

CALCULUS 3

April 8, 2009

3rd TEST

YOUR NAME:

- | | |
|---|---|
| <input type="radio"/> 001 J. KISH (9AM)
<input type="radio"/> 002 T. DENT (10AM)
<input type="radio"/> 003 A. SPINA (11AM) | <input type="radio"/> 004 A. SPINA (12PM)
<input type="radio"/> 005 D. KEYES (1PM) |
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SHOW ALL YOUR WORK

final answers without any supporting work
will receive no credit *even if they are right!*

No calculators allowed.
No cheat-sheets allowed.

Partial credit will be given for any **reasonable amount of work pointing in the right direction** towards the solution of your problem. You will not get any partial credit for memorizing formulas and not knowing how to use them, or for anything you write that is not directly related to the solution of your problem.

If your tests contains **more than one solution or answer** to a problem or part of a problem, and one of them is wrong, then it will be **the wrong one** the one that **counts** for your grading!

DO NOT WRITE INSIDE THIS BOX!

problem	points	score
1	16 pts	
2	12 pts	
3	11 pts	
4	11 pts	
5	17 pts	
6	17 pts	
7	16 pts	
TOTAL	100 pts	

1. [16 pts] Find the minimum distance between the point $P_0(1, 2, 0)$ and the quadric cone $z^2 = x^2 + y^2$.

SOLUTION:

The distance between $P_0(1, 2, 0)$ and $P(x, y, z)$ is

$$d(P_0, P) = \sqrt{(x-1)^2 + (y-2)^2 + (z-0)^2}$$

If the point $P(x, y, z)$ is in the quadric cone, then the distance will be

$$d(P_0, P) = \sqrt{(x-1)^2 + (y-2)^2 + (\sqrt{x^2 + y^2})^2} = \sqrt{(x-1)^2 + (y-2)^2 + (x^2 + y^2)}$$

Minimizing this distance is equivalent to minimize the square of this distance. Therefore, the function we chose to minimize is

$$\begin{aligned} f(x, y) &= \left(d(P_0, P)\right)^2 \\ &= (x-1)^2 + (y-2)^2 + (x^2 + y^2) \\ &= 5 - 2x + 2x^2 - 4y + 2y^2. \end{aligned}$$

The domain is all the real plane, so we only need to look for the critical points.

We look now for the critical points,

$$\begin{cases} \frac{\partial f}{\partial x} = -2 + 4x = 0 \\ \frac{\partial f}{\partial y} = -4 + 4y = 0 \end{cases} \Rightarrow \begin{cases} x^* = \frac{1}{2} \\ y^* = 1 \end{cases}$$

The quadric cone is an unbounded figure. If we find only one critical point, then it must correspond to a minimum. The minimum distance we are looking is then

$$\boxed{d_{\min} = \sqrt{f(x^*, y^*)}}$$

But

$$\sqrt{f(x^*, y^*)} = \sqrt{\left(\frac{1}{2} - 1\right)^2 + (1 - 2)^2 + \left(\left(\frac{1}{2}\right)^2 + 1^2\right)}$$

so

$$\boxed{d_{\min} = \sqrt{\frac{5}{2}}}$$

This can be verified using the second-derivative test. With the second partial derivatives given by

$$\frac{\partial^2}{\partial x^2} f(x, y) = 4, \quad \frac{\partial^2}{\partial y^2} f(x, y) = 4, \quad \frac{\partial^2}{\partial x \partial y} f(x, y) = 0,$$

we compute the discriminant

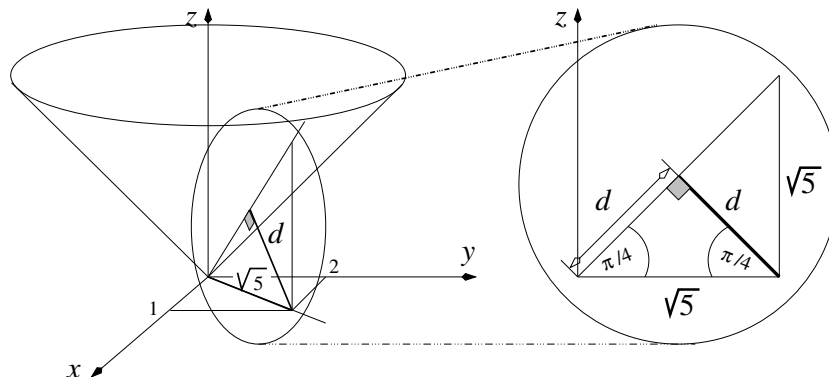
$$D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - f_{xy}^2(x, y) = 16.$$

