Review Sheet for Midterm 3

1. Evaluate

$$\iint_R e^{\frac{y-x}{y+x}}\, dA$$

where R is the triangle bounded by the line $x+y=2$ and the coordinates axes.

2. Find $\displaystyle\iint_R \sqrt{x+y}\, dA$, where R is the parallelogram bounded by the lines

$$x+y=0\,, \quad x+y=1\,, \quad 2x-y=0\,, \quad 2x-y=-3$$

3. Find the area of the region bounded by the curves

$$y=x^4\,, \quad y=2x^4\,, \quad xy=1\,, \quad xy=6$$

4. For the following integrals

(a) sketch the region of integration.

(b) reverse the order of integration and evaluate it.

i. $\displaystyle\int_0^2 \int_{\frac{y}{2}}^1 ye^{x^3}\, dx\, dy$

ii. $\displaystyle\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} (6x+2y^2)\, dy\, dx + \int_1^4 \int_{-\sqrt{x}}^{2-x} (6x+2y^2)\, dy\, dx$

5. Find the volume of the solid that lies above the xy-plane, below $z=x$, and within the triangle whose vertices are $(1,0)$, $(0,2)$, and $(1,2)$.

6. Compute the area of the region enclosed by the rose $r=\sin 2\theta$, given in polar coordinates.

7. Find the volume of the solid below the cone $z=\sqrt{x^2+y^2}$, inside the cylinder $x^2+y^2=2y$, and above the plane $z=0$.

8. Compute

$$\iint_R \sqrt{9-x^2-y^2}\, dA\,,$$

where R is the region in the first quadrant within the circle $x^2+y^2=9$.

9. Consider the parametric surface

$$x=u^2\,, \quad y=v^2\,, \quad z=u+v\,, \quad -\infty<u,v<\infty\,.$$

Find an equation of the tangent plane to the surface at the point $(1,4,3)$.

10. Find the area of the portion of the plane $2x+2y+z=8$ in the first octant.

11. Find the area of the portion of the cone $\mathbf{r}(u,v)=u\cos v\,\mathbf{i}+u\sin v\,\mathbf{j}+u\,\mathbf{k}$, for which $0\le u\le 2v$, and $0\le v\le \frac{\pi}{2}$.

12. Find the volume of the solid bounded by $z=2-x^2$, $z=x^2$, $y=0$, and $y=3$.

13. Find the volume if the tetrahedron bounded by $2x + 3y + z = 6$, $x = 0$, $y = 0$, and $z = 0$.

14. Express the following integral as an equivalent integral with z-integration first, y-integration next, and x-integration last.

$$\int_0^2 \int_0^{\sqrt{4-z^2}} \int_0^{\sqrt{4-y^2-z^2}} f(x, y, z)\, dx\, dy\, dz$$

15. Find the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 4$ and the cylinder $x^2 + y^2 = 1$.

16. Use cylindrical or spherical coordinates to evaluate

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} x\, dz\, dy\, dx$$

17. Use cylindrical or spherical coordinates to evaluate

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{-5+x^2+y^2}^{3-x^2-y^2} dz\, dy\, dx$$

18. Evaluate $\displaystyle\int_C xz\, ds$ along

$$C:\ \mathbf{r}(t) = 6t\,\mathbf{i} + 3\sqrt{2}\,t^2\mathbf{j} + 2t^3\,\mathbf{k}, \quad 0 \le t \le 1.$$

19. Evaluate $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = (y + z)\,\mathbf{i} - x^2\,\mathbf{j} - 4y^2\,\mathbf{k}$, and

$$C:\ \mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^4\,\mathbf{k}, \quad 0 \le t \le 1.$$

20. Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} + xy\,\mathbf{j}$ on a particle that moves around the circle $x^2 + y^2 = 4$ in the counter clockwise direction.

21. Given that $\mathbf{F}(x, y) = 2xy^3\,\mathbf{i} + 3x^2y^2\,\mathbf{j}$ and

$$C:\ \mathbf{r}(t) = \sin t\,\mathbf{i} + (t^2 + 1)\,\mathbf{j}, \quad 0 \le t \le \frac{\pi}{2},$$

(a) find the potential function for $\mathbf{F}$.

(b) find the work done moving a particle along C in the force field $\mathbf{F}$.

