Some Mathematics Behind Photography

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Slow Pitch Colloquium

The University of Colorado at Boulder

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Outline

Main Concept: Photography is both artistic and structured. In order to get good at it, you must master both the creative and mathematical aspects. Here, we'll discuss the latter.

Towards Understanding Color

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- The RYB color system became the foundation of 18^{th} Century color theory, based in the idea of “adding” pigments.

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- In 1704, in a move that would result in ridicule and yet ultimately provide a stable foundation for color theory many years later, - - - - - - - published *Opticks* and provided what is now considered to be the first “additive” color wheel.

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- One query specifically related to the nature of color perception.

Towards Understanding Color

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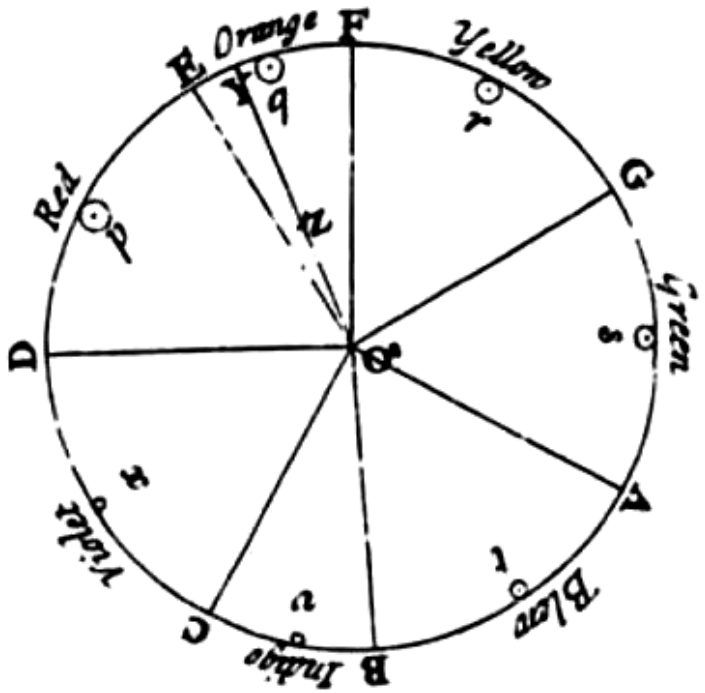

“Are not gross Bodies and Light convertible into one another, and may not Bodies receive much of their Activity from the Particles of Light which enter their Composition?”

Towards Understanding Color

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- But it wasn’t rigorous. So it would remain for 137 years.
- Of course, much was discovered in that time, mostly in regards to specific color perceptions.
- It wasn’t until 1861 that the current wavelength based RGB theory of color took shape, beginning with the production and explanation of the world’s first color photograph by a person named - - - - - - -.

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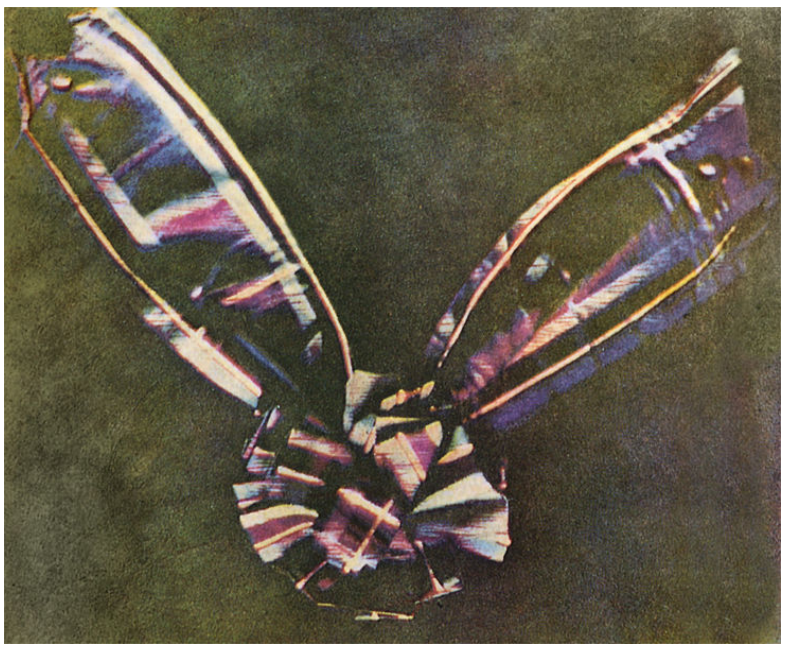
- In 1861, Maxwell took the world's first color photograph.

