Polynomial Representation for the Expected Length of Minimal Spanning Trees

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August 5, 2010

Abstract

In this paper, we investigate the polynomial integrand of an integral formula that yields the expected length of the minimal spanning tree of a graph whose edges are uniformly distributed over the interval $[0, 1]$. In particular, we derive a general formula for the coefficients of the polynomial and use it to express the first few coefficients in terms of the structure of the underlying graph; e.g. number of vertices, edges and cycles.

1 Introduction

In 2002, J.M. Steele [7] derived an integral formula for the expected length of a minimal spanning tree (MST) of a graph with independent edge lengths uniformly distributed over the interval $[0, 1]$. While the formula gives an exact value of the mean length of the MST in terms of the Tutte polynomial of the graph, it yields (at least to us) little intuition of how the MST relates to the structure of the underlying graph.

This provided the motivation for the research project investigated by the Willamette University group of the Willamette Valley REU-RET Consortium for Mathematics Research in the summer of 2008. The authors of this paper were members of that research group and this paper covers the work that began that summer.

The main result of this paper is a general formula for the coefficients of the polynomial integrand in Steele’s formula for the expected length of the MST of a simple, finite, connected graph. For the first seven coefficients of the polynomial, we prove a surprising result that expresses the coefficients in terms of features of the underlying graph; e.g. the number of vertices, edges, and cycles.

The remainder of this paper is organized as follows: In Section 2, we state Steele’s formula, which is written in terms of the Tutte polynomial of the underlying graph. In Section 3, we investigate the integrand of the formula and prove...
that it is a polynomial, expressing the coefficients in terms of characteristics of
the graph. We illustrate our results with an example in Section 4 and examine
the particular case of the complete graph in Section 5.

Throughout this paper, “graph” means a finite simple graph. We adopt the
usual notations: $V(G)$ and $E(G)$ are the vertices and edges of $G$, respectively.
The rank of $G$ is denoted $r(G)$, and $r(G) = |V(G)| - k(G)$, where $k(G)$ is the
number of connected components of $G$.

2 Steele’s Formula

Let $G$ be a graph. We assign independent random lengths $\xi_e$ with uniform
distribution over the interval $[0,1]$ to the edges $e \in E(G)$. The total length of a
minimal spanning tree (MST) of the graph $G$ is denoted by

$$L(G) = \sum_{e \in E(MST(G))} \xi_e.$$ 

We are interested in the expected value of $L(G)$, which we denote by $E[L(G)]$.

Steele’s formula for $E[L(G)]$ is written in terms of the Tutte polynomial of
a graph, which we define next.

Definition 2.1. Let $G$ be a graph, and define $S(G)$ to be the set of spanning
subgraphs of $G$; i.e., subgraphs of $G$ with vertex set $V(G)$ and edge set a subset
of $E(G)$. The Tutte polynomial of $G$ is defined as follows:

$$T(G; x, y) = \sum_{A \in S(G)} (x-1)^{r(G)-r(A)}(y-1)^{|E(A)|-r(A)}.$$

The Tutte polynomial of a graph encodes much information about the graph,
but we will only use the definition above in our analysis and refer the reader to
[1] for more information.

We will use the following result about the Tutte polynomial in the proof of
the main result. The proof is a straightforward calculation from the definition
and so we omit it.

Lemma 2.2. Let $G$ be a connected graph with $n$ vertices and $m$ edges. Then
for values of $(x,y)$ satisfying $(x-1)(y-1) = 1$, we have

(a) $$T(G; x, y) = (x-1)^{n-1} \left( \frac{x}{x-1} \right)^m$$

(b) $$T_x(G; x, y) = (x-1)^{n-2} \left[ \sum_{A \in S(G)} k(A)(y-1)^{|A|} - \left( \frac{x}{x-1} \right)^m \right]$$

We now state Steele’s integral formula for the expected length of the minimal
spanning tree that was proved in [7].
Theorem 2.3. (Steele’s formula) Let $G$ be a connected graph and $T(G; x, y)$ the Tutte polynomial of $G$. Then

$$E[L(G)] = \int_0^1 \frac{1 - t}{t} \frac{T_x \left(G; \frac{1}{x}, \frac{1}{1-t}\right)}{T \left(G; \frac{1}{x}, \frac{1}{1-t}\right)} \, dt$$

(1)

Steele’s formula above has been generalized to the case of an arbitrary edge distribution [5] and to edge distributions that are not necessarily identical [6].