1. Answer: $\boxed{\displaystyle\iint_R e^{\frac{y-x}{y+x}}\,dA = e - \frac{1}{e}}$.

2. Answer: $\boxed{\displaystyle\iint_R \sqrt{x+y}\,dA = \frac{2}{3}}$.

3. Answer: $\boxed{\ln 2}$.

4. Answer:

i. (a) The region is a triangle with vertices $(0, 0, (1, 0)$, and $(1, 2)$.

(b)

$$\int_0^2 \int_{\frac{y}{2}}^1 ye^{x^3}\,dx\,dy = \int_0^1 \int_0^{2x} ye^{x^3}\,dy\,dx = \cdots = \boxed{\frac{2}{3}(e-1)}$$

ii. (a) The region is bounded by $x = y^2$ and $x = 2 - y$. Note that these curves intersect at $(1, 1)$ and $(4, -2)$.

(b)

$$\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} (6x+2y^2)\,dy\,dx + \int_1^4 \int_{-\sqrt{x}}^{2-x} (6x+2y^2)\,dy\,dx = \int_{-2}^1 \int_{y^2}^{2-y} (6x+2y^2)\,dx\,dy = \cdots = \boxed{\frac{99}{2}}$$

5. Answer: $V = \displaystyle\int_0^1 \int_{2-2x}^2 x\,dy\,dx = \cdots = \boxed{\frac{2}{3}}$

6. Answer: $A = 4\displaystyle\int_0^{\frac{\pi}{2}} \int_0^{\sin 2\theta} r\,dr\,d\theta = \cdots = \boxed{\frac{\pi}{2}}$. Note: $\sin^2 2\theta = \frac{1}{2}(1 - \cos 4\theta)$

7. Answer: $V = \displaystyle\int_0^{\pi} \int_0^{2\sin\theta} \int_0^r r\,dz\,dr\,d\theta = \cdots = \boxed{\frac{32}{9}}$. Note: $\sin^3\theta = (1 - \cos^2\theta)\sin\theta$

8. Answer: Use polar coordinates.

$$\iint_R \sqrt{9 - x^2 - y^2}\,dA = \int_0^{\frac{\pi}{2}} \int_0^3 \sqrt{9 - r^2}\,r\,dr\,d\theta = \cdots = \boxed{\frac{9\pi}{2}}$$

.

9. Answer: The point $(1, 4, 3)$ is obtained when $u = 1$ and $v = 2$. Then

$$\mathbf{w} = \frac{\partial \mathbf{r}}{\partial u}(1,2) \times \frac{\partial \mathbf{r}}{\partial v}(1,2)$$

is normal to the surface, where $\mathbf{r}(u,v) = u^2\,\mathbf{i} + v^2\,\mathbf{j} + (u+v)\,\mathbf{k}$. We compute

$$\frac{\partial \mathbf{r}}{\partial u} = 2u\,\mathbf{i} + \mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial v} = 2v\,\mathbf{j} + \mathbf{k}$$

so $\frac{\partial \mathbf{r}}{\partial u}(1,2) = 2\,\mathbf{i} + \mathbf{k}$ and $\frac{\partial \mathbf{r}}{\partial v}(1,2) = u\mathbf{j} + \mathbf{k}$. Thus, $\mathbf{w} = -4\,\mathbf{i} - 2\,\mathbf{j} + 8\,\mathbf{k}$. Therefore, the equation of the tangent plane is $-4(x-1) - 2(y-4) + 8(z-3) = 0$, which simplifies to $\boxed{2x + y - 4z = -6}$.

10. Answer: We have $z = 8 - 2x - 2y$ so $\frac{\partial z}{\partial x} = -2$ and $\frac{\partial z}{\partial y} = -2$. Thus

$$A = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA = 3 \iint_R dA = \boxed{24}$$

where R is the triangle with vertices $(0,0)$, $(4,0)$, and $(0,4)$. Note: R is a triangle with base and height equal to 4, so $\displaystyle\iint_R dA = 8$.

