Review Sheet for Midterm 2

1. Compute the following limits. If a limit does not exist, explain why.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x-y}{x^2+y^2}$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{\sqrt{x^2+y^2}}$

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2}$

2. A point moves along the intersection of the elliptic paraboloid $z = x^2 + 3y^2$ and the plane $y = 1$. At what rate is z changing with respect to x when the point is at $(2, 1, 7)$?

3. Use total differentials to approximate the change in the value of $f(x, y) = \ln\left(\sqrt{1 + xy}\right)$ from $P(0, 2)$ to $Q(-0.09, 1.98)$.

4. Given $f(x, y, z) = \dfrac{x + y}{y + z}$, $P(-1, 1, 1)$, $Q(-0.99, 0.99, 1.01)$,

(a) Find the local linear approximation, L, to the function $f(x, y, z)$ at the point P.

(b) Use part (a) to approximate $f(Q)$.

5. Let $w = xy + yz + zx$ with $x = u^2 - v^2$, $y = u^2 + v^2$, and $z = u^2v^2$. Compute $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$.

6. The sun is melting a rectangular block of ice. When the block's height is 1 ft and the length of each edge of its square base is 2 ft, its height is decreasing at 2 in/h and the length of the edges of the base is decreasing at 3 in/h. What is the rate of change of the volume of the block at that instant?

7. Let $f(x, y, z) = xyz$. Find the directional derivative of this function at the point $P(1, 1, 1)$ in the direction $\langle 1, 1, 1\rangle$.

8. Find a unit vector that is normal at $P(2, 3)$ to the level curve of $f(x, y) = 3x^2y^2 - xy$ through P.

9. Find an equation for the tangent plane and parametric equations for the normal line to the surface $z = \frac{1}{2}x^7y^{-2}$ at $P(2, 4, 4)$.

10. Suppose that three quantities x, y, and z, are constrained by the equation $2x^2 + 3y^2 + z^2 = 20$. The graph of this equation is a surface S in space.

(a) Verify that the point $P(2, 1, 3)$ is a point on S and find the equation of the tangent plane to S at this point.

(b) Near $P(2, 1, 3)$ we can think of z as a function of x and y, $z = f(x, y)$. Without finding $f(x, y)$ explicitly, determine its linear approximation L_f near $x = 2$, $y = 1$.

(c) Approximate the value of z corresponding to $x = 1.97$ and $y = 1.12$.

11. Consider the function $f(x, y) = 2x^3\sqrt{4x + 3y^2}$ and the point $P(1, 2, 8)$.

(a) Find an equation of the tangent plane to the graph of $z = f(x, y)$ at the point P.

(b) Find a normal vector to the (tangent plane of the) graph of $z = f(x, y)$ at the point P.

(c) Approximate $f(1.1, 1.8)$.

12. Find the absolute extrema of f on R, where

$f(x, y) = 5 - 3x + 4y,$ R is the closed triangular region with vertices $(0, 0), (4, 0), (4, 5)$.

13. Find the points on the surface $z^2 = xy + 1$ that are closest to the origin.

1. Solution:

(a) Along then line $y = 0$,

$$\lim_{x\to 0} \frac{x}{x^2} = \lim_{x\to 0} \frac{1}{x},$$

which does not exist. Thus $\lim\limits_{(x,y)\to(0,0)} \dfrac{x-y}{x^2+y^2}$ $\boxed{\text{does not exist}}$.

(b) Covert to polar coordinates.

$$\left|\frac{x^2y^2}{\sqrt{x^2+y^2}}\right| = \left|\frac{r^4\cos^2\theta\sin^2\theta}{r}\right| \le r^3 .$$

As $r \to 0^+$, $r^3 \to 0$, so by the Squeeze Theorem,

$$\left|\frac{x^2y^2}{\sqrt{x^2+y^2}}\right| \to 0 \text{ as } (x,y) \to (0,0) .$$

Thus $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{\sqrt{x^2+y^2}} = \boxed{0}$.

