

CALCULUS 3

March 5, 2008

2nd TEST

YOUR NAME:

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|---|--|
| <input type="radio"/> 001 A. SPINA(9AM)
<input type="radio"/> 002 E. WITTENBORN (10AM) | <input type="radio"/> 003 T. DENT(11AM)
<input type="radio"/> 004 J. WISCONS (12PM) |
| <input type="radio"/> 005 A. SPINA(1PM) | |

SHOW ALL YOUR WORK

No calculators allowed.
No cheat-sheets allowed.

DO NOT WRITE INSIDE THIS BOX!

problem	points	score
1	5 pts	
2	10 pts	
3	15 pts	
4	15 pts	
5	15 pts	
6	15 pts	
7	15 pts	
8	10 pts	
TOTAL	100 pts	

1. [5 pts] Find the natural domain of $\ln(4 - x^2 - y^2)$.

SOLUTION:

$$4 - x^2 - y^2 > 0 \quad \implies \quad x^2 + y^2 < 4$$

that is, all points *inside* the circle of radius 2 with center at the origin.

2. [10 pts] Given the function

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2 + y^2}} & (x, y) \neq (0, 0), \\ 0 & (x, y) = (0, 0); \end{cases}$$

show that it is continuous at $(0, 0)$.

SOLUTION:

To show that the function f is continuous at $(0, 0)$ we must show that $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0$. This we do by writing the function f in polar coordinates,

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} f(x, y) &= \lim_{r \rightarrow 0} f(r \cos \theta, r \sin \theta) \\ &= \lim_{r \rightarrow 0} \frac{r^2 \cos \theta \sin \theta}{r} \\ &= \lim_{r \rightarrow 0} r \cos \theta \sin \theta \\ &= 0. \end{aligned}$$

3. [15 pts] The function $f(x, y)$ is given by

$$f(x, y) = e^{xy}.$$

If $x = r \cos \theta$ and $y = r \sin \theta$ use the *chain rule* to find $\frac{\partial f}{\partial r}(r, \theta)$ and $\frac{\partial f}{\partial \theta}(r, \theta)$. Simplify your answer!

SOLUTION:

$$\begin{aligned} \frac{\partial f}{\partial r}(r, \theta) &= \left[\frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r} \right]_{\substack{x=r \cos \theta \\ y=r \sin \theta}} \\ &= [ye^{xy} \cos \theta + xe^{xy} \sin \theta]_{\substack{x=r \cos \theta \\ y=r \sin \theta}} \\ &= \left[r \cos \theta \sin \theta e^{r^2 \cos \theta \sin \theta} + r \cos \theta \sin \theta e^{r^2 \cos \theta \sin \theta} \right]_{\substack{x=r \cos \theta \\ y=r \sin \theta}} \\ &= 2r \cos \theta \sin \theta e^{r^2 \cos \theta \sin \theta} \\ &= r \sin(2\theta) e^{r^2 \sin(2\theta)/2}; \\ \frac{\partial f}{\partial \theta}(r, \theta) &= \left[\frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta} \right]_{\substack{x=r \cos \theta \\ y=r \sin \theta}} \\ &= [ye^{xy}(-r \sin \theta) + xe^{xy}(r \cos \theta)]_{\substack{x=r \cos \theta \\ y=r \sin \theta}} \\ &= -r^2 \sin^2 \theta e^{r^2 \cos \theta \sin \theta} + r^2 \cos^2 \theta e^{r^2 \cos \theta \sin \theta} \\ &= r^2 (\cos^2 \theta - \sin^2 \theta) e^{r^2 \cos \theta \sin \theta} \\ &= r^2 \cos(2\theta) e^{r^2 \sin(2\theta)/2}. \end{aligned}$$

4. [15 pts] Find the directional derivative of $f(x, y) = \sin(x) \cos(y)$ at the origin in the direction $\theta = \pi/3$ measured counterclockwise with respect to the x -axis.

SOLUTION:

The directional derivative at (x_0, y_0) in the direction of \mathbf{u} is given by

$$D_{\mathbf{u}} = \nabla f(x_0, y_0) \cdot \mathbf{u}.$$

The unit vector in the given direction is

$$\mathbf{u} = \left\langle \cos\left(\frac{\pi}{3}\right), \sin\left(\frac{\pi}{3}\right) \right\rangle = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$$

The gradient of f is

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle \cos(x) \cos(y), -\sin(x) \sin(y) \rangle$$

Therefore

$$D_{\mathbf{u}}f(0, 0) = \langle \cos(0) \cos(0), -\sin(0) \sin(0) \rangle \cdot \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle = \frac{1}{2}.$$

5. [15 pts] The temperature (in degrees Celsius) at a point (x, y) on a metal plate in the xy -plane is

$$T(x, y) = \frac{xy}{1 + x^2 + y^2}.$$

At the point $(1, 1)$ find the **unit** vector in the direction in which the temperature drops most rapidly.

