrt{3}}\big(-\mathbf{i}-\mathbf{j}-\mathbf{k}\big)\,\sqrt{3}\,\mathrm{d}x\,\mathrm{d}y \\ &= -z(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= -(1-x-y)\,\mathrm{d}x\,\mathrm{d}y\,.\end{aligned}$$

Hence

$$\begin{aligned}\iint_{\mathcal{S}} \mathbf{F}\cdot\mathrm{d}\mathbf{S} &= \int_{y=0}^{1}\int_{x=0}^{1-y}(-1+x+y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{y=0}^{1}\left[-x+\frac{1}{2}x^2+xy\right]_{x=0}^{1-y}\mathrm{d}y \\ &= \int_{y=0}^{1}\left(-\frac{1}{2}+y-\frac{1}{2}y^2\right)\mathrm{d}y \\ &= \left[-\frac{1}{2}y+\frac{1}{2}y^2-\frac{1}{6}y^3\right]_{y=0}^{1} \\ &= \frac{1}{6}\,.\end{aligned}$$

Therefore

$$\boxed{\iint_{\mathcal{S}} \mathbf{F}\cdot\mathrm{d}\mathbf{S} = -\frac{1}{6}}$$

11. **[20 pts]** Evaluate the circulation of the vector field $\mathbf{F}(\mathbf{r}) = yz\mathbf{i} + 2xz\mathbf{j} + e^{xy}\mathbf{k}$ along the *counterclockwise (positively-oriented)* circle $x^2+y^2=16$, $z=5$.

SOLUTION

Stokes' Theorem states that if $\mathcal{S}$ is an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve $\mathcal{C}$ with positive orientation, then for any vector field $\mathbf{F}$ whose components have continuous partial derivatives on an open region of $\mathbb{R}^3$ that contains $\mathcal{S}$, then

$$\int_{\mathcal{C}} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \iint_{\mathcal{S}} \operatorname{curl}\mathbf{F}\cdot\mathrm{d}\mathbf{S}\,.$$

We can use Cartesian coordinates to set up the problem and then, if needed, change to more a more appropriate set of coordinates. We take the surface $\mathcal{S}$ to be the circular disk

$$z(x,y) = 5\,, \qquad \text{with} \qquad x^2+y^2 \le 16\,.$$

If the curve $\mathcal{C}$ is positively oriented, then the normal to $\mathcal{S}$ points upwards in the z-direction,

$$\mathbf{n} = \mathbf{k}\,,$$

and the element of surface $\mathrm{d}S$ is given by

$$\mathrm{d}S = \sqrt{\left(\frac{\partial z(x,y)}{\partial x}\right)^2 + \left(\frac{\partial z(x,y)}{\partial x}\right)^2 + 1}\;\mathrm{d}x\,\mathrm{d}y = \mathrm{d}x\,\mathrm{d}y$$

Therefore,

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \iint_{\mathcal{S}} \operatorname{curl}\mathbf{F}\cdot\mathrm{d}\mathbf{S} \\ &= \iint_{\mathcal{S}} \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S \\ &= \iint_{x^2+y^2\le 16} \operatorname{curl}\mathbf{F}\cdot\mathbf{k}\;\mathrm{d}x\,\mathrm{d}y\,.\end{aligned}$$

This last expression tells me that I only need the **k**-component of the curl of **F**, that is,

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ yz & 2xz & e^{xy} \end{vmatrix} = \cdots\mathbf{i} + \cdots\mathbf{j} + z\mathbf{k}$$

Hence

$$\begin{aligned} \int_{\mathcal{C}} \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \iint_{x^2+y^2\leq 16} z(x,y)\ \mathrm{d}x\,\mathrm{d}y \\ &= \int_{\theta=0}^{2\pi}\int_{r=0}^{4} 5\,r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= 5\left(\int_{\theta=0}^{2\pi}\mathrm{d}\theta\right)\left(\int_{r=0}^{4} r\,\mathrm{d}r\right) \\ &= 5\ [\theta]_{\theta=0}^{2\pi}\ \left[\frac{r^2}{2}\right]_{r=0}^{4} \\ &= 80\pi\,. \end{aligned}$$

Therefore

$$\boxed{\int_{\mathcal{C}} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = 80\pi}$$

12. **[20 pts]** Evaluate the flux of the vector field $\mathbf{F} = xy\sin z\,\mathbf{i} + \cos(xz)\mathbf{j} + y\cos z\,\mathbf{k}$ through the surface of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

SOLUTION

> *Gauss' Theorem* states that if $\mathcal{V}$ is a simple solid region, and $\mathcal{S}$ is the boundary surface of $\mathcal{V}$ with positive (outward) orientation, and **F** is a vector field whose component functions have continuous partial derivatives on an open region of $\mathbb{R}^3$ that contains $\mathcal{V}$, then
>
> $$\iint_{\mathcal{S}} \mathbf{F}\cdot\mathrm{d}\mathbf{S} = \iiint_{\mathcal{V}} \operatorname{div}\mathbf{F}\ \mathrm{d}V\,.$$

