

Review Sheet for Midterm 2

1. Compute the following limits. If a limit does not exist, explain why.
 - (a) $\lim_{(x,y) \rightarrow (0,0)} \frac{x-y}{x^2+y^2}$
 - (b) $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2y^2}{\sqrt{x^2+y^2}}$
 - (c) $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3y}{x^6+y^2}$
2. A point moves along the intersection of the elliptic paraboloid $z = x^2 + 3y^2$ and the plane $y = 1$. At what rate is z changing with respect to x when the point is at $(2, 1, 7)$?
3. Use total differentials to approximate the change in the value of $f(x, y) = \ln(\sqrt{1+xy})$ from $P(0, 2)$ to $Q(-0.09, 1.98)$.
4. Given $f(x, y, z) = \frac{x+y}{y+z}$, $P(-1, 1, 1)$, $Q(-0.99, 0.99, 1.01)$,
 - (a) Find the local linear approximation, L , to the function $f(x, y, z)$ at the point P .
 - (b) Use part (a) to approximate $f(Q)$.
5. Let $w = xy + yz + zx$ with $x = u^2 - v^2$, $y = u^2 + v^2$, and $z = u^2v^2$. Compute $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$.
6. The sun is melting a rectangular block of ice. When the block's height is 1 ft and the length of each edge of its square base is 2 ft, its height is decreasing at 2 in/h and the length of the edges of the base is decreasing at 3 in/h. What is the rate of change of the volume of the block at that instant?
7. Let $f(x, y, z) = xyz$. Find the directional derivative of this function at the point $P(1, 1, 1)$ in the direction $\langle 1, 1, 1 \rangle$.
8. Find a unit vector that is normal at $P(2, 3)$ to the level curve of $f(x, y) = 3x^2y^2 - xy$ through P .
9. Find an equation for the tangent plane and parametric equations for the normal line to the surface $z = \frac{1}{2}x^7y^{-2}$ at $P(2, 4, 4)$.
10. Suppose that three quantities x , y , and z , are constrained by the equation $2x^2 + 3y^2 + z^2 = 20$. The graph of this equation is a surface S in space.
 - (a) Verify that the point $P(2, 1, 3)$ is a point on S and find the equation of the tangent plane to S at this point.
 - (b) Near $P(2, 1, 3)$ we can think of z as a function of x and y , $z = f(x, y)$. Without finding $f(x, y)$ explicitly, determine its linear approximation L_f near $x = 2$, $y = 1$.
 - (c) Approximate the value of z corresponding to $x = 1.97$ and $y = 1.12$.

11. Consider the function $f(x, y) = 2x^3\sqrt{4x + 3y^2}$ and the point $P(1, 2, 8)$.

(a) Find an equation of the tangent plane to the graph of $z = f(x, y)$ at the point P .

(b) Find a normal vector to the (tangent plane of the) graph of $z = f(x, y)$ at the point P .

(c) Approximate $f(1.1, 1.8)$.

12. Find the absolute extrema of f on R , where

$f(x, y) = 5 - 3x + 4y$, R is the closed triangular region with vertices $(0, 0)$, $(4, 0)$, $(4, 5)$.

13. Find the points on the surface $z^2 = xy + 1$ that are closest to the origin.

1. Solution:

(a) Along the line $y = 0$,

$$\lim_{x \rightarrow 0} \frac{x}{x^2} = \lim_{x \rightarrow 0} \frac{1}{x},$$

which does not exist. Thus $\lim_{(x,y) \rightarrow (0,0)} \frac{x-y}{x^2+y^2}$ does not exist.

(b) Convert to polar coordinates.

$$\left| \frac{x^2 y^2}{\sqrt{x^2 + y^2}} \right| = \left| \frac{r^4 \cos^2 \theta \sin^2 \theta}{r} \right| \leq r^3.$$

As $r \rightarrow 0^+$, $r^3 \rightarrow 0$, so by the Squeeze Theorem,

$$\left| \frac{x^2 y^2}{\sqrt{x^2 + y^2}} \right| \rightarrow 0 \text{ as } (x, y) \rightarrow (0, 0).$$

Thus $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^2}{\sqrt{x^2 + y^2}} = \boxed{0}$.