SOLUTION:

The direction of maximum increase of the function T at (x_0, y_0) is given by the gradient of T at (x_0, y_0) . The direction of maximum decrease at $(x_0, y_0) = (1, 1)$ is therefore given by the direction of the vector

$$\begin{aligned} -\nabla T(1, 1) &= -\left\langle \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y} \right\rangle_{\substack{x=1 \\ y=1}} \\ &= -\left\langle \frac{y(1-x^2) + y^3}{(1+x^2+y^2)^2}, \frac{x(1-y^2) + x^3}{(1+x^2+y^2)^2} \right\rangle_{\substack{x=1 \\ y=1}} \\ &= -\left\langle \frac{1}{3}, \frac{1}{3} \right\rangle \end{aligned}$$

which is $5\pi/4$ measured counterclockwise from the x -axis.

The unit vector in the direction of maximum temperature decrease is then given by

$$\begin{aligned} \mathbf{u} &= \frac{-\left\langle \frac{1}{3}, \frac{1}{3} \right\rangle}{\sqrt{(1/3)^2 + (1/3)^2}} \\ &= \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle. \end{aligned}$$

6. [15 pts] Find the direction of the tangent line to the curve of intersection of the cone $z = \sqrt{x^2 + y^2}$ and the plane $x + 2y + 2z = 20$ at the point $(4, 3, 5)$. Just any vector along the tangent line will do, you do not need to normalize it!

SOLUTION:

A vector \mathbf{v} having the same direction as that of the tangent line to the curve of intersection of two surfaces, $f(x, y, z) = 0$, and $g(x, y, z) = 0$ at a point (x_0, y_0, z_0) must be perpendicular to the normal vectors to each of the surfaces at that point. Therefore, we can take

$$\mathbf{v} = \nabla f(x_0, y_0, z_0) \times \nabla g(x_0, y_0, z_0).$$

Four our problem,

$$\begin{aligned} z = \sqrt{x^2 + y^2} &\longrightarrow f(x, y, z) = x^2 + y^2 - z^2 = 0, \quad z \geq 0, \\ x + 2y + 2z = 20 &\longrightarrow h(x, y, z) = x + 2y + 2z - 20 = 0, \end{aligned}$$

therefore,

$$\nabla f(x, y, z) = \langle 2x, 2y, -2z \rangle,$$

$$\nabla h(x, y, z) = \langle 1, 2, 2 \rangle,$$

$$\nabla f(4, 3, 5) = \langle 8, 6, -10 \rangle,$$

$$\nabla h(4, 3, 5) = \langle 1, 2, 2 \rangle,$$

and

$$\mathbf{v} = \nabla f(4, 3, 5) \times \nabla g(3, 3, 5) = \langle 8, 6, -10 \rangle \times \langle 1, 2, 2 \rangle = \langle 32, -26, 10 \rangle.$$

7. [15 pts] Find the equation of the plane that is tangent to the paraboloid

$$z = \frac{x^2}{2^2} + \frac{y^2}{3^2}$$

at the point $(2, 3, 2)$.

SOLUTION:

The equation of a plane is given by

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0.$$

The equation of the paraboloid can be written as

$$f(x, y, z) = \frac{x^2}{2^2} + \frac{y^2}{3^2} - z = 0.$$

A vector normal to the paraboloid at $(2, 3, 2)$ is given by the gradient of f at $(2, 3, 2)$,

$$\nabla f(2, 3, 2) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle_{\substack{x=2 \\ y=3 \\ z=2}} = \left\langle \frac{2x}{2^2}, \frac{2y}{3^2}, -1 \right\rangle_{\substack{x=2 \\ y=3 \\ z=2}} = \left\langle 1, \frac{2}{3}, -1 \right\rangle$$

The equation of the tangent plane is then

$$\begin{aligned} 0 &= \nabla f(2, 3, 2) \cdot \langle x - 2, y - 3, z - 2 \rangle \\ &= \left\langle 1, \frac{2}{3}, -1 \right\rangle \cdot \langle x - 2, y - 3, z - 2 \rangle \\ &= (x - 2) + \frac{2}{3}(y - 3) - (z - 2) \end{aligned}$$

or,

$$x + \frac{2}{3}y - z = 1.$$

8. [10 pts] The following statements are either **true** or **false**. If true, then say so and explain why. If false, then say so and give a simple counter-example to show why the statement is false.

(a) If $f(x, y)$ is differentiable at (x_0, y_0) and $\nabla f(x_0, y_0) = \langle 0, 0 \rangle$ then the tangent plane to $z = f(x, y)$ at (x_0, y_0) is parallel to the xy -plane.

(b) Let $\mathbf{r}(t)$ be a vector valued function. Then $\frac{d}{dt} \|\mathbf{r}(t)\| = \|\mathbf{r}'(t)\|$.

SOLUTION:

(a) **True.** If f is differentiable at (x_0, y_0) then the equation of the tangent plane to $z = f(x, y)$ at (x_0, y_0) is

$$\langle 0, 0, -1 \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0 \quad \implies \quad z = z_0 = f(x_0, y_0)$$

which is the equation of a plane parallel to the xy -plane (the variables x and y do not appear in the equation).

(b) **False.**

$$\begin{aligned} \frac{d}{dt} \|\mathbf{r}(t)\| &= \frac{d}{dt} \sqrt{\mathbf{r}(t) \cdot \mathbf{r}(t)} \\ &= \frac{\mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t)}{2\sqrt{\mathbf{r}(t) \cdot \mathbf{r}(t)}} \\ &= \frac{\mathbf{r}(t) \cdot \mathbf{r}'(t)}{\|\mathbf{r}(t)\|}. \end{aligned}$$