11. Answer: We have

$$\frac{\partial \mathbf{r}}{\partial u} = \cos v\, \mathbf{i} + \sin v\, \mathbf{j} + \mathbf{k}$$
$$\frac{\partial \mathbf{r}}{\partial v} = -u \sin v\, \mathbf{i} + u \cos v\, \mathbf{j}\,.$$

Thus,

$$\left|\left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right|\right| = || - u\cos v\, \mathbf{i} - u \sin v\, \mathbf{j} + u\, \mathbf{k}|| = \sqrt{2u^2}\,.$$

Now,

$$A = \iint_R \left|\left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right|\right| du\, dv = \int_0^{\frac{\pi}{2}} \int_0^{2v} \sqrt{2u^2}\, du\, dv = \cdots = \boxed{\frac{\sqrt{2}\pi^3}{12}}.$$

12. Answer: $V = \displaystyle\int_0^3 \int_{-1}^1 \int_{x^2}^{2-x^2} dz\, dx\, dy = \cdots = \boxed{8}.$

13. Answer: $V = \displaystyle\int_0^3 \int_0^{2-\frac{2}{3}x} \int_0^{6-2x-3y} dz\, dy\, dx = \cdots = \boxed{6}.$

14. Answer: This is the 1st octant part of the sphere $x^2 + y^2 + z^2 = 4$, so

$$\int_0^2 \int_0^{\sqrt{4-z^2}} \int_0^{\sqrt{4-y^2-z^2}} f(x,y,z)\, dx\, dy\, dz = \boxed{\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} f(x,y,z)\, dz\, dy\, dx}.$$

15. Answer: $V = 2\displaystyle\int_0^{2\pi} \int_0^1 \int_0^{\sqrt{4-r^2}} r\, dz\, dr\, d\theta = \cdots = \boxed{\frac{32\pi}{3} - 4\pi\sqrt{3}}.$

16. Answer:

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} x\, dz\, dy\, dx = \int_0^{\frac{\pi}{2}} \int_0^3 \int_0^{9-r^2} r^2 \cos\theta\, dz\, dr\, d\theta = \cdots = \boxed{\frac{162}{5}}.$$

17. Answer:

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{-5+x^2+y^2}^{3-x^2-y^2} dz\, dy\, dx = \int_0^{\frac{\pi}{2}} \int_0^2 \int_{r^2-5}^{3-r^2} r\, dz\, dr\, d\theta = \cdots = \boxed{4\pi}.$$

18. Answer:

$$\int_C xz\, ds = \int_0^1 (6t)(2t^3)\, ||6\,\mathbf{i} + 6\sqrt{2}t\,\mathbf{j} + 6t^2\,\mathbf{k}||\, dt = \cdots = 72 \int_0^1 t^4(t^2+1)\, dt = \cdots$$

19. Answer:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 [(t^2+t^4)\,\mathbf{i} - t^2\,\mathbf{j} - 4t^4\,\mathbf{k}]\cdot[\,\mathbf{i} + 2t\,\mathbf{j} + 4t^3\,\mathbf{k}]\,dt = \int_0^1 (t^2+t^4-2t^3-16t^7)\,dt = \cdots$$

20. Answer:

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} (4\cos^2 t\,\mathbf{i} + 4\cos t\sin t\,\mathbf{j})\cdot(-2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j})\,dt = \cdots$$

21. Answer:

(a) Since $\dfrac{\partial}{\partial y}(2xy^3) = 6xy^2$ and $\dfrac{\partial}{\partial x}(3x^2y^2) = 6xy^2$, $\mathbf{F}$ is conservative. If we let f be the potential function for $\mathbf{F}$, we have $\nabla f = \mathbf{F}$, so $\dfrac{\partial f}{\partial x} = 2xy^3$ and $\dfrac{\partial f}{\partial y} = 3x^2y^2$. Integrate $\dfrac{\partial f}{\partial x}$ with respect to x to find that $f(x,y) = x^2y^3 + g(y)$. Since $\dfrac{\partial f}{\partial y} = 3x^2y^2$, we find that $g'(y) = 0$, so $g(y)$ is just a constant K. Thus f is of the form $f(x,y) = x^2y^3 + K$. Any value for K will satisfy the equation $\nabla f = \mathbf{F}$, so we might as well choose $K = 0$. Thus $\boxed{f(x,y) = x^2y^3}$ is a potential function for $\mathbf{F}$ (of course $f(x,y) = x^2y^3 + K$ is as well for any constant K).

(b) $W = \displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \nabla f\cdot d\mathbf{r} = \boxed{f(1, \frac{\pi^2}{4}+1) - f(0,1)}$.