Towards Understanding Color



Our Current Understanding

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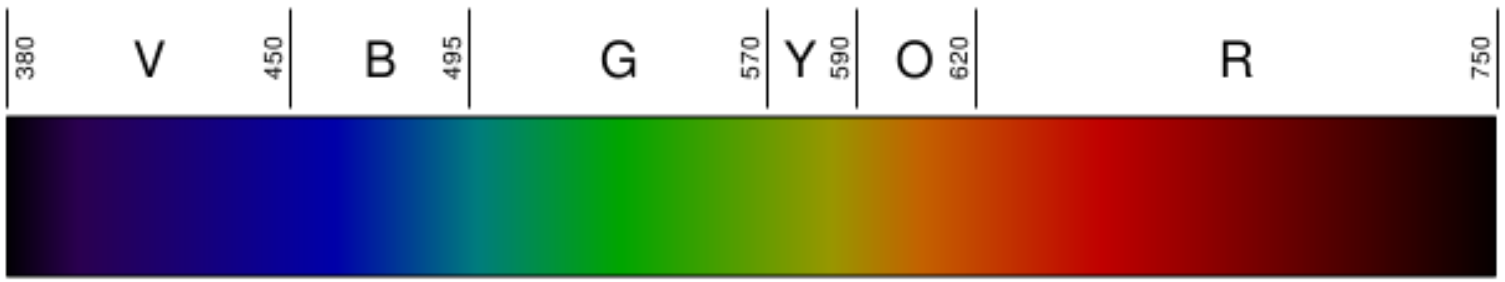
- The first is the wave-based nature of light.



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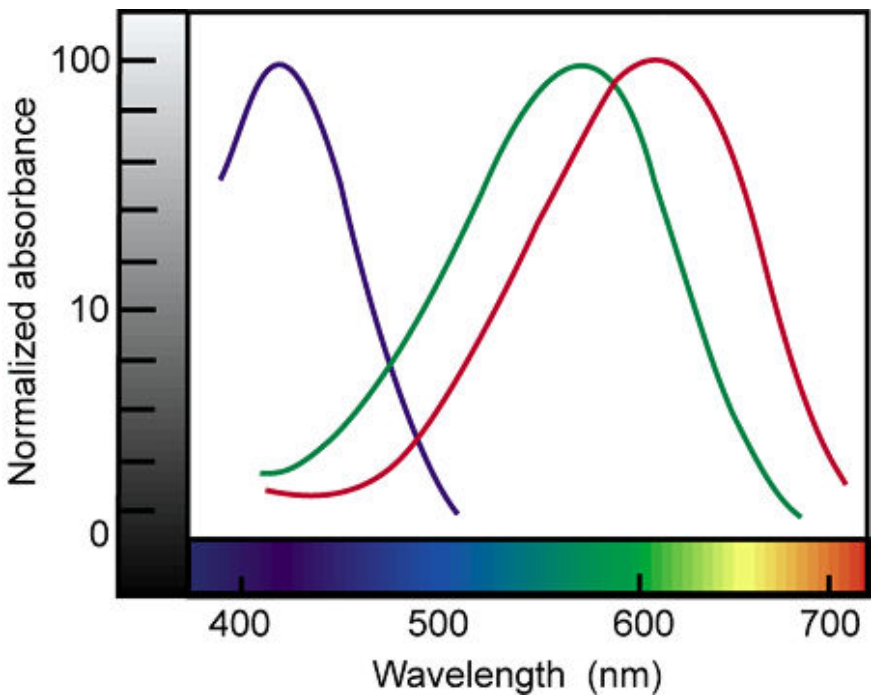
- Our eyes have three types of “cones,” each responding to a specific color or wavelength.



Understanding Perception, or the Lack Thereof

In a regularly functioning human eye, we find the following:

- 40% red cones
- 35% green cones
- 25% blue cones

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When one of these sets of cones fails to function properly, the result is what we call colorblindness.

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- Hence, someone who suffers from “Deuteranomaly” cannot distinguish red from green.

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- Failure for one color results in “dichromacy.”
- Failure for two colors results in “monochromacy.”
- These forms of colorblindness result in more than simply a lack of color perception. ...

