CALCULUS 3

September 17, 2008

1st TEST

YOUR NAME:

- ◯ **001** B. KATZ-MOSES .... (8AM)
- ◯ **002** J. SANDERS .... (9AM)
- ◯ **003** J. NEWHALL .... (10AM)
- ◯ **004** A. SPINA .... (11AM)
- ◯ **005** E. ANGEL .... (12PM)
- ◯ **006** A. SPINA .... (1PM)
- ◯ **007** A. SPINA .... (3PM)

SHOW ALL YOUR WORK

final answers without any supporting work
will receive no credit *even if they are right!*

No calculators allowed.
No cheat-sheets allowed.

Partial credit will be given for any **reasonable amount of work pointing in the right direction** towards the solution of your problem. You will not get any partial credit for memorizing formulas and not knowing how to use them, or for anything you write that is not directly related to the solution of your problem.

If your tests contains **more than one solution or answer** to a problem or part of a problem, and one of them is wrong, then it will be **the wrong one** the one that **counts** for your grading!

DO NOT WRITE INSIDE THIS BOX!

problem	points	score
1	10 pts	
2	10 pts	
3	10 pts	
4	10 pts	
5	10 pts	
6	15 pts	
7	10 pts	
8	15 pts	
9	10 pts	
TOTAL	100 pts	

1. **[10 pts]** If $\mathbf{u} = \mathbf{i}+\mathbf{j}-5\mathbf{k}$, and $\mathbf{v} = 2\mathbf{i}+\mathbf{j}-\mathbf{k}$, write $\mathbf{u}$ as the sum of a vector parallel to $\mathbf{v}$ and a vector orthogonal to $\mathbf{v}$.

SOLUTION:

$$\begin{aligned}
\mathbf{u}_{\parallel} &= \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{v}||^2}\mathbf{v} \\
&= \frac{(1)(2)+(1)(1)+(-5)(-1)}{(2)^2+(1)^2+(-1)^2}\big(2\mathbf{i}+\mathbf{j}-\mathbf{k}\big) \\
&= \frac{8}{6}\big(2\mathbf{i}+\mathbf{j}-\mathbf{k}\big) \\
&= \boxed{\frac{8}{3}\mathbf{i}+\frac{4}{3}\mathbf{j}-\frac{4}{3}\mathbf{k}} \\
\mathbf{u}_{\perp} &= \mathbf{u}-\mathbf{u}_{\parallel} \\
&= \big(\mathbf{i}+\mathbf{j}-5\mathbf{k}\big)-\big(\frac{8}{3}\mathbf{i}+\frac{4}{3}\mathbf{j}-\frac{4}{3}\mathbf{k}\big) \\
&= \boxed{-\frac{5}{3}\mathbf{i}-\frac{1}{3}\mathbf{j}-\frac{11}{3}\mathbf{k}}
\end{aligned}$$

2. **[10 pts]** For what values of a will the vectors $\mathbf{u} = 2\mathbf{i}+4\mathbf{j}-5\mathbf{k}$, and $\mathbf{v} = -4\mathbf{i}-8\mathbf{j}+a\mathbf{k}$, be parallel?

SOLUTION:

1st Way.

$$\mathbf{v}\parallel\mathbf{u} \quad\Leftrightarrow\quad \mathbf{v}=\lambda\mathbf{u} \quad\Leftrightarrow\quad \begin{cases} v_1 = \lambda u_1 \\ v_2 = \lambda u_2 \\ v_3 = \lambda u_3 \end{cases}$$

In our case we have

$$\mathbf{v}\parallel\mathbf{u} \quad\Leftrightarrow\quad \begin{cases} -4 = 2\lambda \\ -8 = 4\lambda \\ a = -5\lambda \end{cases}$$

The first two equations are identically satisfied for $\lambda = -2$, therefore, the third equations yields

$$\boxed{\mathbf{u}\parallel\mathbf{v} \quad\Leftrightarrow\quad a = 10}$$

2nd Way.

$$\mathbf{u}\parallel\mathbf{v} \quad\Leftrightarrow\quad \mathbf{u}\times\mathbf{v} = \mathbf{0}\,.$$

$$\mathbf{0} = \mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & -5 \\ -4 & -8 & a \end{vmatrix} = (4a-40)\mathbf{i}+(20-2a)\mathbf{j}+0\mathbf{k} \quad\Rightarrow\quad \begin{cases} 4a-40 = 0\,, \\ 20-2a = 0\,. \end{cases}$$

Therefore

$$\boxed{\mathbf{u}\parallel\mathbf{v} \quad\Leftrightarrow\quad a = 10}$$

3. **[10 pts]** Parametrize the line segment joining the points $P(1,2,0)$ and $Q(1,3,-1)$.

