

CALCULUS 3

October 15, 2008

2nd TEST

YOUR NAME:

- | | |
|---|--|
| <input type="radio"/> 001 B. KATZ-MOSES (8AM)
<input type="radio"/> 002 J. SANDERS (9AM)
<input type="radio"/> 003 J. NEWHALL (10AM) | <input type="radio"/> 004 A. SPINA (11AM)
<input type="radio"/> 005 E. ANGEL (12PM)
<input type="radio"/> 006 A. SPINA (1PM)

<input type="radio"/> 007 A. SPINA (3PM) |
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SHOW ALL YOUR WORK

final answers without any supporting work
will receive no credit *even if they are right!*

No calculators allowed.
No cheat-sheets allowed.

Partial credit will be given for any **reasonable amount of work pointing in the right direction** towards the solution of your problem. You will not get any partial credit for memorizing formulas and not knowing how to use them, or for anything you write that is not directly related to the solution of your problem.

If your tests contains **more than one solution or answer** to a problem or part of a problem, and one of them is wrong, then it will be **the wrong one** the one that **counts** for your grading!

DO NOT WRITE INSIDE THIS BOX!

problem	points	score
1	10 pts	
2	12 pts	
3	10 pts	
4	10 pts	
5	16 pts	
6	10 pts	
7	10 pts	
8	10 pts	
9	12 pts	
TOTAL	100 pts	

1. [10 pts] Find the domain and range of the function

$$f(x, y) = \sqrt{x^2 - y},$$

identify its level curves, and sketch four of them in a grid of your own, *including* the level curve that forms the boundary of the domain (indicate which one it is).

SOLUTION:

The domain of f is given by

$$\sqrt{x^2 - y} \in \mathbb{R} \Rightarrow 0 \leq x^2 - y \Rightarrow y \leq x^2 \Rightarrow \text{Dom}(f) = \left\{ \begin{array}{l} -\infty < x < +\infty \\ -\infty < y \leq x^2 \end{array} \right.$$

For the range of f we observe that

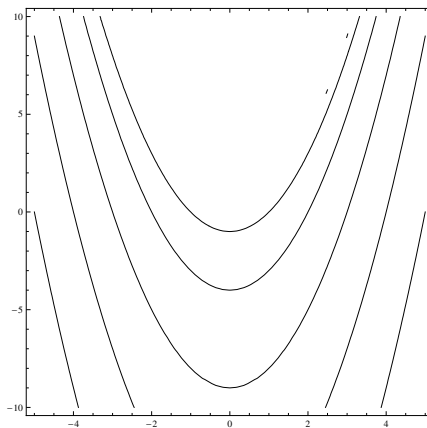
$$f(x, 0) = \sqrt{x^2} = |x|.$$

Therefore,

$$\text{Ran}(f) = [0, \infty)$$

The equation of the level curves is given by

$$f(x, y) = \sqrt{x^2 - y} = k \Rightarrow y = x^2 - k^2$$



2. [12 pts] Let

$$f(x, y) = \frac{\sin(x - y)}{|x| + |y|}, \quad (x, y) \neq (0, 0)$$

Does the limit at $(0, 0)$ exist? If so, find it. If not, show why.

SOLUTION:

1st Way.

$$\lim_{\substack{(x, y) \rightarrow (0, 0) \\ y=0}} \frac{\sin(x - y)}{|x| + |y|} = \lim_{x \rightarrow 0} \frac{\sin(x)}{|x|} = \begin{cases} +1, & \text{if } x \rightarrow 0^+, \\ -1, & \text{if } x \rightarrow 0^-. \end{cases}$$

Therefore the limit $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ doesn't exist. (To disprove the existence of the limit you need one direction along which the limit doesn't exist, or two directions with differing limits.)