3 Integrant in Steele’s Formula

3.1 Polynomial integrand

We begin by showing that the integrand in Steele’s integral formula is a polynomial of degree less than or equal to the number of edges in the graph.

Theorem 3.1. Let $G$ be a connected graph with $n$ vertices and $m$ edges. Then

$$E[L(G)] = \int_0^1 p_m(t) \, dt$$

where $p_m(t)$ is a polynomial of degree $m$.

Proof. For convenience, we let $|A| = |E(A)|$. By Lemma 2.2, we have

$$\frac{1 - t}{t} \frac{T_x \left(G; \frac{1}{x}, \frac{1}{1-t}\right)}{T \left(G; \frac{1}{x}, \frac{1}{1-t}\right)} = \sum_{A \in S(G)} k(A) t^{|A|} (1 - t)^{m - |A|} - 1$$

$$= -1 + \sum_{A \in S(G)} k(A) \sum_{j=0}^{m - |A|} (-1)^{m - |A| - j} \binom{m - |A|}{j} t^{m - j}$$

This establishes the result, but we refine the coefficients further. Define

$$p_m(t) = -1 + \sum_{A \in S(G)} k(A) \sum_{j=0}^{m - |A|} (-1)^{m - |A| - j} \binom{m - |A|}{j} t^{m - j}$$

Let $i = m - j$. Then $m - |A| - j = i - |A|$, so we have

$$p_m(t) = -1 + \sum_{A \in S(G)} k(A) \sum_{m - i = 0}^{m - |A|} (-1)^{i - |A|} \binom{m - |A|}{m - i} t^i.$$
where $S_\ell := \{ A \in S(G) : |E(A)| = \ell \}$. Thus

$$p_m(t) = -1 + \sum_{i=0}^{m} a_i t^i$$

with $a_i$ as above.

In the proof of Theorem 3.1, we derived an initial formula (2) for the coefficients of the polynomial integrand in Steele’s formula for the expected length of the MST. In the next section, we derive an easier working form for the coefficients but we end this section with our first main result on the first three coefficients.

**Theorem 3.2.** Let

$$p_m(t) = -1 + \sum_{i=0}^{m} a_i t^i$$

be the polynomial integrand in Steele’s formula for the expected length of the MST of a connected graph $G$ with $n$ vertices and $m$ edges. Then

$$a_0 = n, \quad a_1 = -m, \quad \text{and} \quad a_2 = 0$$

**Proof.** The set $S_0$ consists of just the single subgraph of $G$ with no edges and $n$ vertices, which has $n$ connected components. Therefore, $\sum_{A \in S_0} k(A) = n$. Next, the set $S_1$ consists of the $m$ spanning subgraphs with just one edge, each of which has exactly $n-1$ connected components. Therefore, $\sum_{A \in S_1} k(A) = m(n-1)$. Lastly, the set $S_2$ consists of $\binom{m}{2}$ spanning subgraphs with two edges, each of which has exactly $n-2$ connected components.

Substituting in these values into formula (2) yields

$$a_0 = \sum_{A \in S_0} k(A) = n, \quad a_2 = -m \sum_{A \in S_1} k(A) + \sum_{A \in S_0} k(A) = -mn + m(n-1) = -m$$

and

$$a_2 = \binom{m}{2} \sum_{A \in S_0} k(A) - (m-1) \sum_{A \in S_1} k(A) + \sum_{A \in S_2} k(A)$$

$$= \binom{m}{2} n - m(m-1)(n-1) + \binom{m}{2} (n-2) = 0.$$ 

This completes the proof.