(c) Along the line $y = x$,

$$\lim_{x \rightarrow 0} \frac{x^4}{x^6 + x^2} = \lim_{x \rightarrow 0} \frac{x^2}{x^4 + 1} = 0.$$

Along the curve $y = x^3$,

$$\lim_{x \rightarrow 0} \frac{x^6}{2x^6} = \frac{1}{2}.$$

We have different limits, so $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 y}{x^6 + y^2}$ does not exist.

2. Solution: Need to find $\frac{\partial z}{\partial x}(2, 1)$. Since, $\frac{\partial z}{\partial x} = 2x$, the answer is 4.

3. Solution:

$$f(x, y) = \ln(\sqrt{1 + xy}) = \frac{1}{2} \ln(1 + xy)$$

$$df = \frac{1}{2} \frac{y}{1 + xy} dx + \frac{1}{2} \frac{x}{1 + xy} dy$$

$$\Delta f \approx \frac{1}{2} \frac{2}{1 + (0)(2)} (-0.09 - 0) + \frac{1}{2} \frac{0}{1 + (0)(2)} (1.98 - 2) = \boxed{-0.09}$$

4. Solution:

(a)

$$L(x, y, z) = f_x(-1, 1, 1)(x + 1) + f_y(-1, 1, 1)(y - 1) + f_z(-1, 1, 1)(z - 1)$$

$$f_x = \frac{1}{y + z} \text{ so } f_x(-1, 1, 1) = \frac{1}{2}$$

$$f_y = \frac{(y + z) - (x + y)}{(y + z)^2} \text{ so } f_y(-1, 1, 1) = \frac{1}{2}$$

$$f_z = (x + y) \frac{-1}{(y + z)^2} \text{ so } f_z(-1, 1, 1) = 0$$

Putting everything together, we get $L(x, y, z) = \frac{1}{2}(x + 1) + \frac{1}{2}(y - 1)$.

(b)

$$f(-0.99, 0.99, 1.01) \approx L(-0.99, 0.99, 1.01) = \frac{1}{2}(-0.99 + 1) + \frac{1}{2}(0.99 - 1) = \boxed{0}$$

5. Solution:

$$\begin{aligned}\frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u} \\ &= (y+z)(2u) + (x+z)(2u) + (y+x)(2uv^2) \\ &= (u^2 + v^2 + u^2v^2)(2u) + (u^2 - v^2 + u^2v^2)(2u) + (2u^2)(2uv^2)\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial v} \\ &= (y+z)(-2v) + (x+z)(2v) + (y+x)(2u^2v) \\ &= (u^2 + v^2 + u^2v^2)(-2v) + (u^2 - v^2 + u^2v^2)(2v) + (2u^2)(2u^2v)\end{aligned}$$

6. Solution: We have $V = x^2y$. We want $\frac{dV}{dt}(2, 1)$.

$$\frac{dV}{dt} = \frac{\partial V}{\partial x} \frac{dx}{dt} + \frac{\partial V}{\partial y} \frac{dy}{dt} = (2xy) \frac{dx}{dt} + x^2 \frac{dy}{dt}$$

$$\frac{dV}{dt}(2, 1) = 2(2)(1) \left(-\frac{3}{12}\right) + (2)^2 \left(-\frac{2}{12}\right) = -\frac{5}{3}$$

Our answer is $\boxed{-\frac{5}{3}\text{ft}^3/\text{h}}$.

7. Solution: Set $\mathbf{u} = \frac{\langle 1, 1, 1 \rangle}{\|\langle 1, 1, 1 \rangle\|} = \frac{1}{\sqrt{3}}\langle 1, 1, 1 \rangle$. We want $\mathbf{D}_{\mathbf{u}}f(1, 1, 1) = \nabla f(1, 1, 1) \cdot \mathbf{u}$. Now, $\nabla f = \langle yz, xz, xy \rangle$, so $\nabla f(1, 1, 1) = \langle 1, 1, 1 \rangle$. Thus $\mathbf{D}_{\mathbf{u}}f(1, 1, 1) = \boxed{\frac{3}{\sqrt{3}}}$.