Other possibilities

$$\lim_{\substack{(x, y) \rightarrow (0, 0) \\ x=0}} \frac{\sin(x - y)}{|x| + |y|} = \lim_{y \rightarrow 0} \frac{\sin(-y)}{|y|} = \begin{cases} -1, & \text{if } x \rightarrow 0^+, \\ +1, & \text{if } x \rightarrow 0^-. \end{cases}$$

$$\lim_{\substack{(x, y) \rightarrow (0, 0) \\ y=x}} \frac{\sin(x - y)}{|x| + |y|} = 0.$$

$$\lim_{\substack{(x, y) \rightarrow (0, 0) \\ y=-x}} \frac{\sin(x - y)}{|x| + |y|} = \lim_{x \rightarrow 0} \frac{\sin(2x)}{|2x|} = \begin{cases} +1, & \text{if } x \rightarrow 0^+, \\ -1, & \text{if } x \rightarrow 0^-. \end{cases}$$

2nd Way. By converting the problem into polar coordinates and using L'Hôpital's rule we get

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x-y)}{|x|+|y|} &= \lim_{r \rightarrow 0} \frac{\sin[r(\cos\theta - \sin\theta)]}{r(|\cos\theta| + |\sin\theta|)} \\ &= \lim_{r \rightarrow 0} \frac{\cos[r(\cos\theta - \sin\theta)](\cos\theta - \sin\theta)}{|\cos\theta| + |\sin\theta|} \\ &= \frac{\cos\theta - \sin\theta}{|\cos\theta| + |\sin\theta|}. \end{aligned}$$

Therefore the limit doesn't exist because this last result is θ -dependent.

3. [10 pts] If $f(x, y) = e^y + y \sin x$, find all the second order partial derivatives of f .

SOLUTION:

We start by finding the first order derivatives

$$\begin{aligned} f_x &= \frac{\partial f}{\partial x} = y \cos x, \\ f_y &= \frac{\partial f}{\partial y} = e^y + \sin x, \end{aligned}$$

and now

$$\begin{aligned} f_{xx} &= \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} (y \cos x) = -y \sin x, \\ f_{yy} &= \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} (e^y + \sin x) = e^y, \\ f_{xy} &= \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} (e^y + \sin x) = \cos x, \\ f_{yx} &= \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \cos x. \end{aligned}$$

4. [10 pts] Given the function $f(x, y) = x^\alpha y^\beta$, where α and β are constants, find the local linear approximation at $P_0(1, 1)$.

SOLUTION:

The local linear approximation to $f(x, y)$ at $P_0(x_0, y_0)$ is given by

$$L(x, y) = f(x_0, y_0) + \left(\frac{\partial f}{\partial x} \right)_{P_0} (x - x_0) + \left(\frac{\partial f}{\partial y} \right)_{P_0} (y - y_0)$$

For our problem we have

$$\frac{\partial f}{\partial x} = \alpha x^{\alpha-1} y^\beta, \quad \text{and} \quad \frac{\partial f}{\partial y} = \beta x^\alpha y^{\beta-1}.$$

Therefore

$$\boxed{f(x, y) \approx L(x, y) = 1 + \alpha(x - 1) + \beta(y - 1)}$$

5. [16 pts] Let f be a differentiable function of two variables, and suppose that $z = f(xy, x^2y^2)$. Show that

$$x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y} = 0.$$

SOLUTION:

$$z = f(u, v), \quad \text{with} \quad u = u(x, y) = xy, \quad v = v(x, y) = x^2y^2.$$

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} = \frac{\partial f}{\partial u} y + \frac{\partial f}{\partial v} 2xy^2 \\ \frac{\partial z}{\partial y} &= \frac{\partial f}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial y} = \frac{\partial f}{\partial u} x + \frac{\partial f}{\partial v} 2x^2y\end{aligned}$$

Therefore,

$$\begin{aligned}x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y} &= x \left[\frac{\partial f}{\partial u} y + \frac{\partial f}{\partial v} 2xy^2 \right] - y \left[\frac{\partial f}{\partial u} x + \frac{\partial f}{\partial v} 2x^2y \right] \\ &= \frac{\partial f}{\partial u} [xy - xy] + \frac{\partial f}{\partial v} [2x^2y^2 - 2x^2y^2] \\ &= 0.\end{aligned}$$

6. [10 pts] Find all points at which the direction of fastest change of the function

$$f(x, y) = x^2 + y^2 - 2x - 4y$$

is $\langle 1, 1 \rangle$.

SOLUTION:

The direction of fastest change is given by the direction of the gradient,

$$\nabla f = \langle 2x - 2, 2y - 4 \rangle.$$

If $\nabla f \parallel \langle 1, 1 \rangle$, then

$$\begin{cases} 2x - 2 = k \times 1 \\ 2y - 4 = k \times 1 \end{cases}$$

for some constant $k \neq 0$. Therefore

$$\begin{cases} 2x - 2 = k \\ 2y - 4 = k \end{cases} \Rightarrow 2x - 2 = 2y - 4 \Rightarrow y = x + 1.$$

That is, all points on the line $\boxed{y = x + 1}$

7. [10 pts] Find an equation for the plane tangent to the surface Given the function

$$z = \ln(x^2 + y^2)$$

at the point $P_0(0, 1, 0)$.