**3.2 Coefficients of the polynomial integrand**

In the previous theorem, the initial formula (2) for the coefficients is easily applied for the cases $\ell = 0, 1, 2$, because for each such $\ell$, the members of $S_\ell$ all have the same number of connected components. When $k(A)$ is non-constant on $S_\ell$, the enumeration becomes more difficult.
Accordingly, we partition the set $S_\ell$ into subsets with different numbers of connected components. This can be achieved by partitioning over the ranks of the members of $S_\ell$ since subgraphs in $S_\ell$ with the same rank have the same number of connected components, namely $n-r$.

Let $k^\ell_r$ be the number of spanning subgraphs of $G$ in $S_\ell$ with rank $r$; i.e. the number of spanning subgraphs of $G$ with $\ell$ edges and $n-r$ connected components. In terms of $k^\ell_r$, formula (2) can be rewritten as

$$a_i = \sum_{\ell=0}^{i} (-1)^{i-\ell} \binom{m-\ell}{m-i} \sum_{r=\ell}^{\ell} k^\ell_r(n-r), \quad (3)$$

where $r_\ell$ is the minimum rank of a graph with $n$ vertices and $\ell$ edges. If $K_q$ is the largest complete graph with $|E(K_q)| < \ell$, then $r_\ell = q$. In other words, $r_\ell$ is the largest integer with $\binom{r_\ell}{2} < \ell$.

We use the fact that $\sum_{\ell=0}^{\ell} k^\ell_r = \binom{n}{r}$ to reduce the number of terms of $k^\ell_r$ by one in (3). The new general expression for the polynomial coefficients $a_i$ for $i \geq 3$ is stated in Theorem 3.4 below. The proof of the theorem requires a couple of combinatorial identities.

**Lemma 3.3.** For integers $m, k, i$ and $n$,

(a) $\binom{m-k}{m-i} \binom{m}{k} = \binom{m}{m-i} \binom{i}{k}$

(b) $\sum_{k=0}^{n} (-1)^{k} \binom{n}{k} k = 0$

**Theorem 3.4.** Let $a_i$ be the coefficients of the polynomial integrand in Steele’s integral formula for the expected length of the MST of a connected graph $G$ with $n$ vertices and $m$ edges. Then for $i \geq 3$

$$a_i = \sum_{\ell=3}^{i} (-1)^{i-\ell} \binom{m-\ell}{m-i} \sum_{r=\ell}^{\ell-1} k^\ell_r(\ell-r).$$

**Proof.** Summing all the terms $k^\ell_r$ for a fixed number of edges $\ell$ yields the total number of spanning subgraphs in $S_\ell$, which equals $\binom{m}{\ell}$. This implies that $k^\ell_R = \binom{m}{\ell} - \sum_{\ell=1}^{\ell-1} k^\ell_r$ and thus from formula (3), we get

$$a_i = \sum_{\ell=0}^{i} (-1)^{i-\ell} \binom{m-\ell}{m-i} \left[ \sum_{r=\ell}^{\ell-1} k^\ell_r(n-r) + \binom{m}{\ell} - \sum_{r=\ell}^{\ell-1} k^\ell_r \right] (n-\ell)$$

$$= \sum_{\ell=0}^{i} (-1)^{i-\ell} \binom{m-\ell}{m-i} \left[ \sum_{r=\ell}^{\ell-1} k^\ell_r(\ell-r) + \binom{m}{\ell} \right] (n-\ell)$$

$$= \sum_{\ell=0}^{i} (-1)^{i-\ell} \binom{m-\ell}{m-i} \sum_{r=\ell}^{\ell-1} k^\ell_r(\ell-r) + \sum_{\ell=0}^{i} (-1)^{i-\ell} \binom{m-\ell}{m-i} \binom{m}{\ell} (n-\ell)$$
The minimum ranks for $\ell = 0, 1, 2$ are $r_0 = 0, r_1 = 1$ and $r_2 = 2$. Therefore, for these values of $\ell$, the summation on $r$ is empty and the $a_i$ reduces to the second summation. This and Lemma 3.3(a) yield

$$a_i = \left[ \sum_{\ell=3}^i (-1)^{i-\ell} \binom{m-\ell}{m-i} \sum_{r=r_\ell}^{\ell-1} k_r^\ell (\ell - r) \right] + \left[ \sum_{\ell=0}^i (-1)^{i-\ell} \binom{m}{m-i} \binom{\ell}{i} (n - \ell) \right]$$

