## Review Sheet for Midterm 1

**1.** Let $\mathbf{v_1} = \langle 1, 1, 1 \rangle$, $\mathbf{v_2} = \langle -1, 1, 1 \rangle$.
Find the angle between $\mathbf{v_1}$ and $\mathbf{v_2}$ in terms of $arccos(x)$.

**2.** A wagon is pulled horizontally by exerting a force of 10 lbs on the handle at an angle of 60°. How much work is done by moving the wagon 50 feet?

**3.** Let $\mathbf{v} = \langle 1, 1, 1 \rangle$, $\mathbf{b} = \langle 2, 2, 0 \rangle$.

(a) Find the orthogonal projection of $\mathbf{v}$ onto $\mathbf{b}$.

(b) Find the component of $\mathbf{v}$ orthogonal to $\mathbf{b}$.

**4.** Show that $\hat{u} \times \hat{u} = \hat{0}$ for any vector $\hat{u}$.

**5.** Let $\hat{v}_1 = \langle 1, 2, 3 \rangle$, $\hat{v}_2 = \langle 3, 2, 1 \rangle$. Find a vector that is orthogonal to both $\hat{v}_1$ and $\hat{v}_2$.

**6.** Find the volume of the tetrahedron defined by $P_1(0, 0, 0)$, $P_2(3, -2, -5)$, $P_3(1, 4, -4)$, $P_4(0, 3, 2)$

**7.** Consider the points $A(4, -2, 3)$, $B(8, -7, 6)$, and $C(6, -4, 6)$.

(a) Find the area of the parallelogram spanned by the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

(b) Find an equation of the plane containing the points $A$, $B$, $C$.

(c) Find an equation of the sphere centered at the origin and tangent to the plane containing the points $A$, $B$, $C$.

**8.** (a) Find the area of the triangle with vertices $A(1, 0, 0)$, $B(0, 1, 0)$, and $C(0, 0, 1)$ and compare it to the area of its projection on the $xy$-plane.

(b) Find the distance from the point $P(14, 4, 8)$ to the plane $6x - 2y + 3z = 2$.

(c) Find an equation of the sphere centered at the point $P(14, 4, 8)$ and touching the plane $6x - 2y + 3z = 2$.

**9.** (a) Find an equation of the plane containing the line $L : \mathbf{r}(t) = (2+3t)\mathbf{i}+(1-2t)\mathbf{j}+(-1+t)\mathbf{k}$ and the point $P(1, 3, 1)$.

(b) Find a parametric equation of the line of intersections of the planes $x - y + 2z = 2$ and $3x + y - z = 4$.

**10.** Consider the points $A(2, -7, 1)$, $B(6, 5, 4)$, $C(6, 2, 3)$, and $D(7, -4, 4)$.

a) Find the area of the parallelogram defined by $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

b) Find the volume of the parallelipiped defined by $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AD}$.

c) Find the distance from the point $D$ to the plane passing through $A$, $B$ and $C$.

**11.** Find the vector component of $\mathbf{v}$ along $\mathbf{b}$ and the vector component of $\mathbf{v}$ orthogonal to $\mathbf{b}$, where $\mathbf{v} = \langle 3, -2, -6 \rangle$ and $\mathbf{b} = \langle 1, -2, 2 \rangle$.

**12.** Find the area of the triangle with vertices $P(1, 1, 0)$, $Q(1, 0, 1)$, and $R(0, 1, 10)$.

**13.** Find parametric equations for the line

(a) through $(3, -1, 8)$ and parallel to $\mathbf{v} = \langle 2, 3, 5 \rangle$.

(b) through $(3, 1, -1)$ and $(3, 2, -6)$.

**14.** Where does the line parallel to $\langle x, y, z \rangle = \langle 1 + 2t, 3t, 5 - 7t \rangle$ and through the point $(0, 2, -1)$ intersect the coordinate planes?

**15.** Where does the line $\langle x, y, z \rangle = \langle 2t, t - 1, -3t \rangle$ intersect the hyperboloid

$$\frac{x^2}{4} + y^2 - \frac{z^2}{9} = 1$$

**16.** Determine whether the lines $L_1$ and $L_2$ are parallel, skew, or intersecting. If they intersect, find the point of intersection.

$$L_1 : x = 1 + t, \quad y = 2 - t, \quad z = 3t$$
$$L_2 : x = 2 - t, \quad y = 1 + 2t, \quad z = 4 + t$$

**17.** Show that the lines $L_1$ and $L_2$ are not paralleland do not intersect one another.

$$L_1 : x = 2 - t, \quad y = 3t, \quad z = 4 + t$$
$$L_2 : x = 5 - 3t, \quad y = t, \quad z = 2 + 3t$$

**18.** Find an equation of the plane

(a) passing through $(-5, 1, 2)$ with normal vector $\mathbf{n} = \langle 3, -5, 2 \rangle$.

