CALCULUS 3

February 4, 2009

1st TEST

YOUR NAME:

- ◯ **001** J. KISH .... (9AM)
- ◯ **002** T. DENT .... (10AM)
- ◯ **003** A. SPINA .... (11AM)
- ◯ **004** A. SPINA .... (12PM)
- ◯ **005** D. KEYES .... (1PM)

SHOW ALL YOUR WORK

final answers without any supporting work
will receive no credit *even if they are right!*
No calculators allowed.
No cheat-sheets allowed.

Partial credit will be given for any **reasonable amount of work pointing in the right direction** towards the solution of your problem. You will not get any partial credit for memorizing formulas and not knowing how to use them, or for anything you write that is not directly related to the solution of your problem.

If your tests contains **more than one solution or answer** to a problem or part of a problem, and one of them is wrong, then it will be **the wrong one** the one that **counts** for your grading!

Make sure you write an arrow on top of vector quantities to differentiate them from scalar quantities (numbers). A word-processor and boldface fonts were used in writing test, but you are writing by hand! Remember that, within the same context, $\vec{v}$ (with the arrow) is a *vector* ($\vec{v} \equiv \mathbf{v}$) and v (without the arrow) is the *norm* of the previous vector ($v \equiv ||\vec{v}|| \equiv ||\mathbf{v}||$). If a vector is the null vector, write an arrow on top of the zero!

DO NOT WRITE INSIDE THIS BOX!

problem	points	score
1	10 pts	
2	10 pts	
3	13 pts	
4	13 pts	
5	13 pts	
6	09 pts	
7	10 pts	
8	10 pts	
9	12 pts	
TOTAL	100 pts	

1. **[10 pts]** Find the points P, Q, and R, at which the line $x = 1 + 2t$, $y = -1 - t$, $z = -t$ meets the three coordinate planes.

SOLUTION:

- xy**–plane**

$$z(t) = 0 \quad \Rightarrow \quad -t = 0 \quad \Rightarrow \quad t = 0\,,$$

$$\boxed{(x_P, y_P, z_P) = \big(x(0), y(0), z(0)\big) = (1, -1, 0)}$$

- yz**–plane**

$$x(t) = 0 \quad \Rightarrow \quad 1 + 2t = 0 \quad \Rightarrow \quad t = -\frac{1}{2}\,,$$

$$\boxed{(x_Q, y_Q, z_Q) = \big(x(-1/2), y(-1/2), z(-1/2)\big) = (0, -1/2, 1/2)}$$

- zx**–plane**

$$y(t) = 0 \quad \Rightarrow \quad -1 - t = 0 \quad \Rightarrow \quad t = -1\,,$$

$$\boxed{(x_R, y_R, z_R) = \big(x(-1), y(-1), z(-1)\big) = (-1, 0, 1)}$$

2. **[10 pts]** Determine whether the line L and plane Π are parallel or intersect? If they intersect, in how many points do they intersect?

$$L : \begin{cases} x(t) &= 1 - t \\ y(t) &= 1 + t \\ z(t) &= 1 - 3t \end{cases} \qquad \Pi : \; 6x - 3y - 3z = 0\,.$$

SOLUTION:

We can answer all the questions together by replacing the $x(t)$, $y(t)$ and $z(t)$ of the line L in the equation of the plane Π and solving for the parameter t. No solution will mean they are parallel; one solution, that they intersect; infinite solutions, that the line is contained in the plane.

Substituting,

$$6(1 - t) - 3(1 + t) - 3(1 - 3t) = 0 \quad \Rightarrow \quad 0 = 0\,,$$

The equation is identically satisfied for all t, therefore the line is contained in the plane (it intersects at infinitely many points).

3. **[13 pts]** Determine whether the lines L_1 and L_2 are parallel, skew, or intersect.

$$\begin{aligned} L_1 : &\quad x = 4 + 2t\,, \quad y = -5 + 4t\,, \quad z = 1 - 3t\,, \\ L_2 : &\quad x = 2 + t\,, \quad y = -1 + 3t\,, \quad z = 2t\,. \end{aligned}$$

If they intersect, find the point of intersection.

SOLUTION:

The vector equations of the lines are

$$\begin{aligned} L_1 : \quad \mathbf{r} &= \;<4, -5, 1> + t <2, 4, -3> &= \mathbf{r}_0^{(1)} + t\mathbf{v}^{(1)}\,, \\ L_2 : \quad \mathbf{r} &= \;<2, -1, 0> + t <1, 3, 2> &= \mathbf{r}_0^{(2)} + t\mathbf{v}^{(2)}\,, \end{aligned}$$

The lines are *not parallel* because $\mathbf{v}^{(1)} \nparallel \mathbf{v}^{(2)}$ ($\mathbf{v}^{(1)} \neq k\mathbf{v}^{(2)}$, the entries of the vectors are not proportional). If the lines intersect, then they intersect at only one point, otherwise they are skew.