8. Solution: $\nabla f(P)$ is normal to the level curve of f through P .

$\nabla f = \langle 6xy^2 - y, 6x^2y - x \rangle$, so $\nabla f(P) = \langle 105, 70 \rangle$. We need a unit vector, so we use $\frac{\langle 105, 70 \rangle}{\|\langle 105, 70 \rangle\|} = \frac{1}{5\sqrt{637}}\langle 105, 70 \rangle = \boxed{\frac{1}{\sqrt{637}}\langle 21, 14 \rangle}$.

9. Solution: The tangent plane at P will have $\mathbf{n} = \langle \frac{\partial z}{\partial x}(P), \frac{\partial z}{\partial y}(P), -1 \rangle$ as a normal vector. Since,

$\frac{\partial z}{\partial x} = \frac{7x^6}{2y^2}$ and $\frac{\partial z}{\partial y} = -\frac{x^7}{y^3}$, we calculate that $\mathbf{n} = \langle 28, -2, -1 \rangle$. Thus the tangent plane at P

has equation $\boxed{28(x-2) - 2(y-4) - 4(z-4) = 0}$. The normal line at P has \mathbf{n} as a direction vector, so the normal line has vector equation $\langle 2, 4, 4 \rangle + t\langle 28, -2, -1 \rangle$. Parametrically, we get

$$\boxed{x = 2 + 28t, y = 4 - 2t, z = 4 - t}$$

10. Answers:

(a) The tangent plane to S at the point P is given by $\boxed{8(x-2) + 6(y-1) + 6(z-3) = 0}$.

(b) The linear approximation of $z = f(x, y)$ near $(2, 1)$ is $\boxed{L_f(x, y) = 3 - \frac{4}{3}(x-2) - (y-1)}$.

(c) $\boxed{z(1.97, 1, 12) \approx 2.92}$.

11. Answers:

(a) Tangent plane is given by $z = 8 + 25(x - 1) + 3(y - 2)$.

(b) A normal vector to the graph of $z = f(x, y)$ at the point P is $\mathbf{n} = \langle -25, -3, 1 \rangle$.

(c) $f(1.1, 1.8) \approx 9.9$.

12. Solution: First find the critical points:

$$\begin{aligned}f_x &= -3 \\f_y &= 4\end{aligned}$$

Since the first partials are never zero, there are no critical points.

Next find extrema on the boundary:

(a) One leg of the triangle is $y = 0$, $0 \leq x \leq 4$. Along this path the function values are

$$f(x, 0) = 5 - 3x, \quad 0 \leq x \leq 4$$

Since this is a decreasing function, the absolute maximum is $f(0, 0) = 5$ and the absolute minimum is $f(4, 0) = -7$.

(b) Another leg of the triangle is $x = 4$, $0 \leq y \leq 5$. Along this path the function values are

$$f(4, y) = -7 + 4y, \quad 0 \leq y \leq 5$$

Since this is an increasing function, the absolute maximum is $f(4, 5) = 13$ and the absolute minimum is $f(4, 0) = -7$.

(c) Along the hypotenuse of the triangle, $y = \frac{5}{4}x$, $0 \leq x \leq 4$. Along this path, the function values are

$$f\left(x, \frac{5}{4}x\right) = 5 - 3x + 4\left(\frac{5}{4}x\right) = 5 + 2x, \quad 0 \leq x \leq 4$$

Since this is an increasing function, the absolute maximum is $f(4, 5) = 13$ and the absolute minimum is $f(0, 0) = 5$.

Therefore the absolute maximum is $f(4, 5) = 13$ and the absolute minimum is $f(4, 0) = -7$.

13. Answer: The points $(0, 0, \pm 1)$ on the surface are closest to the origin.