(b) passing through $(1, 2, 3)$ with normal vector $\mathbf{n} = \langle 15, 9, -12 \rangle$.

(c) passing through $(1, 0, -3)$, $(0, -2, -4)$, $(4, 1, 6)$.

(d) passing through $(2, 1, -3)$, $(5, -1, 4)$, $(2, -2, 4)$.

(e) passing through $(-1, -3, 2)$ and containing the line $\langle x, y, z \rangle = \langle -1 - 2t, 4t, 2 + t \rangle$.

(f) containing the point $(-1, 2, 1)$ and the line of intersection of the planes $x + y - z = 2$ and $2x - y + 3z = 1$.

**19.** Determine whether the planes are parallel, perpendicular, or neither. If neither, find the angle between them.

(a) $x + 4y - 3z = 1, \quad -3x + 6y + 7z = 0$

(b) $2x + 2y - z = 4, \quad -3x - 6y + 3z = 10$

**20.** (a) Determine whether the line $L$ and the plane $\mathcal{P}$ intersect or are parallel.

$$L : x = 7 - 4t, \quad y = 3 + 6t, \quad z = 9 + 5t$$
$$\mathcal{P} : 4x + y + 2z = 17$$

(b) Do the same for these.

$$L : x = 3 + 2t, \quad y = 6 - 5t, \quad z = 2 + 3t$$
$$\mathcal{P} : 3x + 2y - 4z = 1$$

**21.** Derive an equation in $x$, $y$, and $z$ for the plane that contains the point $P_0(x_0, y_0, z_0)$ and is perpendicular to an arrowvector represented by the cartesian vector $\mathbf{n} = \langle a, b, c\rangle$.

**22.** Show the distance between the parallel planes $ax + by + cz = d_1$ and $ax + by + cz = d_2$ is

$$D = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

**23.** Given rectangular coordinates, convert them to cylindrical coordinates and spherical coordinates.

(a) $(0, 2, 0)$

(b) $(1, -1, \sqrt{2})$

**24.** Given cylindrical coordinates, convert them to rectangular coordinates and spherical coordinates.

(a) $(\sqrt{3}, \frac{\pi}{6}, 3)$

(b) $(2, 0, -2)$

**25.** Given spherical coordinates, convert them to cylindrical coordinates and rectangular coordinates.

(a) $(\sqrt{2}, \frac{3\pi}{4}, \pi)$

(b) $(5, 0, 0)$

**26.** Convert the given equation in cylindrical or spherical coordinates to an equivalent equation in cartesian coordinates and identify the surface represented by it.

(a) $\rho = 2\sec\phi$

(b) $r^2 \cos 2\theta = z$

**27.** Match the equations. Match each rectangular equation from the first column with an equivalent cylindrical equation in the second column. Then match each cylindrical equation in the second column with an equivalent spherical equation in the third column.

| | | |
|---|---|---|
| (1) $z = \sqrt{x^2 + y^2}$ | (a) $r(\cos\theta + 2\sin\theta) + 5z = 10$ | (i) $\phi = \frac{\pi}{4}$ |
| (2) $x + 2y + 5z = 10$ | (b) $z = r$ | (ii) $\cos\theta\sin\phi + 2\sin\theta\sin\phi + 5\cos\phi = 10$ |

**28.** Let $\mathbf{r}(t) = \cos(t)\,\mathbf{i} + \sin(t)\,\mathbf{j} + t\,\mathbf{k}$ and $t_0 = \frac{\pi}{4}$. Find the vector $\mathbf{r}'(t_0)$

**29.** Find a parametric equation of the line tangent to the graph of $\mathbf{r}(t) = e^{2t}\,\mathbf{i} - 2\cos(3t)\,\mathbf{j}$ at the point where $t = 1$.