SOLUTION:

The line segment joining the points P (with position vector $\vec{OP}\equiv\mathbf{p}$) and Q (with position vector $\vec{OQ}\equiv\mathbf{q}$) is given by

$$\mathbf{r}(t) = \mathbf{p}+t(\mathbf{q}-\mathbf{p})\,, \qquad t\in[0,1]\,.$$

in our case we have

$$\mathbf{r}(t) = <1,2,0> + t\big(<1,3,-1> - <1,2,0>\big) \quad t\in[0,1]\,,$$

that is,

$$\boxed{\mathbf{r}(t) =< 1,2,0 > +t < 0,1,-1 > \quad t \in [0,1]}$$

or

$$\boxed{\begin{cases} x(t) &= 1\,, \\ y(t) &= 2+t\,, \qquad t \in [0,1] \\ z(t) &= -t\,, \end{cases}}$$

where $< x(t), y(t), z(t) >\equiv \mathbf{r}(t)$.

4. **[10 pts]** Find the equation of the plane through the points $Q_1(1,0,0)$, $Q_2(0,1,0)$, and $Q_3(0,0,1)$.

SOLUTION:

1st Way. The vector equation of a plane through a point P_0 and with normal $\mathbf{n}$ is given by

$$\mathbf{n}\cdot\big(\mathbf{r}-\mathbf{r}_0\big)=0\,,$$

where $\mathbf{r}=\vec{OP} =< x,y,z >$ is the generic position vector, and $\mathbf{r}_0=\vec{OP_o} =< x_0,y_0,z_0 >$ is the position vector of P_0, the reference point in the plane.

We already have one point in the plane–actually three of them, we can chose any.

We only need to find a vector perpendicular to the plane. But a vector perpendicular to the plane has to be perpendicular to every vector in the plane. Two such vectors are

$$\begin{aligned} \mathbf{u} &= \vec{OQ_2}-\vec{OQ_1} &= < 0,1,0 > - < 1,0,0 > &= < -1,1,0 >\,, \\ \mathbf{v} &= \vec{OQ_3}-\vec{OQ_1} &= < 0,0,1 > - < 1,0,0 > &= < -1,0,1 >\,. \end{aligned}$$

Hence, we can take for our normal vector

$$\mathbf{n}=\mathbf{u}\times\mathbf{v}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} =< 1,1,1 >$$

Therefore, the vector equation for the plane is

$$\boxed{< 1,1,1 > \cdot\big(< x,y,z > - < 1,0,0 >\big)=0}$$

or

$$\boxed{< 1,1,1 > \cdot\big(< x,y,z > - < 0,1,0 >\big)=0}$$

or

$$\boxed{< 1,1,1 > \cdot\big(< x,y,z > - < 0,0,1 >\big)=0}$$

depending on what point in the plane you used as reference point, or, in rectangular form,

$$\boxed{x+y+z=1}$$

2nd Way. Replace the coordinatesof the points in the plane into the general plane equation in rectangular form,

$$ax+by+cz=d\,.$$

We obtain the system of linear equations

$$\begin{cases} a(1)+b(0)+c(0) &= d \\ a(0)+b(1)+c(0) &= d \\ a(0)+b(0)+c(1) &= d \end{cases}$$

whose solutions are

$$a=b=c=d\neq 0\,.$$

Take all of them equal to one, for simplicity, and get

$$\boxed{x+y+z=1}$$

5. **[10 pts]** Find the acute angle between the planes $\Pi_1 : x = 7$ and $\Pi_2 : x + y + \sqrt{2}z = -3$.

SOLUTION:

$$\begin{aligned}\angle(\Pi_1, \Pi_2) &= \angle(\mathbf{n}_1, \mathbf{n}_2)\\ &= \arccos\left|\frac{\mathbf{n}_1}{||\mathbf{n}_1||}\cdot\frac{\mathbf{n}_2}{||\mathbf{n}_2||}\right|\\ &= \arccos\left|\frac{<1,0,0>}{1}\cdot\frac{<1,1,\sqrt{2}>}{2}\right|\\ &= \arccos\left(\frac{1}{2}\right)\end{aligned}$$

where the absolute value bars around the dot product are to ensure that we obtain the smallest of the two angles. Therefore

$$\boxed{\angle(\Pi_1, \Pi_2) = \frac{\pi}{3}}$$

6. **[15 pts]** Find the parametric equation (either vector-parametric or rectangular-parametric) for the line in which the planes $x + 2y + z = 1$ and $x - y + 2z = -8$ intersect.

SOLUTION:

1st Way. The vector parametric eqution of a line is given by

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$$

where $\mathbf{r}_0$ is the position vector of a point in the line, and $\mathbf{v}$ is a vector parallel to the line.

We need to find both, $\mathbf{r}_0$ and $\mathbf{v}$.

To find $\mathbf{r}_0$ we solve for the linear system of the two planes. There are infinite solutions (if the planes intersect), we just choose a simple one. Subtracting the second equation from the first we get,

$$\left\{\begin{array}{ccccccc} x & + & 2y & + & z & = & 1\\ x & - & y & + & 2z & = & -8\end{array}\right. \quad\Rightarrow\quad \left\{\begin{array}{ccccccc} & & 3y & - & z & = & 9\\ x & - & y & + & 2z & = & -8\end{array}\right.$$

choose $z = 0$, $y = 3$ in the first equation, and replace in the second equation to obtain $x = -5$. Therefore

$$\mathbf{r}_0 =< -5, 3, 0 >$$

will do the work.