The second sum equals zero by the Binomial Theorem and Lemma 3.3(b). $\square$

The above result gives a general formula for the coefficients of the polynomial integrand in terms of the values $k_\ell^r$. Determining the values of $k_\ell^r$ for large $\ell$ poses a major challenge. We conclude this section with the enumeration for $\ell = 3, \ldots, 6$ and the corresponding coefficients of $p_m(t)$.

**Definition 3.5.** For a connected graph $G$, define

(a) $c_i = \text{number of cycles of size } i \text{ in } G$.

(b) $c_{i,1} = \text{number of cycles of size } i \text{ with one chord}$.

(c) $\bar{c}_{i,1} = \text{number of cycles of size } i \text{ with one chord and one additional edge that is not a chord of the cycle}$.

(d) $k_i = \text{number of complete subgraphs } K_i \text{ in } G$.

(e) $k_{i,j} = \text{number of complete bipartite subgraphs } K_{i,j} \text{ in } G$.

**Lemma 3.6.** For $\ell = 3, \ldots, 6$,

$$\sum_{r=r_\ell}^{\ell-1} k_r^\ell (\ell - r) = \sum_{j=r_\ell}^{\ell} c_j \binom{m-j}{m-\ell} - d_\ell \tag{4}$$

where

$$d_3 = 0, \quad d_4 = 0, \quad d_5 = k_5^3, \quad d_6 = \bar{c}_{4,1} + c_{5,1} + k_{3,2} + 4k_4$$

**Proof.** We show the above result for $\ell = 5$; the other cases are similar in nature. The minimum rank for $\ell = 5$ is $r_5 = 3$ and so the left-hand side of equation (4) is $k_3^5 + 2k_3^2$. The types of subgraphs counted in $k_3^5$ are those with 5 edges and $n - 4$ connected components, which have the form shown in Figure 1(a)-(c). Analogously, there is only one type of subgraph counted in $k_3^2$, which is shown in Figure 1(d). Note that the graphs in Figures 1 and 2 that are a one-clique sum of smaller graphs actually represent families that include subgraphs that are disjoint unions of the summands. For example, 1(a) includes $K_3 \bigcup P_2$.

Now consider the right-hand side of (4). Start with any 3-cycle and choose any other 2 edges in the graph; there are $c_3 \binom{m-3}{2}$ ways to do this. This counts all the types of subgraphs depicted in Figure 1(a) and counts the subgraphs in Figure 1(d) twice. Figure 2 gives a pictorial representation of $c_3 \binom{m-3}{2}$. The subgraphs counted by $c_4 \binom{m-4}{1}$ (start with a 4-cycle and choose any other edge) are of the type shown in Figure 1(b) and Figure 1(d). These are depicted in the right-hand side of Figure 2.
Lastly, \( c_5 \) is the number of 5-cycles, which are shown in Figure 1(c). Therefore,

\[
k_4^3 + 2k_3^5 = c_3\left(\frac{m-3}{2}\right) + c_4\left(\frac{m-4}{1}\right) + c_5 - k_3^5.
\]

While initially Lemma 3.6 appears only to complicate the coefficient formula given in Theorem 3.4, the next lemma shows that when it is applied to the coefficient formula, it actually simplifies it. The proof is a straightforward calculation and so we omit it; the reasoning is analogous to the proof of Lemma 3.6. Although we proved the first equation in Lemma 3.7 for \( i = 3, \ldots , 6 \), we conjecture that it holds in general for all \( i \geq 3 \).