(c) Along the line $y = x$,

$$\lim_{x\to 0} \frac{x^4}{x^6+x^2} = \lim_{x\to 0} \frac{x^2}{x^4+1} = 0 .$$

Along the curve $y = x^3$,

$$\lim_{x\to 0} \frac{x^6}{2x^6} = \frac{1}{2} .$$

We have different limits, so $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3y}{x^6+y^2}$ $\boxed{\text{does not exist}}$.

2. Solution: Need to find $\frac{\partial z}{\partial x}(2,1)$. Since, $\frac{\partial z}{\partial x} = 2x$, the answer is $\boxed{4}$.

3. Solution:

$$f(x,y) = \ln(\sqrt{1+xy}) = \frac{1}{2}\ln(1+xy)$$

$$df = \frac{1}{2}\frac{y}{1+xy}dx + \frac{1}{2}\frac{x}{1+xy}dy$$

$$\Delta f \approx \frac{1}{2}\frac{2}{1+(0)(2)}(-0.09-0) + \frac{1}{2}\frac{0}{1+(0)(2)}(1.98-2) = \boxed{\text{-0.09}}$$

4. Solution:

(a)

$$L(x,y,z) = f_x(-1,1,1)(x+1) + f_y(-1,1,1)(y-1) + f_z(-1,1,1)(z-1)$$

$$f_x = \frac{1}{y+z} \text{ so } f_x(-1,1,1) = \frac{1}{2}$$

$$f_y = \frac{(y+z)-(x+y)}{(y+z)^2} \text{ so } f_y(-1,1,1) = \frac{1}{2}$$

$$f_z = (x+y)\frac{-1}{(y+z)^2} \text{ so } f_z(-1,1,1) = 0$$

Putting everything together, we get $\boxed{L(x,y,z) = \frac{1}{2}(x+1) + \frac{1}{2}(y-1)}$.

(b)

$$f(-0.99, 0.99, 1.01) \approx L(-0.99, 0.99, 1.01) = \frac{1}{2}(-0.99+1) + \frac{1}{2}(0.99-1) = \boxed{0}$$

5. Solution:

$$\begin{aligned}\frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} \\ &= (y+z)(2u) + (x+z)(2u) + (y+x)(2uv^2) \\ &= (u^2+v^2+u^2v^2)(2u) + (u^2-v^2+u^2v^2)(2u) + (2u^2)(2uv^2)\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} \\ &= (y+z)(-2v) + (x+z)(2v) + (y+x)(2u^2v) \\ &= (u^2+v^2+u^2v^2)(-2v) + (u^2-v^2+u^2v^2)(2v) + (2u^2)(2u^2v)\end{aligned}$$

6. Solution: We have $V = x^2y$. We want $\frac{dV}{dt}(2,1)$.

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} = (2xy)\frac{dx}{dt} + x^2\frac{dy}{dt}$$

$$\frac{dV}{dt}(2,1) = 2(2)(1)\left(-\frac{3}{12}\right) + (2)^2\left(-\frac{2}{12}\right) = -\frac{5}{3}$$

Our answer is $\boxed{-\frac{5}{3}\text{ft}^3/\text{h}}$.

7. Solution: Set $\mathbf{u} = \frac{\langle 1,1,1\rangle}{||\langle 1,1,1\rangle||} = \frac{1}{\sqrt{3}}\langle 1,1,1\rangle$. We want $\mathbf{D_u}f(1,1,1) = \nabla f(1,1,1)\cdot\mathbf{u}$. Now, $\nabla f = \langle yz, xz, xy\rangle$, so $\nabla f(1,1,1) = \langle 1,1,1\rangle$. Thus $\mathbf{D_u}f(1,1,1) = \boxed{\frac{3}{\sqrt{3}}}$.

8. Solution: $\nabla f(P)$ is normal to the level curve of f through P.
$\nabla f = \langle 6xy^2 - y, 6x^2y - x\rangle$, so $\nabla f(P) = \langle 105, 70\rangle$. We need a unit vector, so we use $\frac{\langle 105,70\rangle}{||\langle 105,70\rangle||} = \frac{1}{5\sqrt{637}}\langle 105,70\rangle = \boxed{\frac{1}{\sqrt{637}}\langle 21,14\rangle}$.