Therefore

$$D(x^*, y^*) = 16 > 0 \quad \& \quad f(x^*, y^*) = 4 > 0 \quad \Rightarrow \quad \text{relative minimum at } (x^*, y^*).$$

Check. From the figure below we see immediately that

$$2d^2 = (\sqrt{5})^2 \rightarrow d^2 = \frac{5}{2} \rightarrow d = \sqrt{\frac{5}{2}}.$$



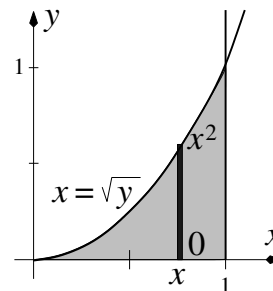
2. [12 pts] Evaluate the integral

$$\int_{y=0}^1 \int_{x=\sqrt{y}}^1 \sqrt{x^3 + 1} \, dx \, dy.$$

SOLUTION:

In the given order the iterated integral is beyond our means. We try therefore a change in the order of integration. We can read the new limits from the figure on the right, and obtain thus

$$\begin{aligned} \int_{y=0}^1 \int_{x=\sqrt{y}}^1 \sqrt{x^3 + 1} \, dx \, dy &= \int_{x=0}^1 \int_{y=0}^{x^2} \sqrt{x^3 + 1} \, dy \, dx \\ &= \int_{x=0}^1 \sqrt{x^3 + 1} [y]_{y=0}^{x^2} \, dx \\ &= \int_{x=0}^1 \sqrt{x^3 + 1} x^2 \, dx \\ &= \left[\frac{2}{9} (1 + x^3)^{3/2} \right]_{x=0}^1 \\ &= \frac{2}{9} (2\sqrt{2} - 1). \end{aligned}$$



Therefore,

$$\boxed{\int_{y=0}^1 \int_{x=\sqrt{y}}^1 \sqrt{x^3 + 1} \, dx \, dy = \int_{x=0}^1 \int_{y=0}^{x^2} \sqrt{x^3 + 1} \, dy \, dx = \frac{2}{9} (2\sqrt{2} - 1)}$$

3. [11 pts] Set-up, in a coordinate system other than the rectangular and appropriate to the geometry of the problem, the integral that gives the surface area of the portion of the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$ cut by the planes $z = h_1$ and $z = h_2$ ($0 < h_1 < h_2 < a$). YOU DO NOT HAVE TO EVALUATE THE INTEGRAL, that is, you do not have to actually find the surface area—we just want an integral that represents the surface area.

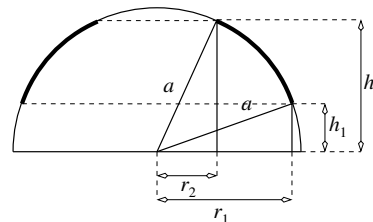
SOLUTION:

1st way. Since the surface in question is a portion of the surface of a sphere, the solution is simpler in spherical coordinates, where

$$dS = a^2 \sin \phi \, d\phi \, d\theta$$

with a the radius of the sphere. In this problem, the range of the angles is given by

$$\mathcal{R} : \begin{cases} \phi \in [\phi_1, \phi_2] = \left[\arccos\left(\frac{h_2}{a}\right), \arccos\left(\frac{h_1}{a}\right) \right] \\ \theta \in [0, 2\pi) \end{cases}$$



Therefore,

$$S = \int_{\theta=0}^{2\pi} \int_{\phi=\phi_2}^{\phi_1} a^2 \sin \phi \, d\phi \, d\theta$$

To evaluate this integral we first integrate over θ ,

$$\begin{aligned} S &= \int_{\theta=0}^{2\pi} \int_{\phi=\phi_2}^{\phi_1} a^2 \sin \phi \, d\phi \, d\theta \\ &= 2\pi a^2 [-\cos \phi]_{\phi=\phi_2}^{\phi_1} \\ &= 2\pi a^2 (\cos \phi_2 - \cos \phi_1) \\ &= 2\pi a^2 \left(\cos \arccos\left(\frac{h_2}{a}\right) - \cos \arccos\left(\frac{h_1}{a}\right) \right) \\ &= 2\pi a (h_2 - h_1). \end{aligned}$$

2nd way. The surface can be described in rectangular coordinates by an expression of the form $z = f(x, y)$, so

$$S = \iint_{\mathcal{R}} \left\| \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} \right\| \, dA_{xy} = \iint_{\mathcal{R}} \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \, dA_{xy}$$

with

$$\mathcal{R} = \{(x, y) : a^2 - h_2^2 \leq x^2 + y^2 \leq a^2 - h_1^2\}$$

and, if needed, rewrite the resulting integral in a more appropriate system of coordinates in order to evaluate it.