**30.** Evaluate $\displaystyle\int_0^1 (e^{2t}\,\mathbf{i} + e^{-t}\,\mathbf{j} + t\,\mathbf{k})\,dt$.

**31.** Solve the vector initial-value problem for $\mathbf{r}(t)$ by integrating (antidifferentiating) and using the initial conditions to find the constants of integration:

$$\mathbf{r}''(t) = 9(\sin t)\mathbf{i} + 9(\cos t)\mathbf{j} + 4\mathbf{k}, \quad \mathbf{r}(0) = 3\mathbf{i} + 4\mathbf{j}, \quad \mathbf{r}'(0) = 2\mathbf{i} - 7\mathbf{j}\,.$$

**32.** Find an arc length parametrization of the curve that has the same orientation as the given parametrization and has $t = 1$ as the reference point:

$$\mathbf{r}(t) = 2t^2\mathbf{i} + t^3\mathbf{j}\,; \quad 1 \leq t \leq 3\,.$$

**33.** Find the arc length of $\mathbf{r}(t) = t^2\mathbf{i} + (\cos t + t\sin t)\mathbf{j} + (\sin t - t\cos t)\mathbf{k}$ for $0 \leq t \leq \pi$.

**34.** Find an arc length parametrization of the curve that has the same orientation as the given parametrization and has $t = 0$ as the reference point:

$$\mathbf{r}(t) = 2\cos^3 t\mathbf{i} + \sin^3 t\mathbf{j}\,; \quad 0 \leq t \leq \frac{\pi}{2}\,.$$

## (Some) Answers and Solutions

**1. Answer:** $\boxed{\theta = \arccos(\frac{1}{3})}$.

**2. Answer:** $\boxed{250 \text{ lb}\cdot\text{ft}}$.

**3.**

(a) **Answer:** $\boxed{\langle 1, 1, 0\rangle}$.

(b) **Answer:** $\boxed{\langle 0, 0, 1\rangle}$.

**4. Solution.** Let $\hat{u} = \langle u_1, u_2, u_3\rangle$. Then,

$$\hat{u}\times\hat{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = (u_2u_3 - u_3u_2)\,\hat{i} - (u_1u_3 - u_3u_1)\,\hat{j} + (u_1u_2 - u_2u_1)\,\hat{k} = 0\,\hat{i} - 0\,\hat{j} + 0\,\hat{k} = \hat{0}.$$

□

**5. Hint:** Consider $\hat{v}_1 \times \hat{v}_2$.

**6. Hint:** See problem #28 from section 12.4 in the book.

**7. Solution.** (a) We have

$$\begin{aligned}\overrightarrow{AB} &= \langle 4, -5, 3\rangle \\ \overrightarrow{AC} &= \langle 2, -2, 3\rangle\end{aligned} \quad \overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -5 & 3 \\ 2 & -2 & 3 \end{vmatrix} = \langle -9, -6, 2\rangle, \quad |\overrightarrow{AB}\times\overrightarrow{AC}| = \sqrt{81+36+4} = \sqrt{121} = 1$$

**Answer:** $\boxed{\text{Area}(\overrightarrow{AB}, \overrightarrow{AC}) = |\overrightarrow{AB}\times\overrightarrow{AC}| = 11}$.

(b) Since $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are two vectors in the plane, $\mathbf{n} = \overrightarrow{AB}\times\overrightarrow{AC}$. Taking $A$ as a reference point we get

$$-9(x-4) - 6(y+2) + 2(z-3) = 0,$$

**Answer:** The equation of the plane is $\boxed{9x + 6y - 2z = 18}$.

(c) Radius of this sphere is equal to the distance from this plane to the origin. Since

$$\text{comp}_{\mathbf{n}}\,\overrightarrow{OA} = \frac{\langle 4, -2, 3\rangle \bullet \langle -9, -6, 2\rangle}{|\langle -9, -6, 2\rangle|} = \frac{-18}{11},$$

the radius is $r = \text{dist}(O, A) = |\,\text{comp}_{\mathbf{n}}\,\overrightarrow{OA}| = 18/11$.

**Answer:** The equation of the sphere is $\boxed{x^2 + y^2 + z^2 = \frac{18^2}{11^2}}$.

□

8. **Solution.** (a) Since the area of the triangle $\triangle ABC$ is half the area of the parallelogram spanned by $\overrightarrow{AB}$ and $\overrightarrow{AC}$, we get:

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \langle 0,1,0\rangle - \langle 1,0,0\rangle = \langle -1,1,0\rangle, \qquad \overrightarrow{AC} = \langle -1,0,1\rangle$$

$$\overrightarrow{AB}\times\overrightarrow{AC} = \left\|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{matrix}\right\| = \langle 1,1,1\rangle, \qquad |\overrightarrow{AB}\times\overrightarrow{AC}| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

$$\text{Area}(\triangle ABC) = \frac{1}{2}\text{Area}(\overrightarrow{AB},\overrightarrow{AC}) = \frac{\sqrt{3}}{2}.$$

The area of its "shadow" (or *projection*) on the $xy$-plane is

$$\text{Area}(\triangle ABO) = \frac{1}{2}\text{Area}(\overrightarrow{AB},\overrightarrow{A0}) = \frac{1}{2}(\mathbf{k}\text{-th component of the vector } \overrightarrow{AB}\times\overrightarrow{AC}) = \frac{1}{2}.$$

**Answer:** The area is $\boxed{\text{Area}(\triangle ABC) = \sqrt{3}/2}$; it is $\boxed{\sqrt{3}\text{ times larger}}$ than the area of its "shadow".