Understanding Perception, or the Lack Thereof

Prevalence of color blindness

	Men	Women	Total	References
Overall	—	—	—	
Overall (United States)	—	—	1.30%	[19]
Red-green (Overall)	7 to 10%	—	—	[20][19]
Red-green (Caucasians)	8%	—	—	[21]
Red-green (Asians)	5%	—	—	[21]
Red-green (Africans)	4%	—	—	[21]
Monochromacy	—	—	—	
Rod monochromacy (disfunctional, abnormally shaped or no cones)	0.00001%	0.00001%	—	[22]
Dichromacy	2.4%	0.03%	—	[22]
Protanopia (L-cone absent)	1% to 1.3%	0.02%	—	[19][22]
Deuteranopia (M-cone absent)	1% to 1.2%	0.01%	—	[19][22]
Tritanopia (S-cone absent)	0.001%	0.03%	—	[22]
Anomalous Trichromacy	6.3%	0.37%	—	[22]
Protanomaly (L-cone defect)	1.3%	0.02%	—	[22]
Deuteranomaly (M-cone defect)	5.0%	0.35%	—	[22]
Tritanomaly (S-cone defect)	0.01%	0.01%	—	[22]

Understanding Perception, or the Lack Thereof

Are you colorblind? (let's hope this works on the projector)

- The Ishihara color test can probe for cone disfunction.
- In the following example, you should see a 74.
- Various cone disfunctions will result in seeing either a 21 or no number at all.

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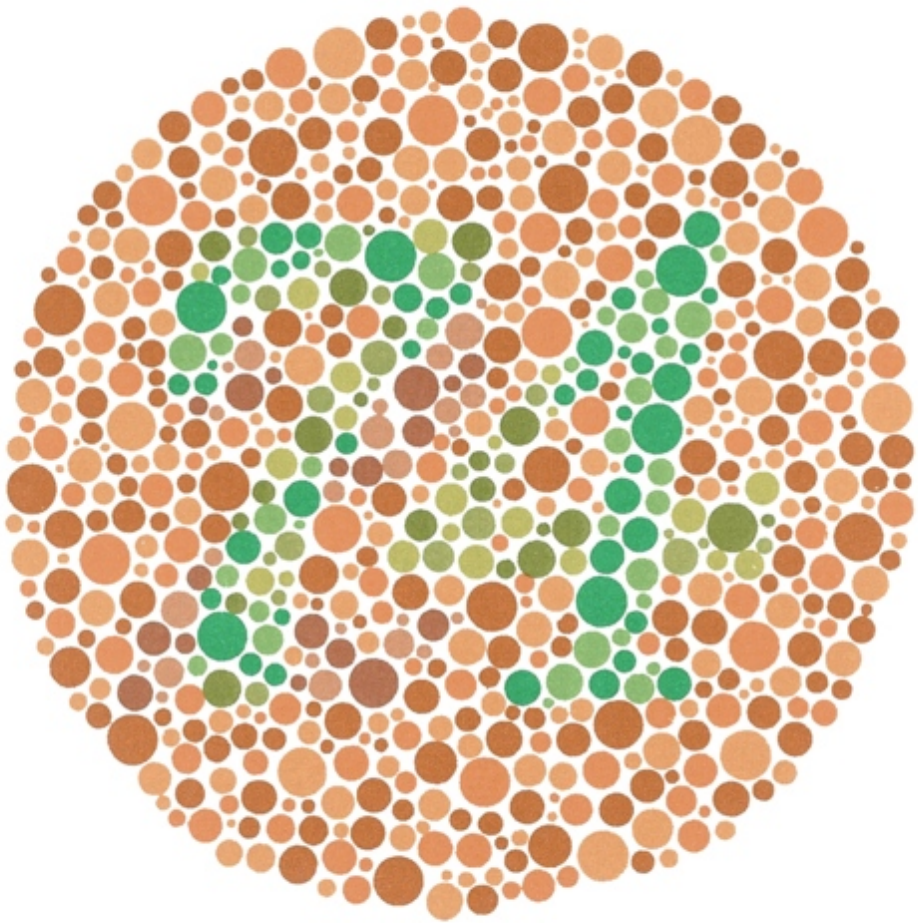