The equation of the surface in rectangular coordinates is

$$z = f(x, y) = \sqrt{a^2 - x^2 - y^2}$$

and the surface area

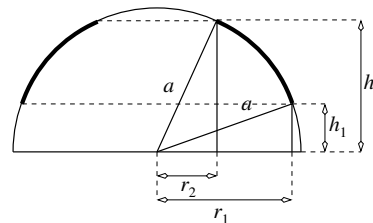
$$\begin{aligned} S &= \iint_{\mathcal{R}} \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \, dA_{xy} \\ &= \iint_{\mathcal{R}} \sqrt{\left(\frac{x}{\sqrt{a^2 - x^2 - y^2}}\right)^2 + \left(\frac{y}{\sqrt{a^2 - x^2 - y^2}}\right)^2 + 1} \, dA_{xy}, \end{aligned}$$

that is,

$$S = \iint_{\mathcal{R}} \frac{a}{\sqrt{a^2 - x^2 - y^2}} \, dA_{xy}$$

At this point is obvious that the evaluation is better done in polar coordinates. Either form the expression for \mathcal{R} , or from the figure of the right we get

$$r_1 = \sqrt{a^2 - h_1^2}, \quad \text{and} \quad r_2 = \sqrt{a^2 - h_2^2}.$$



The integration limits in polar coordinates will be

$$\begin{aligned} r_2 &\leq r < r_1, \\ 0 &\leq \theta < 2\pi, \end{aligned}$$

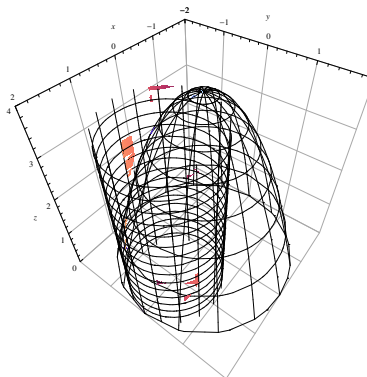
and the integral takes the form

$$S = \iint_{\mathcal{R}} \frac{a}{\sqrt{a^2 - x^2 - y^2}} \, dA_{xy} = \int_{\theta=0}^{2\pi} \int_{r=\sqrt{a^2-h_2^2}}^{\sqrt{a^2-h_1^2}} \frac{a}{\sqrt{a^2 - r^2}} r \, dr \, d\theta$$

Evaluating the integral we get

$$\begin{aligned} S &= \int_{\theta=0}^{2\pi} \int_{r=\sqrt{a^2-h_2^2}}^{\sqrt{a^2-h_1^2}} \frac{a}{\sqrt{a^2 - r^2}} r \, dr \, d\theta \\ &= \int_{\theta=0}^{2\pi} \left[-\sqrt{a^2 - r^2} \right]_{r=\sqrt{a^2-h_2^2}}^{\sqrt{a^2-h_1^2}} \, d\theta \\ &= \int_{\theta=0}^{2\pi} a [h_2 - h_1] \, d\theta \\ &= 2\pi a (h_2 - h_1). \end{aligned}$$

4. [11 pts] **Set-up, in a coordinate system other than the rectangular and appropriate to the geometry of the problem,** the integral that gives the volume of the solid that is bounded on top by the paraboloid $z = 4 - x^2 - y^2$, laterally by the cylinder $x^2 + y^2 = 2x$ and on bottom by the xy -plane, as shown in the figure on the side **YOU DO NOT HAVE TO EVALUATE THE INTEGRAL**, that is you do not have to actually find the volume—we just want an integral that *represents* the volume.



SOLUTION:

The problem is most easily solved in *cylindrical coordinates*, given by the transformation

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases} \quad \text{and} \quad dV = r \, dr \, dz \, d\theta.$$

In cylindrical coordinates, the equations of the paraboloid and the cylinder take the form

$$\begin{aligned} z = 4 - x^2 - y^2 &\longrightarrow z = 4 - r^2 \\ x^2 + y^2 = 2x &\longrightarrow r = 2 \cos \theta \end{aligned}$$

The boundaries of the region \mathcal{V} in cylindrical coordinates are given by

$$\mathcal{V} : \begin{cases} 0 \leq z \leq 4 - r^2, \\ 0 \leq r \leq 2 \cos \theta, \\ -\pi/2 \leq \theta \leq \pi/2. \end{cases}$$

and the volume by

$$V = \iiint_{\mathcal{V}} dV = \int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{2\cos\theta} \int_{z=0}^{4-r^2} r \, dz \, dr \, d\theta$$