To find whether they intersect or not we need to use *different parameters* for the lines, therefore we rewrite the equations as

$$\begin{aligned} L_1 : &\quad x = 4 + 2t\,, \quad y = -5 + 4t\,, \quad z = 1 - 3t\,, \\ L_2 : &\quad x = 2 + s\,, \quad y = -1 + 3s\,, \quad z = 2s\,. \end{aligned}$$

and look for a solution of the system of equations

$$\begin{aligned} 4+2t &= 2+s \\ -5+4t &= -1+3s \\ 1-3t &= 2s \end{aligned}$$

The first of the equations yields

$$s = 2 + 2t\,.$$

We replace this expression for s in the second or third equation, the third seems to be the easiest, and obtain

$$1 - 3t = 2s\Big|_{s=2+2t} = 2(2+2t) \qquad \Rightarrow \qquad t = -\frac{3}{7}\,\cdot$$

Replacing this in our expression for s we get

$$s = (2+2t)\Big|_{t=-3/7} = \frac{8}{7}\,\cdot$$

But we still have the second equation which we have not touched at all. Replacing the values for t and s just obtained in the second equation we get

$$(-5+4t)\Big|_{t=-3/7} = (-1+3s)\Big|_{t=8/7} \qquad \Rightarrow \qquad -\frac{47}{7} = \frac{17}{7} \qquad !!!$$

There is no solution, therefore *the lines are skew*.

4. **[13 pts]** Find the equation of the plane through the points $Q_1(1,0,0)$, $Q_2(0,0,1)$, and parallel to the line whose vector parametric equation is given by $\mathbf{r} =< t, t, t >$.

SOLUTION:

The vector equation of a plane through a point P_0 and with normal $\mathbf{n}$ is given by

$$\mathbf{n} \cdot \big(\mathbf{r} - \mathbf{r}_0\big) = 0\,,$$

where $\mathbf{r} = \vec{OP} =< x, y, z >$ is the generic position vector, and $\mathbf{r}_0 = \vec{OP_o} =< x_0, y_0, z_0 >$ is the position vector of P_0, the reference point in the plane.

We already have one point in the plane–actually two of them. We can chose any.

We only need to find a vector perpendicular to the plane. But a vector perpendicular to the plane has to be perpendicular to

$$\mathbf{u} = \vec{Q_1Q_2} =< 0,0,1 > - < 1,0,0 >=< -1,0,1 >$$

and to

$$\mathbf{v} =< 1,1,1 >$$

the direction vector of the line ($\mathbf{r} =< t,t,t >= t < 1,1,1 >$).

One such normal vector can be obtained through

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} =< -1, 2, -1 >$$

Therefore, using Q_1 as reference point, the vector equation for the plane is

$$< -1,2,-1 > \cdot \big(< x,y,z > - < 1,0,0 > \big) = 0$$

or, in rectangular form,

$$\boxed{x - 2y + z = 1}$$

5. **[13 pts]** Consider the planes $\Pi_1 : \; x + z = 1$ and $\Pi_2 : \; y + z = 1$.

(a) Are the planes parallel, perpendicular or neither? If neither, find the angle between them.

(b) If your answer from part (a) was "parallel," find the distance between the two planes. Otherwise, find parametric equations for the line of intersection of the two planes.

SOLUTION:

(a) Two vectors perpendicular to the planes Π_1 and Π_2 are

$$\mathbf{n}_1 = <1, 0, 1> \qquad \text{and} \qquad \mathbf{n}_2 = <0, 1, 1>$$

These vectors are *not parallel*, therefore *the planes intersect*. The angle of intersection θ is found from the expression

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|}$$

from which we get

$$\cos\theta = \frac{|<1,0,1> \cdot <0,1,1>|}{\|<1,0,1>\|\,\|<0,1,1>\|} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2} \qquad \Rightarrow \qquad \theta = \frac{\pi}{3}\,.$$

(b) The direction vector $\mathbf{v}$ of the line of intersection L has to be perpendicular to $\mathbf{n}_1$ and $\mathbf{n}_2$. An easy way to obtain one such a vector is

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = <-1, -1, 1> \ .$$

Points $\mathbf{r}_0$ common to both planes can be easily obtained by inspection. Setting $z = 0$ in both equations we get

$$\mathbf{r}_0^{(a)} = <1, 1, 0> \ .$$

Setting $x = y = 0$ in the equations for the plane we obtain also

$$\mathbf{r}_0^{(b)} = <0, 0, 1> \ .$$

Using either of these points we obtain

$$\mathbf{r} = <1,1,0> + t <-1,-1,1> \qquad \longrightarrow \qquad \begin{cases} x = 1 - t \\ y = 1 - t \\ z = t \end{cases}$$

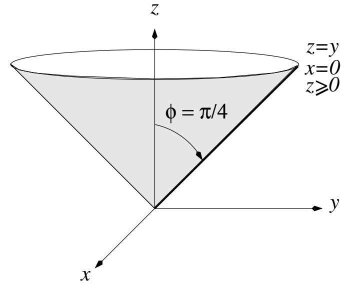
or

$$\mathbf{r} = <0,0,1> + t <-1,-1,1> \qquad \longrightarrow \qquad \begin{cases} x = -t \\ y = -t \\ z = 1 + t \end{cases}$$

Both equations represent the same line (replace t by $1 + t$ in the first equation and you obtain the second).