(b) From the equation of the plane $\Pi$ we can see that its normal vector $\mathbf{n} = \langle 6,-2,3\rangle$. Also, the point $P_0(0,-1,0)$ is clearly a point on this plane. Then

$$\text{dist}(P,\Pi) = \left|\text{comp}_{\mathbf{n}}\overrightarrow{P_0P}\right| = \left|\frac{\langle 14,5,8\rangle \bullet \langle 6,-2,3\rangle}{|\langle 6,-2,3\rangle|}\right| = \left|\frac{84-10+24}{\sqrt{6^2+(-2)^2+(3)^2}}\right| = \frac{98}{\sqrt{49}} = 14.$$

**Answer:** The distance from the point $P(14,4,8)$ to the plane $6x-2y+3z=2$ is $\boxed{14}$.

(c) From part (b) we know that the distance from $P$ to the plane $\Pi$ is $14$, and so the *radius* of the sphere *touching* $\Pi$ and centered at $P$ is $14$.

**Answer:** The equation of the sphere centered at the point $P(14,4,8)$ and touching the plane $6x-2y+3z=2$ is $\boxed{(x-14)^2+(y-4)^2+(z-8)^2 = 14^2}$.

□

9. **Solution.**

(a) The point on $L$ corresponding to $t=0$ is $P_0(2,1,-1)$ and the *direction vector* of $L$ is $\mathbf{v} = \langle 3,-2,1\rangle$ (it is the vector of coefficients of $t$.) Then $\overrightarrow{P_0P} = \langle(-1,2,2\rangle$ and $\mathbf{v}$ are two vectors in the plane and we can get the normal vector of the plane by taking their cross-product:

$$\mathbf{n} = \overrightarrow{P_0P}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 2 \\ 3 & -2 & 1 \end{vmatrix} = \langle 6,7,-4\rangle$$

and so the equation of the plane is

$$6(x-2)+7(y-1)-4(z+1) = 0 \qquad \text{or} \qquad 6x+7y-4z = 23$$

**Answer:** The equation of the plane containing the line $L$ and the point $P$ is $\boxed{6x+7y-4z=23}$.

(b) Taking the cross-product of the normal vectors of the planes, $\mathbf{n}_1 = \langle 1, -1, 2\rangle$ and $\mathbf{n}_2 = \langle 3, 1, -1\rangle$, we get the direction vector of the line of their intersection,

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 3 & 1 & -1 \end{vmatrix} = \langle -1, 7, 4\rangle.$$

To find a reference point on this line, we need to find a solution to the following system of equations:

$$\begin{cases} x - y + 2z = 2 \\ 3x + y - z = 4 \end{cases} \qquad \text{adding the equations, we obtain} \qquad 4x + z = 6$$

so we can take $x = 0$, $z = 6$, $y = x + 2z - 2 = 10$, and we get

$$\mathbf{r}(t) = \langle 0, 10, 6\rangle + \langle -1, 7, 4\rangle t \qquad \text{or, in components,} \qquad \begin{aligned} x &= \quad - \ t \\ y &= 10 + 7t \\ z &= \ 6 + 4t \end{aligned}$$

**Answer:** The line of intersection of two given planes is $\boxed{\mathbf{r}(t) = \langle 0, 10, 6\rangle + \langle -1, 7, 4\rangle t}$.

□

**10. Solution.**

(a) First we calculate the vectors:

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \langle 6, 5, 4\rangle - \langle 2, -7, 1\rangle = \langle 4, 12, 3\rangle$$
$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \langle 6, 2, 3\rangle - \langle 2, -7, 1\rangle = \langle 4, 9, 2\rangle$$

Then we use the property of the *cross-product*:

$$\text{Area}(\overrightarrow{AB}, \overrightarrow{AC}) = \left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 12 & 3 \\ 4 & 9 & 2 \end{matrix} \right\| = |\langle 24 - 27, 12 - 8, 36 - 48\rangle|$$
$$= |\langle -3, 4, -12\rangle| = \sqrt{9 + 16 + 144} = \sqrt{169} = 13$$