To find this volume we need

$$\int \cos^2 \theta \, d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4}, \quad \int \cos^4 \theta \, d\theta = \frac{3\theta}{8} + \frac{\sin 2\theta}{4} + \frac{\sin 4\theta}{32}.$$

Computing the iterated integrals we get

$$\begin{aligned} V &= \int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{2\cos\theta} \int_{z=0}^{4-r^2} r \, dz \, dr \, d\theta \\ &= \int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{2\cos\theta} (4-r^2) r \, dr \, d\theta \\ &= \int_{\theta=-\pi/2}^{\pi/2} \left[2r^2 - \frac{1}{4}r^4 \right]_{r=0}^{2\cos\theta} d\theta \\ &= \int_{\theta=-\pi/2}^{\pi/2} (8\cos^2\theta - 4\cos^4\theta) \, d\theta \\ &= \left[8\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right) - 4\left(\frac{3\theta}{8} + \frac{\sin 2\theta}{4} + \frac{\sin 4\theta}{32}\right) \right]_{\theta=-\pi/2}^{\pi/2} \\ &= \left[\frac{5}{2}\theta + \sin 2\theta - \frac{1}{8}\sin 4\theta \right]_{\theta=-\pi/2}^{\pi/2} \\ &= \frac{5\pi}{2}. \end{aligned}$$

5. [17 pts] Evaluate

$$I = \iiint_{\mathcal{G}} \sqrt{x^2 + y^2 + z^2} \, dV$$

where \mathcal{G} is the solid between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$, and inside $z = \sqrt{x^2 + y^2}$.

SOLUTION:

The solid is that between the the concentric spheres of radius 1, 2 and the and the right cone with vertex at the center of the spheres. The boundaries of the region \mathcal{G} in spherical coordinates are given by

$$\mathcal{G}: \begin{cases} 1 \leq \rho \leq 2, \\ 0 \leq \phi \leq \frac{\pi}{4}, \\ 0 \leq \theta \leq 2\pi. \end{cases}$$

Therefore

$$V = \iiint_{\mathcal{G}} \sqrt{x^2 + y^2 + z^2} \, dV = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/4} \int_{\rho=1}^2 \rho \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

Evaluating the integral, we get

$$\begin{aligned} I &= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/4} \int_{\rho=1}^2 \rho^3 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/4} \left[\frac{\rho^4}{4} \right]_{\rho=1}^2 \sin \phi \, d\phi \, d\theta \\ &= \frac{15}{4} \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/4} \sin \phi \, d\phi \, d\theta \\ &= \frac{15}{4} \int_{\theta=0}^{2\pi} [-\cos \phi]_{\phi=0}^{\pi/4} \, d\theta \end{aligned}$$

$$\begin{aligned}
 &= \frac{15}{4} \left(1 - \frac{\sqrt{2}}{2}\right) \int_{\theta=0}^{2\pi} d\theta \\
 &= \frac{15}{4} \left(1 - \frac{\sqrt{2}}{2}\right) 2\pi.
 \end{aligned}$$

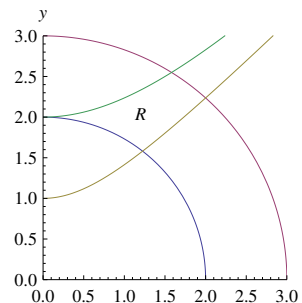
Therefore,

$$\boxed{I = \frac{15}{2} \pi \left(1 - \frac{\sqrt{2}}{2}\right)}$$

6. [17 pts] Evaluate the integral

$$\iint_{\mathcal{R}} 8xy \, dA_{xy}$$

over the region \mathcal{R} in the first quadrant enclosed by the curves $x^2 + y^2 = 4$, $x^2 + y^2 = 9$, $y^2 - x^2 = 1$, and $y^2 - x^2 = 4$.



