Line Integrals - work, circulation

Line Integrals of functions: Let C be a piecewise smooth curve from P to Q, smoothly parameterized by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle, a \le t \le b$. Let $f(x,y,z)$ be a real valued function. Then the line integral along C of $f(x,y,z)$ with respect to arc length, s is
$\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\,\|\mathbf{r}'(t)\|\,dt.$

The line integral with respect to x, y or z is given as follows:

$$\int_C f(x,y,z)dx = \int_a^b f(x(t),y(t),z(t))x'(t)dt$$

$$\int_C f(x,y,z)dy = \int_a^b f(x(t),y(t),z(t))y'(t)dt$$

$$\int_C f(x,y,z)dz = \int_a^b f(x(t),y(t),z(t))z'(t)dt$$

Line Integrals of Vector Fields; Work: Let $\mathbf{F}(x,y,z) = \langle f(x,y,z), g(x,y,z), h(x,y,z)\rangle$ be a vector field and suppose f, g and h are continuous. Then the work done by the vector field on a particle moving along C from P to Q is $\int_C \mathbf{F}\cdot d\mathbf{r}$.

⋆ Strategy for Computing $\int_C \mathbf{F}\cdot d\mathbf{r}$ - 2-Space⋆

First, check if $\mathbf{F} = \langle f(x,y), g(x,y)\rangle$ is conservative; that is, check if $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 0$. If this is the case, use partial integration to find a potential function, ϕ s.t. $\nabla\phi = \mathbf{F}$. If $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, then $\int_C \mathbf{F}\cdot d\mathbf{r} = \phi(x_2,y_2) - \phi(x_1,y_1)$.

If F is not conservative, then check if C is closed; that is, does C begin and end at the same point? If C is closed, then we can use **Green's Theorem**. This theorem says that if C is closed and is the boundary of the planer region R(and other assumptions about $\mathbf{F}$ and C, what are they?) then
$\int_C \mathbf{F}\cdot d\mathbf{r} = \int\int_R \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} dA.$

Finally, if $\mathbf{F}$ is not conservative and C is not closed, then we have to do this the long way: parametrize C as above and $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(\mathbf{t}))\cdot\mathbf{r}'(t)dt$
$= \int_a^b \langle f(x(t),y(t)), g(x(t),y(t))\rangle \cdot \langle x'(t), y'(t)\rangle dt.$

⋆ Strategy for Computing $\int_C \mathbf{F}\cdot d\mathbf{r}$ - 3-Space⋆

If C is the boundary of an oriented surface (that is, C is closed) σ, then we can use **Stokes' Theorem**, which states that under suitable conditions (what are they?)
$\int_C \mathbf{F}\cdot d\mathbf{r} = \int\int_\sigma \text{curl}\mathbf{F}\cdot\mathbf{n}dS.$
This quantity is called the circulation of $\mathbf{F}$ around C.

If C is not the boundary of an oriented surface (C is not closed), then we do it the old fashion way using the parametrization $\mathbf{r}(t)$. **Incidentally**, if C is a curve in 3-space, then the Fundamental Theorem for Line Integrals still holds. That is, if ϕ is a three-variable function s.t. $\nabla\phi = \mathbf{F}$, then $\int_C \mathbf{F}\cdot d\mathbf{r} = \phi(\text{end}) - \phi(\text{beginning})$. However, finding such a potential function is usually not practical in this case.

Surface Integrals-flux

Surface Integrals of functions: Let σ be a smooth surface, smoothly parametrized by
$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle, a \le u \le b, c \le v \le d$. Let $f(x,y,z)$ be a real valued function. Then the surface integral of $f(x,y,z)$ over σ is $\int\int_\sigma f(x,y,z)dS$.
One way to calculate this is using the formula
$\int\int_\sigma f(x,y,z)dS = \int\int_R f(x(u,v),y(u,v),z(u,v))\,\|\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}\|\,dA$

If the surface is the graph of a function, say σ is the graph of $z = g(x,y)$, then we can use an even "simpler" formula:
$\int\int_\sigma f(x,y,z)dS = \int\int_R f(x,y,z(x,y))\sqrt{(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 + 1}dydx$

Likewise, if σ is the graph of $y = g(x,z)$ then

$$\int\int_\sigma f(x,y,z)dS = \int\int_R f(x,y(x,z),z)\sqrt{(\frac{\partial y}{\partial x})^2 + (\frac{\partial y}{\partial z})^2 + 1}dxdz$$

Surface Integrals of Vector Fields; Flux: Let $\mathbf{F}(x,y,z) = \langle f(x,y,z), g(x,y,z), h(x,y,z)\rangle$ be a vector field and suppose f, g and h are nice. If we orient σ (that is, choose normals), then the flux of $\mathbf{F}$ across σ is $\int\int_\sigma \mathbf{F}\cdot\mathbf{n}dS$.

⋆ Strategy for Computing $\int\int_\sigma \mathbf{F}\cdot\mathbf{n}dS$ ⋆

Don't use this method, however, what is the analogue to the Fundamental Theorem of Line Integrals for surface integrals? The answer is **Stokes' Theorem**. If we found another vector field, $\mathbf{G}$ s.t. curl$\mathbf{G}$=$\mathbf{F}$ then Stokes' Theorem says that $\int\int_\sigma \mathbf{F}\cdot\mathbf{n}dS = \int\int_\sigma \text{curl}\mathbf{G}\cdot\mathbf{n}dS = \int_C \mathbf{G}\cdot d\mathbf{r}$, where C is the oriented boundary of σ. The difficulty with this method is finding a suitable $\mathbf{G}$.

First, check if σ is closed; that is, is σ the boundary of some simple solid, G? In this case we can use the **Divergence Theorem**. This theorem says that under suitable conditions (what are they?)
$\int\int_\sigma \mathbf{F}\cdot\mathbf{n}dS = \int\int\int_G \text{div}\mathbf{F}dV.$

If σ is not the boundary of some simple solid, then **check if σ is the graph of some function, say** $z = g(x,y)$. In this case, rewrite the equation $z = f(x,y)$ as a level set $G(x,y,z) = 0$ as a level set for $G(x,y,z) = z - f(x,y)$. Then (recalling the gradients are normal to level sets) we can use $\mathbf{n} = \nabla G$ and use the formula
$\int\int_\sigma \mathbf{F}\cdot\mathbf{n}dS = \int\int_R \mathbf{F}(x,y,g(x,y))\cdot\nabla G dA.$

If all else fails, there is no shame in simply rolling up your sleeves writing down a parametrization, $\mathbf{r}(u,v)$, for σ (as above) and using the good ol' formula $\int\int_\sigma \mathbf{F}\cdot\mathbf{n}dS = \int\int_R \mathbf{F}(\mathbf{r}(u,v))\cdot(\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v})dA.$