

# CALCULUS 3

April 9, 2008

## 3rd TEST

**YOUR NAME:**

- 001** A. SPINA .....(9AM)                       **003** T. DENT .....(11AM)  
 **002** E. WITTENBORN ..... (10AM)                       **004** J. WISCONS ..... (12PM)  
 **005** A. SPINA .....(1PM)

**SHOW ALL YOUR WORK**

No calculators allowed.  
No cheat-sheets allowed.

**DO NOT WRITE INSIDE THIS BOX!**

problem	points	score
<b>1</b>	14 pts	
<b>2</b>	10 pts	
<b>3</b>	19 pts	
<b>4</b>	19 pts	
<b>5</b>	19 pts	
<b>6</b>	19 pts	
<b>TOTAL</b>	100 pts	

1. [14 pts] Locate all relative maxima, relative minima, and saddle points, if any, of the function

$$f(x, y) = xy + \frac{2}{x} + \frac{4}{y}.$$

**SOLUTION:**

First we solve the equations  $\nabla f(x) = \mathbf{0}$ , that is

$$\begin{cases} \frac{\partial f}{\partial x} = y - \frac{2}{x^2} = 0 \Rightarrow y = \frac{2}{x^2} \\ \frac{\partial f}{\partial y} = x - \frac{4}{y^2} = 0 \Rightarrow x = \frac{4}{y^2} \end{cases} \Rightarrow x = x^4 \Rightarrow \begin{cases} x_c = 1 \\ y_c = 2 \end{cases}$$

after replacing  $y$  from the first equation into the second, and solving for  $x$  ( $x = 0$  is not a root of the problem). The second derivatives are

$$\begin{aligned} f_{xx} &= \frac{4}{x^3} & f_{xx}(1, 2) &= 4 \\ f_{yy} &= \frac{8}{y^3} & f_{yy}(1, 2) &= 1 \\ f_{xy} &= 1 & f_{xy}(1, 2) &= 1 \end{aligned} \longrightarrow$$

We have

$$f_{xx}(x_c, y_c) = 4 > 0 \quad \text{and} \quad f_{xx}(x_c, y_c)f_{yy}(x_c, y_c) - f_{xy}^2(x_c, y_c) = 3 > 0$$

Therefore we have a relative minimum at  $(1, 2)$ .

2. [10 pts] Let  $V = \int_{x=0}^{\pi} \int_{y=0}^{\pi} \sin(\sqrt{xy}) \, dy \, dx$

- (a) Give a description of  $V$  as a volume;  
 (b) Give an argument as to why  $0 \leq V \leq \pi^2$ .

**SOLUTION:**

- (a) The surface given by  $z = \sin(\sqrt{xy})$  is a dome-shaped surface over the interval  $[0, \pi] \times [0, \pi]$ .  
 (b) The dome-shaped surface is contained in a box of sides  $\pi$  and height 1, therefore

$$\begin{aligned} 0 \leq \sin(\sqrt{xy}) \leq 1 &\implies \int_{x=0}^{\pi} \int_{y=0}^{\pi} 0 \, dy \, dx \leq V \leq \int_{x=0}^{\pi} \int_{y=0}^{\pi} 1 \, dy \, dx \\ &\implies 0 \leq V \leq \pi^2 \end{aligned}$$

3. [19 pts] Find the surface area of the portion of the surface  $z = xy$  that lies inside the cylinder  $x^2 + y^2 = 16$ .

**SOLUTION:**

- parametrizing the problem in cartesian coordinates.

The parametrization of the surface in cartesian coordinates is

$$\mathbf{r}(x, y) = \langle x, y, xy \rangle \quad (x, y) \in \mathcal{R},$$

where the region  $\mathcal{R}$  is given by those points in the  $xy$ -plane inside the circle  $x^2 + y^2 = 16$ , which is a circle of radius 4 with center at  $(x_C, y_C) = (0, 0)$ .

