MATH 2400

Final Exam Review, Part 1
Fall 2008

This is a review for the "cumulative part" of the final. You should know how to do these problems, but note that this is by no means a "complete list" of what you should know.

(1) Find all vectors of the form $\langle k, k+1, k+2\rangle$ that are orthogonal to $\langle 1, -1, 3\rangle$.

(2) Find two unit vectors perpendicular to both $\langle 0, 3, -2\rangle$ and $\langle -1, 1, 1\rangle$.

(3) Let $\vec{r_1}(t) = \langle x_1(t), y_1(t)\rangle$ and $\vec{r_2}(t) = \langle x_2(t), y_2(t)\rangle$ be smooth functions. Show that

$$\frac{d}{dt}\left[\vec{r_1}(t) \cdot \vec{r_2}(t)\right] = \vec{r_1}(t) \cdot \frac{d}{dt}\left[\vec{r_2}(t)\right] + \frac{d}{dt}\left[\vec{r_1}(t)\right] \cdot \vec{r_2}(t)$$

(4) Evaluate the following limits, or show that they do not exist.

(a) $\displaystyle\lim_{t\to 0}\left(e^t\mathbf{i} + \frac{\sin(t)}{t}\mathbf{j} + \mathbf{k}\right)$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^{x^2+y^2} - 1}{x^2 + y^2}$

(5) Let $\vec{r}(t) = t\mathbf{i} + t^2\mathbf{j}$. Calculate:

(a) $\displaystyle\int_0^2 \vec{r}(t)\ dt$

(b) $\displaystyle\int_0^2 \|\vec{r}(t)\|\ dt$

(6) Find an equation for the plane tangent to the surface $\vec{r}(u,v) = \langle u^2,\ u - v,\ ve^u \rangle$ at the point $(1, 1, 0)$.

(7) Find a unit vector normal to the surface $x^2 - y^2 + z^4 = 4$ at the point $(2, 1, -1)$.

(8) Give a parameterization of the line tangent to the curve of intersection of the surfaces $x^2 - y^2 + z^4 = 4$ and $x^2 + zy = z^2 + 2y$ at the point $(2, 1, -1)$.

(9) Find $\partial z/\partial x$ and $\partial z/\partial y$ for each of the following.

(a) $z = x^2 e^y + \dfrac{y^2}{x+1}$

(b) $z^2 + xy^2 = \tan(x + z)$

(c) $z = uv^2 - u^3$, where $u = x + y$ and $v = 3y - x$.

(10) (a) In what direction is $f(x, y) = x - 2y^2 + x^2 y$ increasing most rapidly at the point $P(1, 2)$?

(b) What is the derivative of f at this point, in the direction of the origin?

(11) Find and classify all critical points of $f(x, y) = ye^x + 2y^2 + x$.

(12) Evaluate the multiple integrals.

(a) $\int_0^1 \int_{x^2}^{\sqrt{x}} (x+y)\, dy\, dx$

(b) $\iint_R \frac{1}{x^2+y^2+1}\, dA$ where R is the area inside the circle $x^2+y^2=a^2$ ($a>0$.)

(c) $\int_0^1 \int_y^1 \sin(x^2)\, dx\, dy$

(d) $\iiint_G x^2\, dV$, where G is the region bound by the unit sphere.

(e) $\int_{-1}^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-x^2-z^2}}^{\sqrt{1-x^2-z^2}} \left(\frac{1}{x^2+y^2+z^2}\right) dy\, dx\, dz$

(13) Find the surface area of the part of the surface $z=100-x^2-y^2$ that is above the x-y plane.

(14) Find the volume of the solid bound above by the plane $z=4x$ and below by the paraboloid $z=x^2+y^2$.

Answers (possibly wrong):

1. $\langle -5/3, -2/3, 1/3\rangle$
2. $\langle -1/3, -2/3, 2/3\rangle$ and $\langle 1/3, 2/3, -2/3\rangle$
4. (a) $\mathbf{i}+\mathbf{j}+\mathbf{k}$; (b) Does not exist; (c) 1
5. (a) $\langle 2, 8/3\rangle$; (b) $2\sqrt{5}/3$
6. Many possible answers. One is: $ex-2ey-2z=-e$
7. One possibility is $\langle 2/\sqrt{6}, -1/\sqrt{6}, -1/\sqrt{6}, \rangle$
8. One possibility is $\vec{r}(t)=\langle 2-3t, 1+5t, -1-t\rangle$
9. (a) $\partial z/\partial x = 2xe^y - y^2/(x+1)^2$, $\partial x/\partial y = x^2e^y + 2y/(x+1)$
   (b) $\frac{\partial z}{\partial x} = \frac{\sec^2(x+z)-y^2}{2z-\sec^2(x+z)}$, $\frac{\partial z}{\partial y} = \frac{2xy}{\sec^2(x+z)-2z}$
   (c) $\frac{\partial z}{\partial x} = -16xy$, $\frac{\partial z}{\partial y} = 24y^2-8x^2$
10. (a) In the direction of $\langle 5, -7\rangle$; (b) $9/\sqrt{5}$
11. Saddle point at $(\ln 2, -1/2)$
12. (a) $3/10$; (b) $\pi\ln(a^2+1)$, (c) $(1-\cos 1)/2$; (d) $-2\pi/15$; (e) 4π
13. $50\pi\sqrt{10}/3$
14. 4π