Math 4230, Fall 2012
Take-home midterm
Due Friday, November 2, 2012

You may use your book and class notes for this exam, and you may use Maple (or Mathematica) if you wish. You may not consult with anyone else, but you may ask me questions about the exam. Please show all your work, and also write and sign the honor code pledge: "On my honor, as a University of Colorado at Boulder student, I have neither given nor received unauthorized assistance on this work."

1. Let $\alpha : I \to \mathbf{R}^3$ be a unit-speed curve whose torsion $\tau(s)$ is nonzero for all s. Suppose that you know the binormal vector $B(s)$ for all s. Show that you can recover the curvature function $\kappa(s)$ and the absolute value of the torsion $\tau(s)$. (Hint: when in doubt, differentiate! Show that the entire Frenet apparatus can be expressed in terms of $B(s)$ and its derivatives.)

2. Let α be a unit-speed curve with curvature $\kappa_\alpha(s) > 0$. Recall from Exercise 1.3.28 that there is a unique circle β with the property that

$$\beta(0) = \alpha(0), \qquad \beta'(0) = \alpha'(0), \qquad \beta''(0) = \alpha''(0).$$

This is the *osculating circle* to α at $s = 0$, and it has curvature equal to $\kappa_\alpha(0)$.

Now suppose that the torsion τ_α of α is nonzero. We can construct the *osculating helix* to α at $s = 0$ as follows: let $p_0 = \alpha(0), \kappa_0 = \kappa_\alpha(0), \tau_0 = \tau_\alpha(0)$, and let $\{T_0, N_0, B_0\}$ denote the Frenet frame of α at $s = 0$. Set

$$a = \frac{\kappa_0}{\kappa_0^2 + \tau_0^2}, \qquad b = \frac{\tau_0}{\kappa_0^2 + \tau_0^2}, \qquad c = \sqrt{a^2 + b^2} = \frac{1}{\sqrt{\kappa_0^2 + \tau_0^2}},$$

and define vectors

$$\begin{aligned} \mathbf{v}_1 &= -N_0 \\ \mathbf{v}_2 &= \frac{a}{c} T_0 - \frac{b}{c} B_0 \\ \mathbf{v}_3 &= \frac{b}{c} T_0 + \frac{a}{c} B_0. \end{aligned}$$

(a) Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ form an oriented, orthonormal basis for $\mathbb{R}^3$. ("Oriented" means that $\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{v}_3$, $\mathbf{v}_2 \times \mathbf{v}_3 = \mathbf{v}_1$, and $\mathbf{v}_3 \times \mathbf{v}_1 = \mathbf{v}_2$.)

Now consider the curve

$$\beta(s) = (p_0 + aN_0) + a \cos\left(\frac{s}{c}\right) \mathbf{v}_1 + a \sin\left(\frac{s}{c}\right) \mathbf{v}_2 + \frac{bs}{c} \mathbf{v}_3.$$

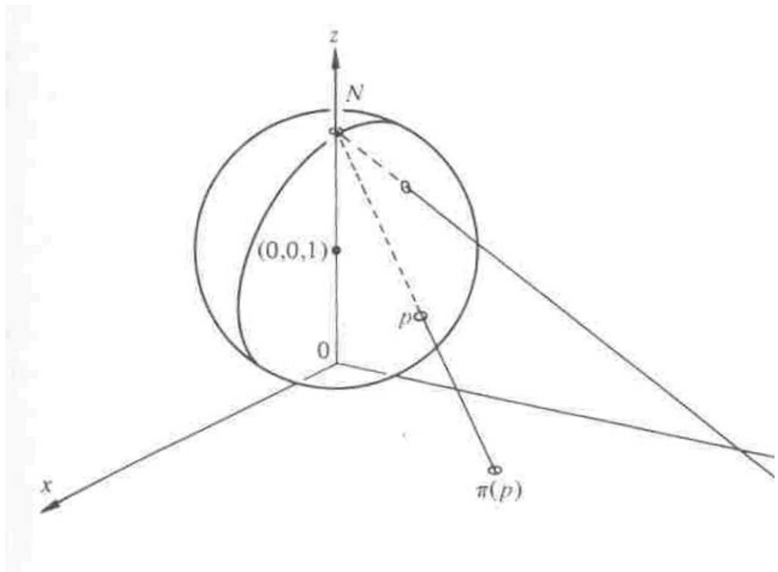
(b) Show that $\beta(0) = \alpha(0)$.
(c) Check that β is a unit-speed curve, and show that the Frenet frame of β at $s = 0$ is $\{T_0, N_0, B_0\}$.
(d) Show that β is a helix with curvature κ_0 and torsion τ_0.

Just for fun: I put a Maple worksheet called "osculatinghelix.mw" on my web site to compute and plot the osculating helix of any curve at any given point. Try it on your favorite space curves and see what you get!

3. One way to define a system of coordinates for the sphere S^2, given by $x^2 + y^2 + (z-1)^2 = 1$, is to consider the so-called *stereographic projection*

$$\pi : S^2 - \{NP\} \to \mathbb{R}^2$$

which carries a point $p = (x, y, z)$ of the sphere minus the north pole $NP = (0, 0, 2)$ onto the intersection of the xy plane with the straight line which connects NP to p. Let $(u, v) = \pi(x, y, z)$, where $(x, y, z) \in S^2 - \{NP\}$ and (u, v) is in the xy plane.