To find a $\mathbf{v}$, since $\mathbf{v}$ has to be perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$ we can take

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2 & 1\\ 1 & -1 & 2\end{vmatrix} =< 5, -1, -3 >$$

Therefore, the equation of the line we obtain this way is

$$\boxed{\mathbf{r}(t) =< -5, 3, 0 > +t < 5, -1, -3 > \qquad \text{or} \qquad \left\{\begin{array}{lcl} x(t) & = & -5 + 5t\\ y(t) & = & 3 - t\\ z(t) & = & -3t\end{array}\right.}$$

2nd Way. Solving the linear system above with a little more care we get

$$\left\{\begin{array}{rcrcrcr} x & + & 2y & + & z & = & 1 \\ x & - & y & + & 2z & = & -8 \end{array}\right. \quad \Rightarrow \quad \left\{\begin{array}{rcrcrcr} x & & & + & \frac{5}{3}z & = & -5 \\ & & y & - & \frac{1}{3}z & = & 3 \end{array}\right.$$

from which we can write an equation for the line in the form

$$\boxed{\left\{\begin{array}{rcl} x(\tau) & = & -5-\frac{5}{3}\tau \\ y(\tau) & = & 3+\frac{1}{3}\tau \\ z(\tau) & = & \tau \end{array}\right. \quad \text{or} \quad \left\{\begin{array}{rcl} x(s) & = & -5-5s \\ y(t) & = & 3+s \\ z(t) & = & 3s \end{array}\right.}$$

NOTE: The "times" t, τ, and s in the above boxed equations are not the same!

$$t=-s\,, \quad \text{and} \quad s=3\tau\,.$$

The equations only look different, but they represent the same line.

7. **[10 pts]** Given the equation $y^2-x^2-z^2=1$, that represents a quadric surface in standard form, in the grid below
   - describe the traces in words;
   - either in the grid below or in a set of axes of your own if this grid is confussing, sketch the surface.

SOLUTION:

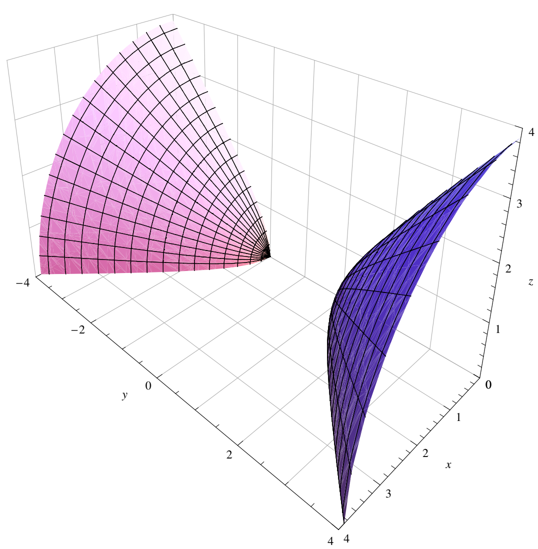
The traces yield

$$x=0 \quad \rightarrow \quad \left\{\begin{array}{l} y^2-z^2=1 \\ x=0 \end{array}\right. \qquad \text{hyperbola}$$

$$y=0 \quad \rightarrow \quad \left\{\begin{array}{l} -x^2-z^2=1 \\ y=0 \end{array}\right. \qquad \text{no intersection}$$

$$z=0 \quad \rightarrow \quad \left\{\begin{array}{l} y^2-x^2=1 \\ z=0 \end{array}\right. \qquad \text{hyperbola}$$

Therefore the quadric represents an *hyperboloid of two sheets* with axis along the y–axis.



8. **[15 pts]** Write an equation for the hyperboloid $2x^2+2y^2-4z^2=1$ in
   (a) cylindrical coordinates;
   (b) in spherical coordinates.

SOLUTION:

(a)

$$2x^2+2y^2-4z^2=1 \qquad \Rightarrow \qquad 2\left(x^2+y^2\right)-4z^2=1$$

therefore

$$\boxed{2r^2-4z^2=1}$$

(b)

$$\begin{aligned} 2x^2 + 2y^2 - 4z^2 = 1 \quad &\Rightarrow \quad 2\left(x^2 + y^2 + z^2\right) - 6z^2 = 1 \\ &\Rightarrow \quad 2\rho^2 - 6\rho^2 \cos^2\phi = 1 \end{aligned}$$

therefore

$$\boxed{2\rho^2\left(1 - 3\cos^2\phi\right) = 1}$$

9. **[10 pts]** Rewrite the spherical equation $\rho \sin\phi = 1$ in rectangular coordinates.

SOLUTION:

$$\begin{aligned} \rho\sin\phi = 1 \quad &\Rightarrow \quad \rho^2 \sin^2\phi = 1 \\ &\Rightarrow \quad \rho^2\left(1 - \cos^2\phi\right) = 1 \\ &\Rightarrow \quad \rho^2 - \rho^2\cos^2\phi = 1\,. \\ &\Rightarrow \quad \left(x^2 + y^2 + z^2\right) - z^2 = 1 \end{aligned}$$

Therefore

$$\boxed{x^2 + y^2 = 1}$$