SOLUTION:

The integrand is not complicated. On the other hand the domain of integration in the x and y variables has some complications. Let us look for a transformation whose grid-lines will match the given region of integration. The change of coordinates

$$\begin{cases} u = y^2 + x^2 \\ v = y^2 - x^2 \end{cases} \Rightarrow \begin{cases} x = \sqrt{\frac{u-v}{2}} \\ y = \sqrt{\frac{u+v}{2}} \end{cases}$$

will simplify the domain of integration which, in the new variables, is given by

$$\mathcal{S} : \begin{cases} u \in [4, 9], \\ v \in [1, 4]. \end{cases}$$

To transform the integral to the new variables we write

$$\iint_{\mathcal{R}} f(x, y) \, dA_{xy} = \iint_{\mathcal{S}} f(x(u, v), y(u, v)) \frac{dA_{xy}}{dA_{uv}} \, dA_{uv} = \iint_{\mathcal{S}} F(u, v) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, dA_{uv}.$$

The Jacobian of the transformation is given by

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2\sqrt{2}}(u-v)^{-1/2} & -\frac{1}{2\sqrt{2}}(u-v)^{-1/2} \\ \frac{1}{2\sqrt{2}}(u+v)^{-1/2} & \frac{1}{2\sqrt{2}}(u+v)^{-1/2} \end{vmatrix} = \frac{1}{4\sqrt{u^2 - v^2}},$$

and the integrand in the new variables is given by

$$8x(u, v)y(u, v) = 8\sqrt{\frac{u-v}{2}}\sqrt{\frac{u+v}{2}} = 4\sqrt{u^2 - v^2}.$$

Therefore, the integral in the new variables will be

$$\begin{aligned}
 \iint_{\mathcal{R}} 8xy \, dA_{xy} &= \iint_{\mathcal{S}} 4\sqrt{u^2 - v^2} \frac{1}{4\sqrt{u^2 - v^2}} \, dA_{uv} \\
 &= \int_{u=4}^9 \int_{v=1}^4 \, dv \, du.
 \end{aligned}$$

Therefore,

$$\boxed{\iint_{\mathcal{R}} 8xy \, dA_{xy} = 15}$$

7. [16 pts] The following statements are either **true** or **false**. If true, then say so and explain why. If false, then say so and give a simple counter-example to show why the statement is false.

NOTE: Just *one lucky* example doesn't prove a statement right, but it can prove it false!

(a) $\int_{x=0}^1 \int_{y=x^2}^x f(x, y) \, dy \, dx = \int_{y=0}^1 \int_{x=\sqrt{y}}^y f(x, y) \, dx \, dy;$

- (b) Let $z = f(x, y)$. Then, from the surface area formula

$$S(\sigma) = \iint_{\mathcal{R}} \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \, dA_{uv},$$

where σ is parametrized by $\mathbf{r}(u, v)$, $(u, v) \in \mathcal{R}$ we can derive the surface area formula

$$S(\sigma) = \iint_{\mathcal{R}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA_{xy}.$$

- (c) If f has a relative maximum at (x_0, y_0, z_0) , then $f_x(x_0, y_0) = 0$.
 (d) If $f(r, \theta)$ is a constant function and the area of the region \mathcal{S} is twice that of region \mathcal{R} , then

$$\iint_{\mathcal{S}} f(r, \theta) r \, dr \, d\theta = 2 \iint_{\mathcal{R}} f(r, \theta) r \, dr \, d\theta.$$

SOLUTION:

- (a) **False!**

The correct reversal of the iterated integrals is

$$\boxed{\int_{x=0}^1 \int_{y=x^2}^x f(x, y) \, dy \, dx = \int_{y=0}^1 \int_{x=y}^{\sqrt{y}} f(x, y) \, dx \, dy}$$

- (b) **True!**

The first of the equations is the general equation for the surface area of a surface given in parametric. The parameters are now (x, y) and the surface σ is parametrized by $\mathbf{r}(x, y) = \langle x, y, z(x, y) \rangle$. Evaluating the cross product in the first integral we get

$$\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{\partial z}{\partial x} \\ 0 & 1 & \frac{\partial z}{\partial y} \end{vmatrix} = \left\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right\rangle$$

and the norm of this vector is

$$\left\| \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} \right\| = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}.$$

- (c) **False!**

If (x_0, y_0) is a boundary point, then $f_x(x_0, y_0)$ need not be zero and may even be undefined.

(d) **True!**

If $f(r, \theta) = c$, then

$$\begin{aligned}\iint_{\mathcal{S}} f(r, \theta) r \, dr \, d\theta &= \iint_{\mathcal{S}} c r \, dr \, d\theta \\ &= c \iint_{\mathcal{S}} r \, dr \, d\theta \\ &= c \operatorname{area}(\mathcal{S}) \\ &= 2c \operatorname{area}(\mathcal{R}) \\ &= 2c \iint_{\mathcal{R}} r \, dr \, d\theta \\ &= 2 \iint_{\mathcal{R}} c r \, dr \, d\theta \\ &= 2 \iint_{\mathcal{R}} f(r, \theta) r \, dr \, d\theta.\end{aligned}$$