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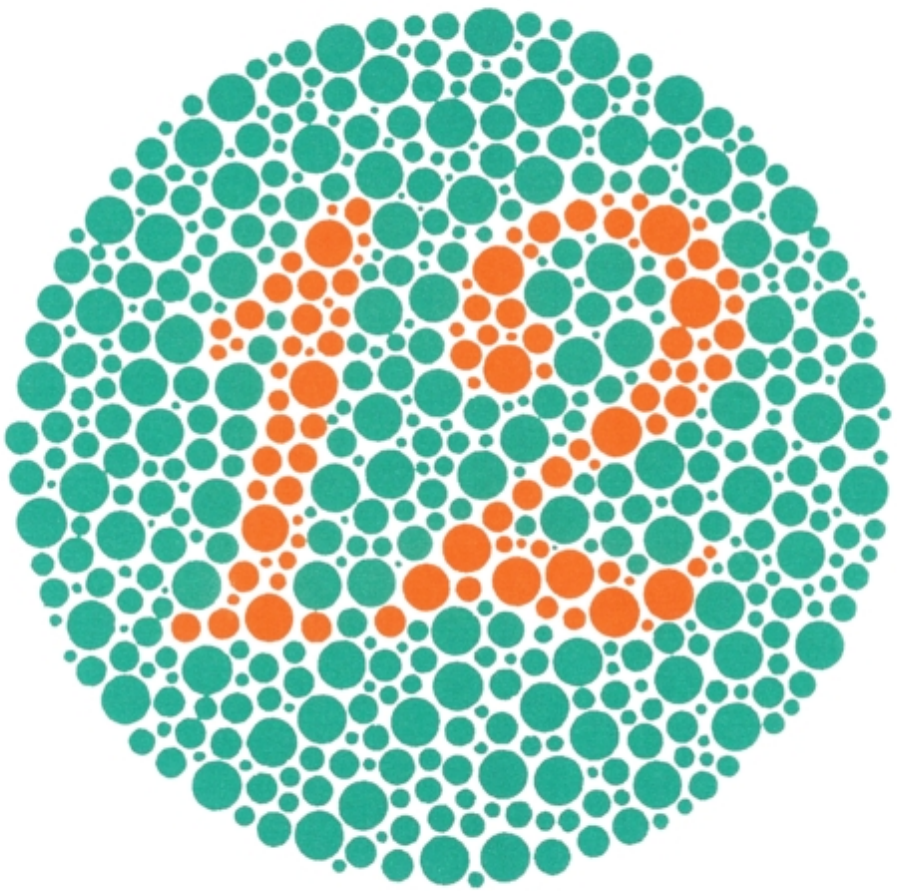

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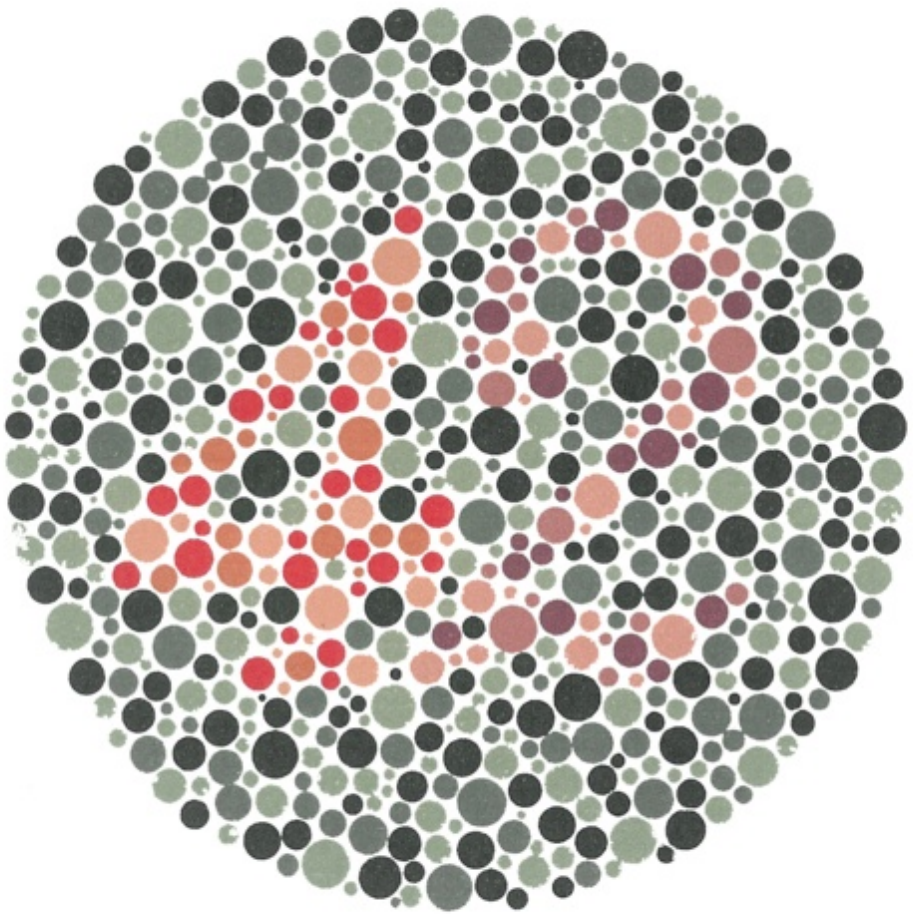