**Lemma 3.7.** For \( i = 3, \ldots , 6 \),

\[
\sum_{\ell = 3}^{i} (-1)^{i-\ell} \left( \frac{m-\ell}{m-i} \right) \sum_{j=\ell}^{\ell} c_j \left( \frac{m-j}{m-\ell} \right) = c_i
\]

and thus

\[
a_i = c_i - \sum_{\ell = 3}^{i} (-1)^{i-\ell} \left( \frac{m-\ell}{m-i} \right) d_{\ell}
\]  

Finally, we derive representations for the coefficients \( a_3 \) through \( a_6 \) in terms of the structure of the underlying graph \( G \). The proof of the theorem is a direct application of Lemmas 3.6 and 3.7 to the general coefficient formula given in Theorem 3.4.

**Theorem 3.8.** Let

\[
p_m(t) = -1 + \sum_{i=0}^{m} a_i t^i
\]
be the polynomial integrand in Steele’s formula for the expected length of the MST of a connected graph \( G \) with \( n \) vertices and \( m \) edges. Then

\[
\begin{align*}
a_3 &= c_3, & a_4 &= c_4, & a_5 &= c_5 - k_3^5, & a_6 &= c_6 + 2k_4 - c_{5,1} - k_{3,2}.
\end{align*}
\]

## 4 Application of Results

In this section, we apply Theorems 3.2 and 3.8 to the complete bipartite graph \( K_{3,2} \) in order to derive the expected length of the minimal spanning tree of \( K_{3,2} \).

**Proposition 4.1.** Let \( p_m(t) \) be the polynomial integrand in Steele’s formula for the complete bipartite graph, \( K_{3,2} \) shown below.

\[
\begin{align*}
\text{Then} \quad p_m(t) &= 4 - 6t + 3t^4 - t^6 \\
\text{and} \quad \mathbb{E}[L(K_{3,2})] &= \int_0^1 (4 - 6t + 3t^4 - t^6) \, dt = 4 - 3 + \frac{3}{5} - \frac{1}{7} = \frac{51}{35}.
\end{align*}
\]

**Proof.** For \( K_{3,2} \), \( n = 5 \) and \( m = 6 \). By Theorem 3.2, we have \( a_0 = 5 \), \( a_1 = -6 \), and \( a_2 = 0 \). Next, we apply Theorem 3.8. \( K_{3,2} \) has no 3-cycles, so \( a_3 = 0 \). The graph has three 4-cycles, so \( a_4 = 3 \). For the coefficient \( a_5 \), we note that there are no 5-cycles and also no \( k_3^5 \)-type subgraphs (a 4-cycle with a chord) either, so \( a_5 = 0 \). Lastly, for \( a_6 \), there are no 6-cycles, no \( K_4 \) subgraphs, no \( c_{5,1} \)-type subgraphs (since there are no 5-cycles), and one \( k_{3,2} \)-type subgraph (the entire graph). Therefore, \( a_6 = -1 \) and we get

\[
p_m(t) = -1 + 5 - 6t + 3t^4 - t^6.
\]

\( \square \)

## 5 The Complete Graph

The MST problem on \( K_n \) has been studied extensively. Frieze [3] proved that

\[
\lim_{n \to \infty} \mathbb{E}[L(K_n)] = \zeta(3) = \sum_{i=1}^{\infty} i^{-3} = 1.202 \ldots
\]

In [8], Steele extended this result to general edge distributions and Janson [4] proved a central limit theorem for \( L(K_n) \).

We apply our results to the complete graph and derive exact formulas in terms of the number of vertices \( n \) for the first seven coefficients of the polynomial integrand in Steele’s formula.
Theorem 5.1. Let $p_m(t) = 1 + \sum_{i=1}^{m} a_i t^i$ be the polynomial integrand in Steele’s formula for the complete graph on $n$ vertices, denoted by $K_n$. Then

$$a_0 = n, \quad a_1 = -\binom{n}{2}, \quad a_2 = 0, \quad a_3 = \binom{n}{3}$$

$$a_4 = 3 \binom{n}{4}, \quad a_5 = 12 \binom{n}{5} - 6 \binom{n}{4}, \quad a_6 = 60 \binom{n}{6} - 60 \binom{n}{5} - 2(n - 5) \binom{n}{4}$$

Proof. For the complete graph on $n$ vertices, the number of edges $m = \binom{n}{2}$ and the number of cycles of length $j$ is given by

$$c_j = \frac{1}{2} \binom{n}{j} (j - 1)!$$

In addition, $k_3^2 = 2c_4, c_{5,1} = 5c_5, k_4 = \binom{n}{4}$ and $k_{3,2} = \binom{n}{5} \binom{5}{2}$.