The surface area is given by

$$S = \iint_{\mathcal{R}} \left\| \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} \right\| \, dx \, dy$$

with the cross product evaluated as follows

$$\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & y \\ 0 & 1 & x \end{vmatrix} = \langle -y, -x, 1 \rangle$$

and its length given by

$$\left\| \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} \right\| = \sqrt{1 + x^2 + y^2}.$$

Therefore

$$S = \iint_{\mathcal{R}} \left\| \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} \right\| dx dy = \iint_{\mathcal{R}} \sqrt{1 + x^2 + y^2} dx dy.$$

We go now to polar coordinates, where the integral takes a simpler form and becomes simpler to evaluate,

$$\begin{aligned} \iint_{\mathcal{R}} \sqrt{1 + x^2 + y^2} dx dy &= \int_{\theta=0}^{2\pi} \int_{r=0}^4 \sqrt{1 + r^2} r dr d\theta \\ &= \int_{\theta=0}^{2\pi} \left[ \frac{1}{3} (1 + r^2)^{3/2} \right]_{r=0}^4 d\theta \\ &= \frac{17\sqrt{17} - 1}{3} \int_{\theta=0}^{2\pi} d\theta \\ &= \frac{2(17\sqrt{17} - 1)\pi}{3}. \end{aligned}$$

- **parametrizing the problem in polar coordinates.**

The parametrization of the surface in cartesian coordinates is

$$\mathbf{r}(r, \theta) = \langle r \cos \theta, r \sin \theta, r^2 \cos \theta \sin \theta \rangle \quad (r, \theta) \in \mathcal{R}, \quad \text{where} \quad \mathcal{R} : \begin{cases} 0 \leq r \leq 4, \\ 0 \leq \theta \leq 2\pi. \end{cases}$$

The surface area is given by

$$S = \iint_{\mathcal{R}} \left\| \frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} \right\| dr d\theta$$

with the cross product evaluated as follows

$$\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & 2r \cos \theta \sin \theta \\ -r \sin \theta & r \cos \theta & r^2 (-\sin^2 \theta + \cos^2 \theta) \end{vmatrix} = \langle -r^2 \cos \theta, -r^2 \sin \theta, r \rangle$$

and its length given by

$$\left\| \frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} \right\| = \sqrt{r^4 \cos^2 \theta + r^4 \sin^2 \theta + r^2} = \sqrt{1 + r^2} r.$$

Therefore

$$\begin{aligned} S &= \iint_{\mathcal{R}} \left\| \frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} \right\| dr d\theta \\ &= \iint_{\mathcal{R}} \sqrt{1 + r^2} r dr d\theta \\ &= \int_{\theta=0}^{2\pi} \int_{r=0}^4 \sqrt{1 + r^2} r dr d\theta \\ &= \int_{\theta=0}^{2\pi} \left[ \frac{1}{3} (1 + r^2)^{3/2} \right]_{r=0}^4 d\theta \\ &= \frac{17\sqrt{17} - 1}{3} \int_{\theta=0}^{2\pi} d\theta \\ &= \frac{2(17\sqrt{17} - 1)\pi}{3}. \end{aligned}$$

4. [19 pts] Find the volume of the solid that is bounded above and below by the sphere  $x^2 + y^2 + z^2 = 9$  and inside the cylinder  $x^2 + y^2 = 4$ .

**SOLUTION:**

The problem is most easily solved in *cylindrical coordinates*, given by the transformation

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases} \quad \text{and} \quad dV = r \, dr \, dz \, d\theta.$$

In cylindrical coordinates, the equations of the sphere and the cylinder take the form

$$\begin{aligned} x^2 + y^2 + z^2 = 9 &\longrightarrow r^2 + z^2 = 9 \\ x^2 + y^2 = 4 &\longrightarrow r = 2 \end{aligned}$$

The boundaries of the region  $\mathcal{V}$  in cylindrical coordinates are given by

$$\mathcal{V} : \begin{cases} -\sqrt{9-r^2} \leq z \leq +\sqrt{9-r^2}, \\ 0 \leq r \leq 2, \\ 0 \leq \theta \leq 2\pi, \end{cases}$$

Therefore

$$\begin{aligned} V &= \int_{\theta=0}^{2\pi} \int_{r=0}^2 \int_{z=-\sqrt{9-r^2}}^{+\sqrt{9-r^2}} r \, dz \, dr \, d\theta \\ &= \int_{\theta=0}^{2\pi} \int_{r=0}^2 2\sqrt{9-r^2} r \, dr \, d\theta \\ &= \int_{\theta=0}^{2\pi} \left[ -\frac{2}{3} (9-r^2)^{3/2} \right]_{r=0}^2 d\theta \\ &= \frac{2}{3} (27 - 5\sqrt{5}) \int_{\theta=0}^{2\pi} d\theta \\ &= \frac{4\pi}{3} (27 - 5\sqrt{5}). \end{aligned}$$