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- As far as we know, this only happens in women. ...

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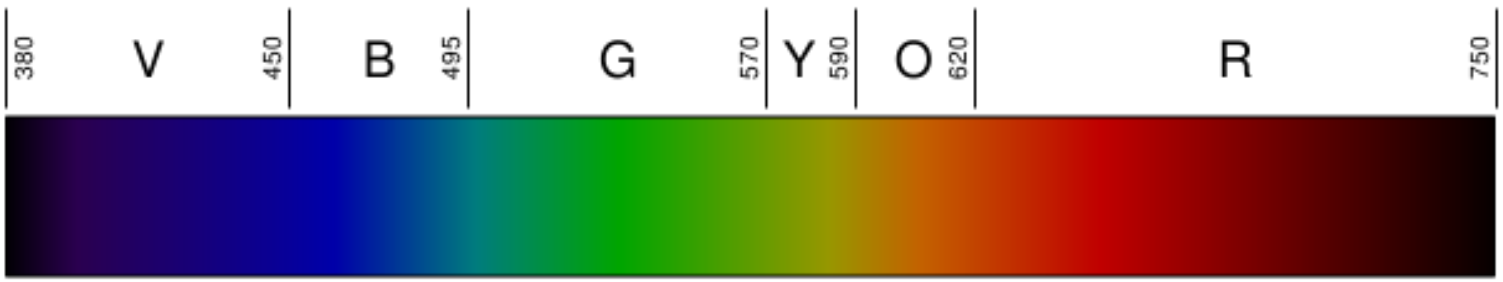
- It arises in ways you may not imagine.
- Color filtration in B&W photography is an example.



see red filter example

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- Film and digital sensors are different.

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- Why??

Color in Photography

- If we prohibit any light with a wavelength below 750nm from passing through the lens, we can record an infrared image.
- “Infrared” can mean a few things. In this case, it’s just a range of light and doesn’t have much to do with heat.
- Why??
- Certain objects reflect a high amount of infrared range light: namely, plants that are undergoing photosynthesis.

see infrared example

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- The principal variables that are base-2 are:
  - Film Speed
  - Shutter Speed
  - Lens Aperture / F-Stop
- Learning how to operate a camera comes down to figuring out how to change these variables in unison.

Film Speed

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Main Idea (2): If the intensity of light stays the same and the sensitivity of the film is doubled, the image should be recorded in half the time.

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- 800 is twice as sensitive as 400.

Film Speed

Consider film speeds as a function $FS(x) = 2^x \cdot 100$.

In a photographic language...

$x = 0$	$\leftrightarrow$	ISO $= 100$	"standard"
$x = 1$	$\leftrightarrow$	ISO $= 200$	1 "stop" above standard
$x = 2$	$\leftrightarrow$	ISO $= 400$	2 "stops" above standard
$x = 3$	$\leftrightarrow$	ISO $= 800$	3 "stops" above standard
		$\ldots$	

In general, a "stop" is a factor of 2 (... or a factor of $\frac{1}{2}$).

Film Speed

Question:

You take a photo with ISO 100 film and the exposure time you need to properly expose the film is $\frac{1}{8}$ seconds. But, you'd like to freeze the action in your photo and take it at $\frac{1}{256}$ seconds instead and the only factor you can change is film speed. Which film speed do you use?