**Answer:** The area of the parallelogramm is $\boxed{\text{13 square units}}$.

(b) The vector $\overrightarrow{AD} = \langle 5, 3, 3\rangle$, so

$$\text{Volume}(\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}) = \left\| \begin{matrix} 5 & 3 & 3 \\ 4 & 12 & 3 \\ 4 & 9 & 2 \end{matrix} \right\| = \left|\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC})\right| = |\langle 5, 3, 3\rangle \cdot \langle -3, 4, -12\rangle|$$
$$= |-15 + 12 - 36| = |-39| = 39.$$

**Answer:** The volume of the parallelipiped is $\boxed{\text{39 cubic units}}$.

(c) Since Volume(parallelipiped) = Area(base) × height,

$$d(D,\Pi) = \text{height} = \frac{\text{Volume}(\mathbf{AB},\mathbf{AC},\mathbf{AD})}{\text{Area}(\mathbf{AB},\mathbf{AC})} = \frac{39}{13} = 3$$

Alternatively,

$$d(D,\Pi) = \left|\text{comp}_{\mathbf{AB}\times\mathbf{AC}}\,\mathbf{AD}\right| = \left|\frac{(\mathbf{AB}\times\mathbf{AC})\cdot\mathbf{AD}}{|\mathbf{AB}\times\mathbf{AC}|}\right| = \left|\frac{\langle -3,4,-12\rangle\cdot\langle 5,3,3\rangle}{|\langle -3,4,-12\rangle|}\right| = \left|\frac{-39}{13}\right| = 3$$

**Answer:** The distance from the point $D$ to the $ABC$-plane is $\boxed{\text{3 units}}$.

□

**11. Answer:** Component of $\mathbf{v}$ along $\mathbf{b}$ is $\boxed{\langle -\frac{5}{9}, \frac{10}{9}, -\frac{10}{9}\rangle}$.

Component of $\mathbf{v}$ orthogonal to $\mathbf{b}$ is $\boxed{\langle \frac{32}{9}, -\frac{28}{9}, -\frac{44}{9}\rangle}$.

**12. Hint:** $\text{Area} = \frac{1}{2}\,||\overrightarrow{PQ}\times\overrightarrow{PR}||$.

**13.**

(a) **Answer:** $\boxed{x = 3+2t,\ y = -1+3t,\ z = 8+5t}$.

(b) **Answer:** $\boxed{x = 3,\ y = 1+t,\ z = -1-5t}$.

**14. Solution.** The line in question is given by $x = 2t$, $y = 2+3t$, $z = -1-7t$. The line will intersect the $xy$-plane when $z = 0$. Solving $0 = -1-7t$, we get that the line intersects the $xy$-plane when $t = -\frac{1}{7}$ which corresponds to the point $\boxed{(-\frac{2}{7}, \frac{11}{7}, 0)}$. Following the same process, we find that the line intersects the $xz$-plane when $y = 0$ which is found to be $\boxed{(-\frac{4}{3}, 0, \frac{11}{3})}$. Finally, the line intersects the $yz$-plane when $x = 0$ which is found to be $\boxed{(0, 2, -1)}$. □

**15.**

**16.**

**17.**

**18.**

**19.**

**20.**

**21.**

**22.**

**23.**

(a) **Answer:** $\boxed{(2, \frac{\pi}{2}, 0)_{\text{C}}\ ,\ (2, \frac{\pi}{2}, \frac{\pi}{2})_{\text{S}}}$

(b) **Answer:** $\boxed{(\sqrt{2}, \frac{7\pi}{4}, \sqrt{2})_{\text{C}}\ ,\ (2, \frac{7\pi}{4}, \frac{\pi}{4})_{\text{S}}}$

**24.**

(a) **Answer:** $\boxed{(\frac{3}{2}, \frac{\sqrt{3}}{2}, 3)_{\text{R}}\ ,\ (2\sqrt{3}, \frac{\pi}{6}, \frac{\pi}{6})_{\text{S}}}$

(b) **Answer:** $\boxed{(2, 0, -2)_{\text{R}}\ ,\ (2\sqrt{2}, 0, \frac{3\pi}{4})_{\text{S}}}$

**25.**

(a) **Answer:** $\boxed{(0, 0, -\sqrt{2})_{\text{R}}\ ,\ (0, \frac{3\pi}{4}, -\sqrt{2})_{\text{C}}}$

(b) **Answer:** $\boxed{(0, 0, 5)_{\text{R}}\ ,\ (0, 0, 5)_{\text{C}}}$

**26.** **Answer:** $\boxed{\text{(1),(b),(i)}}$ are equivalent as are $\boxed{\text{(2),(a),(ii)}}$

**27.**

**28.**

**29.**

**30.**

**31.**

**32.**

**33.**

**34.**