9. Solution: The tangent plane at P will have $\mathbf{n} = \langle \frac{\partial z}{\partial x}(P), \frac{\partial z}{\partial y}(P), -1\rangle$ as a normal vector. Since, $\frac{\partial z}{\partial x} = \frac{7x^6}{2y^2}$ and $\frac{\partial z}{\partial y} = -\frac{x^7}{y^3}$, we calculate that $\mathbf{n} = \langle 28, -2, -1\rangle$. Thus the tangent plane at P has equation $\boxed{28(x-2) - 2(y-4) - 4(z-4) = 0}$. The normal line at P has $\mathbf{n}$ as a direction vector, so the normal line has vector equation $\langle 2,4,4\rangle + t\langle 28,-2,-1\rangle$. Parametrically, we get $\boxed{x = 2+28t,\ y = 4-2t,\ z = 4-t}$.

10. Answers:

(a) The tangent plane to S at the point P is given by $\boxed{8(x-2)+6(y-1)+6(z-3)=0}$.

(b) The linear approximation of $z = f(x,y)$ near $(2,1)$ is $\boxed{L_f(x,y) = 3 - \frac{4}{3}(x-2) - (y-1)}$.

(c) $\boxed{z(1.97, 1, 12) \approx 2.92}$.

11. Answers:

(a) Tangent plane is given by $\boxed{z = 8 + 25(x-1) + 3(y-2)}$.

(b) A normal vector to the graph of $z = f(x,y)$ at the point P is $\boxed{\mathbf{n} = \langle -25, -3, 1\rangle}$.

(c) $\boxed{f(1.1, 1.8) \approx 9.9}$.

12. Solution: First find the critical points:

$$\begin{aligned} f_x &= -3 \\ f_y &= 4 \end{aligned}$$

Since the first partials are never zero, there are no critical points.

Next find extrema on the boundary:

(a) One leg of the triangle is $y = 0$, $0 \le x \le 4$. Along this path the function values are

$$f(x,0) = 5 - 3x, \qquad 0 \le x \le 4$$

Since this is a decreasing function, the absolute maximum is $f(0,0) = 5$ and the absolute minimum is $f(4,0) = -7$.

(b) Another leg of the triangle is $x = 4$, $0 \le y \le 5$. Along this path the function values are

$$f(4,y) = -7 + 4y, \qquad 0 \le y \le 5$$

Since this is an increasing function, the absolute maximum is $f(4,5) = 13$ and the absolute minimum is $f(4,0) = -7$.

(c) Along the hypotenuse of the triangle, $y = \frac{5}{4}x$, $0 \le x \le 4$. Along this path, the function values are

$$f\left(x, \frac{5}{4}x\right) = 5 - 3x + 4\left(\frac{5}{4}x\right) = 5 + 2x, \qquad 0 \le x \le 4$$

Since this is an increasing function, the absolute maximum is $f(4,5) = 13$ and the absolute minimum is $f(0,0) = 5$.

Therefore the absolute maximum is $f(4,5) = \boxed{13}$ and the absolute minimum is $f(4,0) = \boxed{-7}$.

13. Solution: We can minimize the square of the distance. The square of the distance to the origin is

$$\begin{aligned} d^2 &= x^2 + y^2 + z^2 \\ &= x^2 + y^2 + (xy + 1) \end{aligned}$$

To find critical points, set the first derivatives equal to zero:

$$\begin{aligned} (d^2)_x &= 2x + y = 0 \\ (d^2)_y &= 2y + x = 0 \end{aligned}$$

which implies $y = -2x$ and $x = -2y$. The only critical points occur at $x = 0$, $y = 0$. Now use the second derivative test to confirm this is a minimum:

$$\begin{aligned} (d^2)_{xx} &= 2 > 0 \\ (d^2)_{yy} &= 2 \\ (d^2)_{xy} &= 1 \\ D &= 3 > 0 \end{aligned}$$

Therefore the points on the surface $\boxed{(0, 0, \pm 1)}$ are closest to the origin.