Math 4230, Fall 2012
Take-home Final Exam
Due Monday, December 17, 2012, 7:00 P.M.

You may use your book and class notes for this exam, and you may use Maple (or Mathematica) if you wish. You may not consult with anyone else, but you may ask me questions about the exam. Please show all your work, and also write and sign the honor code pledge: "On my honor, as a University of Colorado at Boulder student, I have neither given nor received unauthorized assistance on this work."

1. Prove that a curve α in a regular surface $M \subset \mathbb{R}^3$ is a straight line in $\mathbb{R}^3$ if and only if α is both a geodesic and an asymptotic curve.

2. Let α be an asymptotic curve in a regular surface M with curvature $\kappa_\alpha > 0$, parametrized by arc length s.

(a) Prove that the binormal $B_\alpha(s)$ of α is normal to the surface along α (i.e., $B_\alpha(s) = \pm U$ at each point of α), and conclude that $S(\alpha'(s)) = \tau_\alpha(s)N_\alpha(s)$, where $\tau_\alpha(s)$ is the torsion of α and $N_\alpha(s)$ is the Frenet normal vector to α.

(b) Show that at each point of α, the surface has Gauss curvature $K(\alpha(s)) = -\tau_\alpha(s)^2$. (Hint: consider the matrix of S with respect to the orthonormal basis $(T_\alpha(s), N_\alpha(s))$. The result of part (a) tells you what the first column of this matrix is, and then the symmetry of S gives you enough information to compute $K = \det(S)$.)

(c) Use part (b) to find the Gauss curvature of the helicoid. (Hint: first you need to figure out what the asymptotic curves are. Recall from Problem #5 on the midterm that:

- The mean curvature $H(p)$ of a surface at a point p is zero if and only if the asymptotic directions are orthogonal.
- If M contains a straight line, then this line must be an asymptotic curve.

Since the helicoid is a minimal surface (i.e., $H = 0$ everywhere) and made up of lines, this tells you right away what the asymptotic curves are: if you use the parametrization

$$\mathbf{x}(u, v) = (av\cos(u), av\sin(u), bu),$$

then the v-parameter curves (which are lines) must be asymptotic. Since this also happens to be an orthogonal parametrization, the other asymptotic curves must be the u-parameter curves, which are helices. And since you need curves with $\kappa_\alpha > 0$, the lines won't work and you need to use the helices!)

3. Suppose that a regular surface M can be given an orthogonal parametrization for which the coordinate curves are *unparametrized* geodesics. (This just means that you should *not* assume that the coordinate parametrizations

$$\alpha(u) = \mathbf{x}(u, v_0), \qquad \beta(v) = \mathbf{x}(u_0, v)$$

have unit speed.) Show that the Gauss curvature of M is identically zero.

4. Let $M \subset \mathbb{R}^3$ be a regular surface homeomorphic to a sphere, and suppose that $\alpha : I \to M$ is a simple, closed geodesic which divides M into two regions A and B. Let $G : M \to S^2$ be the Gauss map of M. Prove that $G(A) \subset S^2$ and $G(B) \subset S^2$ have the same area. (Hint: You will need the Gauss-Bonnet theorem, and Prop. 3.1.8 may be useful.)

5. Suppose that we tile a soccer ball with pentagons and hexagons, using h hexagons and p pentagons, in such a way that each vertex is shared between three faces. (If you look at a standard soccer ball, you will see that this is indeed the case.) Prove that $p = 12$. (h, on the other hand, can have infinitely many different values!) What changes if, instead of a sphere, we tile a torus with hexagons and pentagons?

6. Let M be a compact, orientable surface of genus 2. Stand M on end, as shown below. Define a vector field V on M by taking V at each point p to be the projection of the vector $(0, 0, 1)$ onto the tangent plane of M at p. Find the singularities of V, determine their indices, and show directly that the sum of the indices is $\chi(M)$.

7. **(a)** Let P be an oriented geodesic n-polygon in a regular surface M; i.e., P is homeomorphic to a disk, and the boundary of P consists of n geodesic segments. Show that

$$\iint_P K\,dA = 2\pi - \sum_{j=1}^{n} \epsilon_j = (2-n)\pi + \sum_{j=1}^{n} \iota_j,$$

where ϵ_j, ι_j are the exterior and interior angles of P, respectively.

(b) Let M be the hyperbolic plane H of Example 5.4.5. Let P_n be a geodesic n-polygon with $n \geq 3$, whose n vertices are on the boundary of H (and hence not actually *in* H). Use part (a) to find the area of P_n.

(c) Use part (b) to show that the area of H is infinite.