SOLUTION:

If a surface is given by the expression $f(x, y, z) = 0$ then a vector normal to the surface at P_0 (with position vector \mathbf{r}_0) is given by

$$\mathbf{n} = \nabla f|_{\mathbf{r}=\mathbf{r}_0}$$

For our problem we can take

$$f(x, y, z) = \ln(x^2 + y^2) - z,$$

which yields

$$\begin{aligned}\mathbf{n} &= \nabla f|_{\mathbf{r}=\mathbf{r}_0} \\ &= \left\langle \frac{2x}{x^2 + y^2}, \frac{2y}{x^2 + y^2}, -1 \right\rangle \Big|_{(x,y,z)=(0,1,0)} \\ &= \langle 0, 2, -1 \rangle\end{aligned}$$

The equation for the tangent plane at the point P_0 is then

$$\boxed{\langle 0, 2, -1 \rangle \cdot \langle x, y - 1, z \rangle = 0 \quad \text{or} \quad 2(y - 1) - z = 0}$$

8. [10 pts] Find a parameterization for the line normal to the graph of $y = \sin(x)$ at the point $(\pi, 0)$.

SOLUTION:

Let

$$f(x, y) = y - \sin x.$$

The gradient of f , ∇f , is perpendicular to the level curves of f , so we can take

$$\mathbf{v}_N = \nabla f|_{\mathbf{r}=\mathbf{r}_0} = \langle -\cos x, 1 \rangle|_{(x,y)=(\pi,0)} = \langle 1, 1 \rangle.$$

The equation for the line perpendicular to the level curve is

$$\mathbf{r}_N = \mathbf{r}_0 + t\mathbf{v}_N.$$

Therefore

$$\langle x, y \rangle_N = \langle \pi, 0 \rangle + t \langle 1, 1 \rangle \quad \text{or} \quad \begin{cases} x_N(t) = \pi + t \\ y_N(t) = t \end{cases} \quad \text{or} \quad x_N - y_N = \pi$$

9. [12 pts] Find the direction of the tangent line to the curve of intersection of the surfaces

$$9x^2 + 4y^2 + 4z^2 = 41$$

and

$$2x^2 - y^2 + 3z^2 = 10$$

at the point $P_0(1, 2, 2)$.

SOLUTION:

Let

$$f(x, y, z) = 9x^2 + 4y^2 + 4z^2 - 41 = 0,$$

and

$$g(x, y, z) = 2x^2 - y^2 + 3z^2 - 10 = 0.$$

A vector \mathbf{v} having the same direction as that of the tangent line to the curve of intersection of two surfaces, $f(x, y, z) = 0$, and $g(x, y, z) = 0$ at a point $P_0(x_0, y_0, z_0)$ must be perpendicular to the normal vectors to each of the surfaces at that point. Therefore, we can take

$$\mathbf{v} = \nabla f(x_0, y_0, z_0) \times \nabla g(x_0, y_0, z_0),$$

or any other vector parallel to this one. In our case

$$\begin{aligned} \mathbf{v} &= \nabla f(x_0, y_0, z_0) \times \nabla g(x_0, y_0, z_0) \\ &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \Big|_{P_0} \times \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right\rangle \Big|_{P_0} \\ &= \langle 18x, 8y, 8z \rangle \Big|_{P_0} \times \langle 4x, -2y, 6z \rangle \Big|_{P_0} \\ &= \langle 18, 16, 16 \rangle \times \langle 4, -4, 12 \rangle \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 18 & 16 & 16 \\ 4 & -4 & 12 \end{vmatrix} \\ &= 2 \times 4 \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 9 & 8 & 8 \\ 1 & -1 & 3 \end{vmatrix} \\ &= 8(32\mathbf{i} - 19\mathbf{j} - 17\mathbf{k}). \end{aligned}$$

Dividing this result by 8 (to get smaller numbers) we obtain

$$\mathbf{u}_v = \frac{32\mathbf{i} - 19\mathbf{j} - 17\mathbf{k}}{\sqrt{32^2 + 19^2 + 17^2}}$$