We begin by computing the divergence of the vector field, which is always simpler to evaluate in Cartesian coordinates. If the volume integral of the divergence calls for another coordinate system, we can do that later.

$$\begin{aligned} \operatorname{div}\mathbf{F} &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \\ &= \frac{\partial}{\partial x}\left[xy\sin z\right] + \frac{\partial}{\partial y}\left[\cos(xz)\right] + \frac{\partial}{\partial z}\left[y\cos z\right] \\ &= y\sin z - y\sin z \\ &= 0\,. \end{aligned}$$

Therefore,

$$\boxed{\iint_{\mathcal{S}} \mathbf{F}\cdot\mathrm{d}\mathbf{S} = \iiint_{\mathcal{V}} \operatorname{div}\mathbf{F}\ \mathrm{d}V = 0}$$

13. **[10 pts]** The **Laplacian** of a twice-differentiable scalar function f is defined to be

$$\nabla^2 f \equiv \operatorname{div}(\operatorname{grad} f) = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\,.$$

Let $\dfrac{\partial f}{\partial n} \equiv (\boldsymbol{\nabla} f)\cdot\mathbf{n}$ denote the directional derivative of the scalar function f in the direction of the outer unit normal vector $\mathbf{n}$ to the surface $\mathcal{S}$ that bounds the region $\mathcal{T}$. Use the above definition of the Laplacian and Gauss' Theorem to show the following integration formula

$$\iint_{\mathcal{S}} \frac{\partial f}{\partial n}\, \mathrm{d}S = \iiint_{\mathcal{T}} \nabla^2 f \;\mathrm{d}V\,.$$

SOLUTION.

$$\begin{aligned} \iint_{\mathcal{S}} \frac{\partial f}{\partial n}\, \mathrm{d}S &= \iint_{\mathcal{S}} \boldsymbol{\nabla} f\cdot\mathbf{n}\, \mathrm{d}S \\ &= \iiint_{\mathcal{T}} \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla} f)\, \mathrm{d}V \\ &= \iiint_{\mathcal{T}} \nabla^2 f \;\mathrm{d}V\,. \end{aligned}$$

A SHORT TABLE OF USEFUL INTEGRALS

1. $\displaystyle\int \sin x \; \mathrm{d}x = -\cos x$
2. $\displaystyle\int \cos x \; \mathrm{d}x = \sin x$
3. $\displaystyle\int \tan x \; \mathrm{d}x = -\ln|\cos x|$
4. $\displaystyle\int \cot x \; \mathrm{d}x = \ln|\sin x|$
5. $\displaystyle\int \sin^2 x \; \mathrm{d}x = \frac{x}{2} - \frac{\sin 2x}{4}$
6. $\displaystyle\int \cos^2 x \; \mathrm{d}x = \frac{x}{2} + \frac{\sin 2x}{4}$
7. $\displaystyle\int \tan^2 x \; \mathrm{d}x = x + \tan x$
8. $\displaystyle\int \cot^2 x \; \mathrm{d}x = -x - \cot x$
9. $\displaystyle\int \sin^3 x \; \mathrm{d}x = -\frac{3\cos x}{4} + \frac{\cos 3x}{12}$
10. $\displaystyle\int \cos^3 x \; \mathrm{d}x = \frac{3\sin x}{4} + \frac{\sin 3x}{12}$
11. $\displaystyle\int \tan^3 x \; \mathrm{d}x = \ln|\cos x| + \frac{\sec^2 x}{2}$
12. $\displaystyle\int \cot^3 x \; \mathrm{d}x = -\ln|\sin x| - \frac{\csc^2 x}{2}$
13. $\displaystyle\int \sin^n x \; \mathrm{d}x = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x \; \mathrm{d}x$
14. $\displaystyle\int \cos^n x \; \mathrm{d}x = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x \; \mathrm{d}x$
15. $\displaystyle\int \tan^n x \; \mathrm{d}x = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x \; \mathrm{d}x$
16. $\displaystyle\int \cot^n x \; \mathrm{d}x = \frac{-1}{n-1}\cot^{n-1} x - \int \cot^{n-2} x \; \mathrm{d}x$
17. $\displaystyle\int x \;\sin x \; \mathrm{d}x = \sin x - x\cos x$
18. $\displaystyle\int x^n \sin x \; \mathrm{d}x = -x^n \cos x + n\int x^{n-1} \cos x \; \mathrm{d}x$
19. $\displaystyle\int x \;\cos x \; \mathrm{d}x = \cos x + x\sin x$
20. $\displaystyle\int x^n \cos x \; \mathrm{d}x = x^n \sin x - n\int x^{n-1} \sin x \; \mathrm{d}x$
21. $\displaystyle\int e^{ax} \sin bx \; \mathrm{d}x = \frac{e^{ax}}{a^2+b^2}\,(a\sin bx - b\cos bx)$
22. $\displaystyle\int e^{ax} \cos bx \; \mathrm{d}x = \frac{e^{ax}}{a^2+b^2}\,(a\cos bx + b\sin bx)$