(a) Show that $\pi^{-1} : \mathbb{R}^2 \to S^2$ is given by

$$\begin{aligned} x &= \frac{4u}{u^2 + v^2 + 4} \\ y &= \frac{4v}{u^2 + v^2 + 4} \\ z &= \frac{2(u^2 + v^2)}{u^2 + v^2 + 4} \end{aligned}$$

(Hint: Parametrize the line passing through the points $(0, 0, 2)$ and $(u, v, 0)$, and find the point (other than $(0, 0, 2)$) where this line intersects the sphere.)
(b) We can regard π^{-1} as a parametrization $\mathbf{x} : \mathbb{R}^2 \to S^2$ whose image is the entire sphere minus the north pole. Show that this parametrization is *conformal:* i.e., it satisfies the conditions

$$E = G, \qquad F = 0.$$

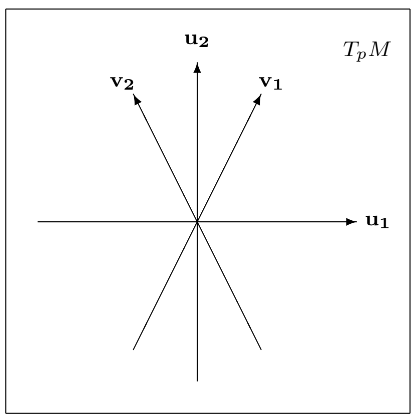
(c) Use Gauss's formula (i.e., do NOT express K in terms of ℓ, m, n) to compute the Gauss curvature of the sphere from this parametrization. (Hint: it had better be equal to 1, right?)
(d) Use this parametrization to plot (most of) the sphere in Maple. What do the coordinate curves look like?

4. Let $\alpha : [a, b] \to \mathbb{R}^3$ be a unit-speed curve. A *tubular surface* of radius r about α is the surface with parametrization

$$\mathbf{x}(u, v) = \alpha(u) + r[\cos(v)N(u) + \sin(v)B(u)],$$

$a \leq u \leq b,\ 0 \leq v \leq 2\pi$, where $N(u), B(u)$ are the Frenet normal and binormal vectors to α, respectively.
(a) Show that, for r sufficiently small, $\mathbf{x}$ defines a regular surface. Exactly how small does r need to be?
(b) Show that the area of $\mathbf{x}$ is $2\pi r(b - a)$. (Hint: the area of a parametrized surface $\mathbf{x} : U \to \mathbb{R}^3$ is $\iint_U \sqrt{EG - F^2}du\, dv$. Use the Frenet equations for α to compute $\mathbf{x}_u, \mathbf{x}_v$ and E, F, G.)
(c) Apply the result of part (b) to compute the area of a torus of revolution. (Hint: what should the curve α be?)

5. A tangent vector $\mathbf{v} \in T_pM$ is called an *asymptotic direction* if $k(\mathbf{v}) = 0$; i.e., if the normal curvature of M in the direction of $\mathbf{v}$ is zero. A curve α in M is called an *asymptotic curve* if $\alpha'(t)$ always points in an asymptotic direction. Show that:
(a) There are no asymptotic directions at p unless $K(p) \leq 0$, and if p is a hyperbolic point (i.e., $K(p) < 0$), then there are two linearly independent asymptotic directions.
(b) At a hyperbolic point p, the principal directions $\mathbf{u}_1, \mathbf{u}_2$ bisect the asymptotic directions $\mathbf{v}_1, \mathbf{v}_2$, as in the diagram below.



(c) At a hyperbolic point p, $H(p) = 0$ if and only if the asymptotic directions are orthogonal.
(d) If M contains a straight line in $\mathbb{R}^3$, then this line must be an asymptotic curve in M.
(Hint: Use Euler's formula for parts **(a)** - **(c)**.)

6. Let $\mathbf{x}$ be a parametrization of a regular surface $M \subset \mathbb{R}^3$ whose coordinate curves are lines of curvature. (Recall that for such a parametrization, $F = m = 0$.)

(a) Show that the principal curvatures of M are given by $k_1 = \frac{l}{E},\ k_2 = \frac{n}{G}$.

(b) Now suppose that k_1, k_2 are constants. Use the Codazzi equations to show that either $k_1 = k_2$ or $k_1k_2 = 0$. (This result can be used to show that any regular surface with constant principal curvatures is either part of a plane, sphere, or cylinder.)

7. Let $\alpha : I \to M$ be a regular curve in a surface M, parametrized by arc length. Let $\beta : I \to S^2$ be the image of α under the Gauss map of M; i.e.,

$$\beta(s) = G(\alpha(s)),$$

where $G : M \to S^2$ is the Gauss map of M. (Note that β is not necessarily parametrized by arc length!) β is called the *spherical image* of α.

(a) Show that if M contains no points where $K = 0$, then β is a regular curve on S^2.

(b) If α is a principal curve in M, with curvature κ_α at the point $p = \alpha(0) \in M$, then

$$\kappa_\alpha = |k_n| \kappa_\beta,$$

where $k_n = k(\alpha'(0))$ is the normal curvature of M at p in the direction of $\alpha'(0)$, and κ_β is the curvature of β at the point $G(p) = \beta(0) \in S^2$. (Hint: Start by computing κ_β according to the formula in Theorem 1.4.5.)