5. [19 pts] Use cylindrical or spherical coordinates (whichever turns out to be the most appropriate) to evaluate the integral

$$\mathcal{I} = \int_{x=0}^1 \int_{y=0}^{\sqrt{1-x^2}} \int_{z=0}^1 \sin(x^2 + y^2) \, dz \, dy \, dx.$$

**SOLUTION:**

Cylindrical coordinates seem to be the most appropriate for our present problem. The change of coordinates to cylindrical coordinates is accomplished through the transformation equations

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases} \quad \text{and} \quad dV = r \, dr \, d\theta \, dz.$$

In cylindrical coordinates the integral takes the form

$$\mathcal{I} = \iiint_{\mathcal{V}} \sin(r^2) r \, dr \, d\theta \, dz.$$

In order to find the region of integration  $\mathcal{V}$  in spherical coordinates, analyzing the limits for the  $z$ -coordinate, we first observe that

$$0 \leq z \leq 1$$

that is, the region of integration  $\mathcal{V}$  is contained between the planes  $z = 0$ , and  $z = 1$ . There is no change in the  $z$ -limits when going from cartesian to cylindrical coordinates. Analyzing the limits of the  $y$ -coordinate of the  $xy$ -projection of the solid  $\mathcal{V}$  we get

$$\begin{aligned} 0 \leq y \leq \sqrt{1-x^2} &\Rightarrow y^2 \leq 1-x^2 \quad \text{and} \quad y \geq 0 \\ &\Rightarrow x^2 + y^2 \leq 1 \quad \text{and} \quad y \geq 0. \end{aligned}$$

which together with the limits for the  $x$ -coordinate,  $0 \leq x \leq 1$  give

$$x^2 + y^2 \leq 1 \quad \text{and} \quad x, y \geq 0.$$

This implies that

$$0 \leq r \leq 1 \quad \text{and} \quad 0 \leq \theta \leq \frac{\pi}{2}.$$

The given integral can be evaluated as

$$\begin{aligned} \mathcal{I} &= \int_{\theta=0}^{\pi/2} \int_{z=0}^1 \int_{r=0}^1 \sin(r^2) r \, dr \, dz \, d\theta \\ &= \int_{\theta=0}^{\pi/2} \int_{z=0}^1 \left[ -\frac{1}{2} \cos(r^2) \right]_{r=0}^1 \, dz \, d\theta \\ &= \frac{1}{2} (1 - \cos(1)) \int_{\theta=0}^{\pi/2} \int_{z=0}^1 \, dz \, d\theta \\ &= \frac{1}{2} (1 - \cos(1)) \int_{\theta=0}^{\pi/2} \, d\theta \\ &= \frac{\pi}{4} (1 - \cos(1)) \end{aligned}$$

6. [19 pts] Using an appropriate transformation of coordinates, evaluate the integral

$$\iint_{\mathcal{R}} y \sin(xy) \, dA_{\mathcal{R}}$$

over the region  $\mathcal{R}$  in the first quadrant enclosed by  $xy = 1$ ,  $xy = 4$ ,  $y = 1$ ,  $y = 4$ .

**SOLUTION:**

There seem to be no problem with the integrand itself. On the other hand the domain of integration in the  $x$  and  $y$  variables is complicated. We see that the coordinate change that will simplify the integral is given by

$$\begin{cases} u = xy, \\ v = y, \end{cases} \quad \Rightarrow \quad \begin{cases} x = u/v, \\ y = v, \end{cases}$$

from which we can compute the Jacobian of the transformation

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}.$$

In the new coordinates, the domain of integration  $\mathcal{S}$  will be given by

$$\mathcal{S} : \begin{cases} u \in [1, 4], \\ v \in [1, 4]. \end{cases}$$

Therefore, the integral in the new variables will be

$$\begin{aligned} \iint_{\mathcal{R}} y \sin(xy) \, dA_{\mathcal{R}} &= \iint_{\mathcal{S}} y(u, v) \sin(x(u, v) y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, dA_{\mathcal{S}} \\ &= \iint_{\mathcal{S}} v \sin(u) \frac{1}{v} \, dA_{\mathcal{S}} \\ &= \int_{u=1}^4 \int_{v=1}^4 \sin(u) \, dv \, du \\ &= 3 \int_{u=1}^4 \sin(u) \, du \\ &= 3 [-\cos(u)]_{u=1}^4 \\ &= 3 (\cos(1) - \cos(4)). \end{aligned}$$