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A: ISO 3200

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- *Cough cough*... *lazy* base-2

$$\frac{1}{4000}s,\ \frac{1}{2000}s,\ \frac{1}{1000}s,\ \frac{1}{500}s,\ \frac{1}{250}s,\ \frac{1}{125}s,\ \frac{1}{60}s,\ \frac{1}{30}s,\ \frac{1}{15}s,\ \frac{1}{8}s,\ \frac{1}{4}s,\ \frac{1}{2}s,\ 1s$$
$$1s,\ 2s,\ 4s,\ 8s,\ 15s,\ 30s$$

Shutter Speed

Question:

You've been taking photos all day using ISO 100 film with a shutter speed of $\frac{1}{125}$ second. You run out of film and can only find ISO 400 film to replace it. What shutter speed should you now be using?

I.e., how many "stops" are required?

Do it in your head...

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- Let’s take a small detour...

Focal Length

See and Explain Focal Length Examples

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Example: Using a 20mm lens, any shutter speed longer than $\frac{1}{20}$s requires a tripod. Okay... about $\frac{1}{15}$s.

Example: Using a 300mm lens, any shutter speed longer than $\frac{1}{300}$s requires a tripod. Okay... about $\frac{1}{250}$s.

Lens Aperture / F-Stop

This last variable takes a while to learn...

- Your eye has an iris, opening and closing to allow more or less light (respectively) into your eye.

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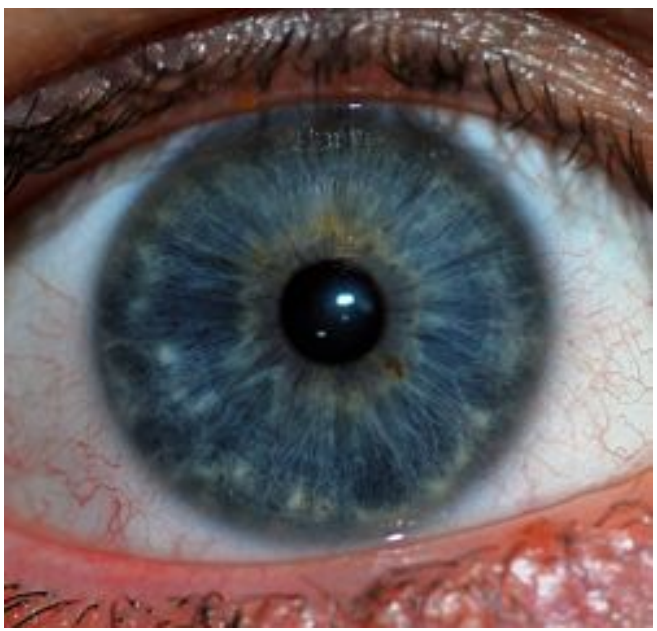
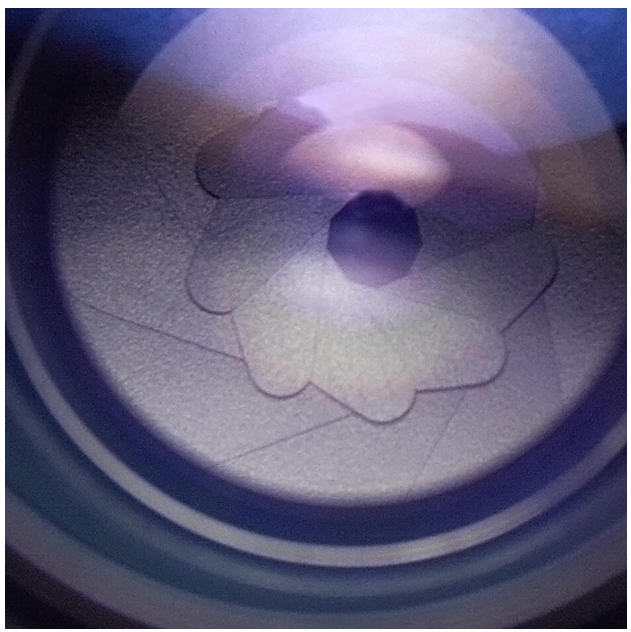
This last variable takes a while to learn...

- Your eye has an iris, opening and closing to allow more or less light (respectively) into your eye.
- Camera lenses also have an iris. ...