6. **[09 pts]** In 3-space, consider the portion of the line $z = y$ (in the yz–plane) for which $z \geq 0$, and revolve it about the z-axis. Sketch the resulting surface and write its equation in spherical coordinates.

SOLUTION:



From the figure we get the equation in spherical coordinates,

$$\phi = \frac{\pi}{4}\cdot$$

7. **[10 pts]** Write an equation for the paraboloid $z = -2x^2 - 2y^2$ in
   (a) cylindrical coordinates;
   (b) in spherical coordinates.

SOLUTION:

(a) In cylindrical coordinates

$$z = -2x^2 - 2y^2 = -2\left(x^2 + y^2\right) .$$

Therefore,

$$\boxed{z = -2r^2}$$

(b) In spherical coordinates

$$\begin{aligned} z = -2x^2 - 2y^2 = -2\left(x^2 + y^2\right) \quad &\Rightarrow \quad \rho\cos\phi = -2\left(\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta\right) \\ &\Rightarrow \quad \rho\cos\phi = -2\rho^2\sin^2\phi\,. \end{aligned}$$

We also have

$$z \le 0 \quad \Rightarrow \quad \phi \in \left[\frac{\pi}{2}, \pi\right) .$$

Therefore

$$\boxed{\rho = -\frac{1}{2}\cos\phi\csc^2\phi\,, \quad \phi \in \left[\frac{\pi}{2}, \pi\right)}$$

8. **[10 pts]** Rewrite the spherical equation $\rho = 2\cos\phi$ in rectangular coordinates.

SOLUTION:

$$\begin{aligned} \rho = 2\cos\phi \quad &\Rightarrow \quad \rho^2 = 2\rho\cos\phi \\ &\Rightarrow \quad x^2 + y^2 + z^2 = 2z \end{aligned}$$

Therefore, completing squares,

$$\boxed{x^2 + y^2 + (z-1)^2 = 1}$$

9. **[12 pts]** The following statements are either **true** or **false**. If true, then say so and explain why. If false, then say so and explain why or give a *counter-example* to show why the statement is false.

Note on the points for the problem: 3 points each part, 1 point for correct *true* or *false* answer, 2 points for justification or counter-example.

   (a) If $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$, then $\dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|} = 1$.
   (b) The cross product of two unit vectors is a unit vector.
   (c) If $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ then $\mathbf{u}$ and $\mathbf{v}$ are perpendicular.
   (d) If $\mathbf{u}$ and $\mathbf{v}$ are any two vectors, then $|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\,\|\mathbf{v}\|$.

SOLUTION:

(a) **False!** If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, using the definition for the dot product we get

$$\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|} = \frac{\|\mathbf{a}\|\,\|\mathbf{b}\|\,\cos\theta}{\|\mathbf{a}\|\,\|\mathbf{b}\|} = \cos\theta \neq 1\,,$$

and this cosine is, in general, different from one.

(b) **False!** If $\mathbf{u}$ and $\mathbf{v}$ are unit vectors, then

$$\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\|\,|\sin\theta| = |\sin\theta|$$

where θ is the angle between the two vectors. Since $0 \le |\sin\theta| \le 1$,

$$0 \le \|\mathbf{u}\times\mathbf{v}\| \le 1\,.$$

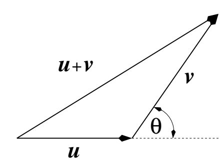
Another way: take for $\mathbf{u}$ and $\mathbf{v}$ any two of the coordinate vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. Their cross product is the null vector, that has zero length.

(c) **True!**

1st way.
From the figure on the right we see that the expression we are given,

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

is just Pythagoras theorem. But Pythagoras theorem works only when when $\theta = \pi/2$, that is, a right angle, therefore $\mathbf{u}$ and $\mathbf{v}$ are perpendicular.



2nd way.
Using the cosine theorem and the figure above and on the right we have

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\|\mathbf{u}\|\,\|\mathbf{v}\|\,\cos\theta\,.$$

But the expression we are given amounts to

$$\cos\theta = 0$$

therefore

$$\cos\theta = 0 \qquad \Rightarrow \qquad \theta = \frac{\pi}{2} \qquad \Rightarrow \qquad \mathbf{u}\perp\mathbf{v}\,.$$

3rd way.
Using $\|\mathbf{a}\|^2 = \mathbf{a}\cdot\mathbf{a}$ and expanding we get

$$\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} + 2\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u}\cdot\mathbf{v}$$

therefore

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \qquad \Rightarrow \qquad \mathbf{u}\cdot\mathbf{v} = 0 \qquad \Rightarrow \qquad \mathbf{u}\perp\mathbf{v}\,.$$

(d) **True!** Using the definition of dot product

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\,\cos\theta$$

Since $0 \le |\cos\theta| \le 1$ then

$$|\mathbf{u}\cdot\mathbf{v}| = \|\mathbf{u}\|\,\|\mathbf{v}\|\,|\cos\theta| \le \|\mathbf{u}\|\,\|\mathbf{v}\|\,.$$