Numerical calculation of $E[L(K_n)]$ had led to the famous conjecture that the convergent sequence is also monotone increasing and concave. This problem was raised at the conference Mathematics and Computer Science II at Versailles in 2002 but no proof has been found. Clearly, our results alone will not answer this question as we have only derived exact formulas for the first seven coefficients. But our results give a hint that there may be a pattern to the coefficients of the polynomial integrand in Steele’s formula for the complete graph, which if true, would answer the conjecture.

We end this section with a result that factors the polynomial integrand in Steele’s formula for $K_n$, with one of the factors a polynomial of degree less than or equal to the number of edges of $K_{n-1}$.

Theorem 5.2. Let $p_m(t)$ be the polynomial integrand in Steele’s formula for the expected length of the MST of the complete graph on $n$ vertices denoted by $K_n$. Then

$$p_m(t) = (1 - t)^{n-1} q(t),$$

where $q(t)$ is a polynomial of degree less than or equal to $\binom{n-1}{2}$.

Proof. As in the proof of Theorem 3.1, we have

$$p_m(t) = \sum_{A \in S(G)} k(A)t^{|A|} (1 - t)^{m-|A|} - 1,$$

where $m = \binom{n}{2}$.

We factor out $(1 - t)^{n-1}$ to get

$$p_m(t) = (1 - t)^{n-1} \left[ \sum_{A \in S(G)} k(A)t^{|A|} (1 - t)^{\binom{n}{2} - |A| - (n-1)} - (1 - t)^{1-n} \right].$$
Note that \( \binom{n}{2} - (n - 1) = \binom{n-1}{2} \). Now the sum ranges over spanning subgraphs of size (in edges) from 0 to \( \binom{n}{2} \). We split it into two sums as follows:

\[
p_m(t) = (1 - t)^{n-1} \left[ \sum_{|A| \leq \binom{n-1}{2}} k(A)t^{|A|}(1 - t)^{\binom{n-1}{2} - |A|} \right. \\
+ \left. \sum_{|A| > \binom{n-1}{2}} k(A)t^{|A|}(1 - t)^{\binom{n}{2} - |A| - (n-1)} - (1 - t)^{1-n} \right]
\]

Clearly, the first sum over \( |A| \leq \binom{n-1}{2} \) is a polynomial of degree at most \( \binom{n-1}{2} \).

Call it \( q_1(t) \).

For the second sum, we sum over possible number of edges \( i > \binom{n-1}{2} \) and count the number of subgraphs with \( i \) edges, which for the complete graph is \( \binom{n}{2} \). Furthermore, for all spanning subgraphs of \( K_n \) with \( i > \binom{n-1}{2} \) edges, the number of connected components is 1. Therefore, we have

\[
\sum_{|A| > \binom{n-1}{2}} k(A)t^{|A|}(1 - t)^{\binom{n}{2} - |A| - (n-1)} - (1 - t)^{1-n} = (1 - t)^{1-n} \sum_{i=0}^{\binom{n}{2}} \binom{\binom{n}{2}}{i} t^i(1 - t)^{\binom{n}{2} - i} - \sum_{i=0}^{\binom{n-1}{2}} \binom{\binom{n}{2}}{i} t^i(1 - t)^{\binom{n-1}{2} - i}
\]

By the Binomial Theorem, the first sum equals 1 and the second sum, call it \( q_2(t) \), is a polynomial of degree at most \( \binom{n-1}{2} \).

We now have

\[
p_m(t) = (1-t)^{n-1}(q_1(t) + (1-t)^{1-n} + q_2(t) - (1-t)^{1-n}) = (1-t)^{n-1}(q_1(t) + q_2(t)),
\]

where both \( q_1(t) \) and \( q_2(t) \) are polynomials of degree less than or equal to \( \binom{n-1}{2} \).

This completes the proof.

\[\Box\]

References


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