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  where f is focal length of the lens and D is the diameter of the pupil at the given setting.
- Main idea: How much light a lens lets in is a function of both the lens length and the pupil size. ...

Lens Aperture / F-Stop

- These numbers would be nice, except pupil area is a function of aperture radius squared.

$$f\# = \frac{f}{D} = \frac{f}{2R} = \frac{1}{2} \cdot \frac{1}{R}$$

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- We'll start at $f\# = 1$, written as $f1.0$. This means $f = D$, focal length is equal to pupil diameter.
- If $f = 2D$ then we get $f2.0$. Unfortunately, halving the diameter results in $1/4$ the original amount of light coming through the pupil.

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$$n = m \cdot 2^{A/2}$$

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- This is why f-numbers appear to be so random...

$f1$ $f1.4$ $f2$ $f2.8$ $f4$ $f5.6$ $f8$ $f11$ $f16$ $f22$ $f32$

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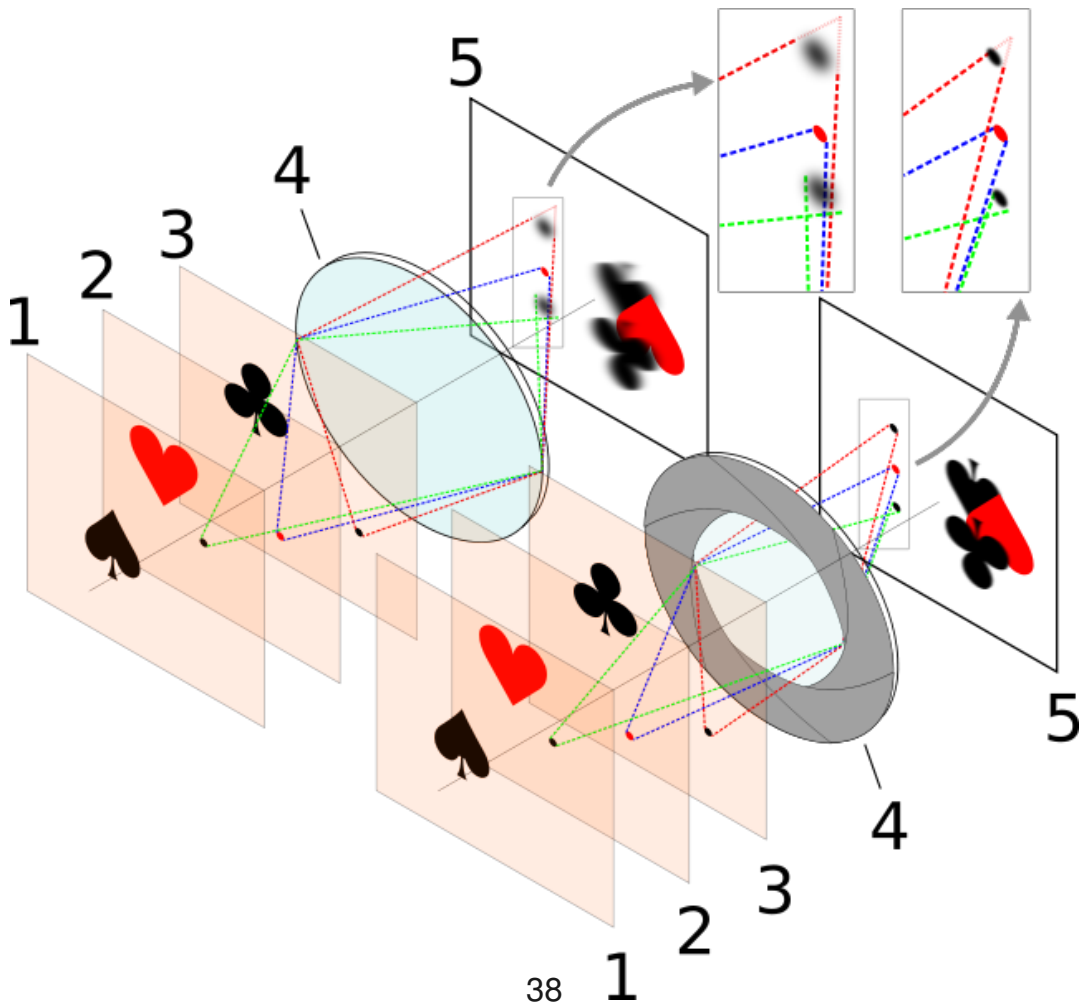
$f1$ $f1.4$ $f2$ $f2.8$ $f4$ $f5.6$ $f8$ $f11$ $f16$ $f22$ $f32$

- Generally, as a lens gets longer, the f-numbers that lens is capable of offering get higher.

Lens Aperture / F-Stop

Q: What is the advantage of apertures/f-stops?

A: Depth of Field



Lens Aperture / F-Stop

see examples

Putting it all Together

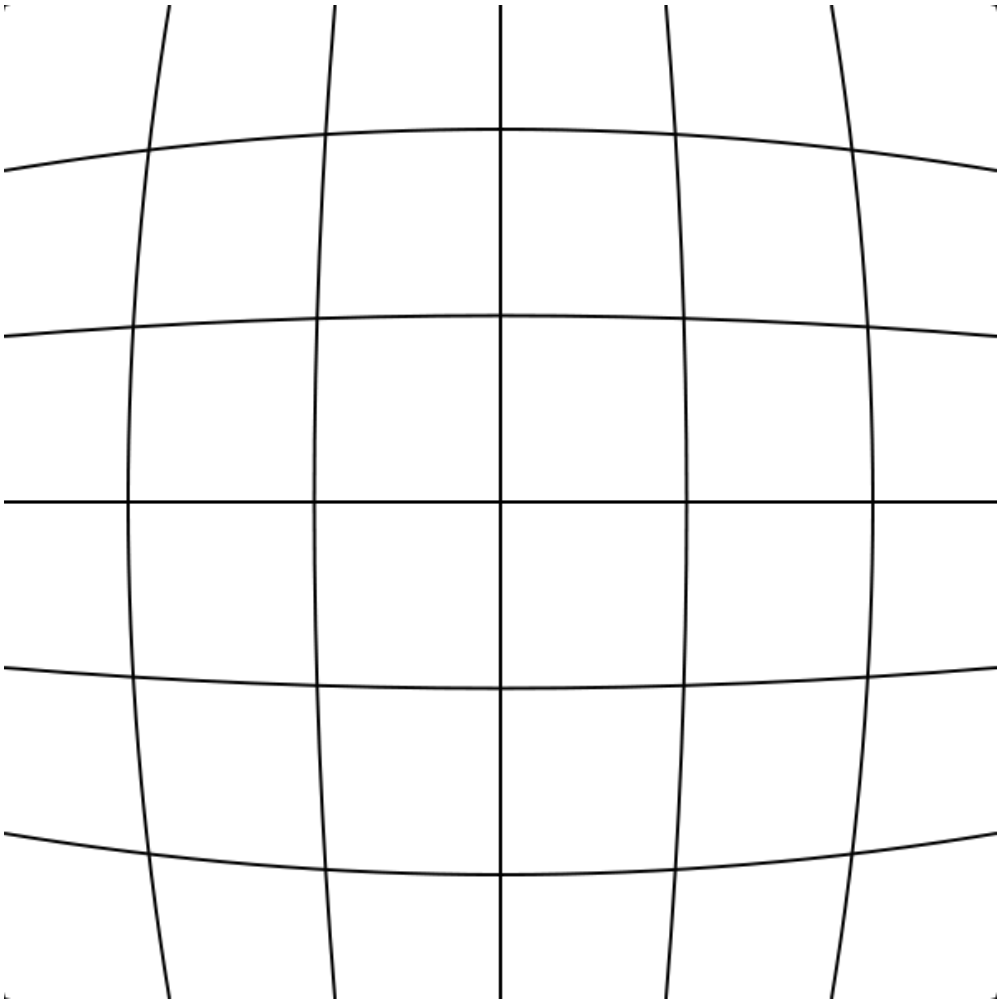
$$\text{shutter speed} \uparrow \quad \Rightarrow \quad \text{f-number} \downarrow \quad \text{and} \quad \text{DOF} \downarrow$$

$$\text{shutter speed} \downarrow \quad \Rightarrow \quad \text{f-number} \uparrow \quad \text{and} \quad \text{DOF} \uparrow$$

$$\text{DOF} \uparrow \quad \Rightarrow \quad \text{f-number} \uparrow \quad \text{and} \quad \text{shutter speed} \downarrow$$

$$\text{DOF} \downarrow \quad \Rightarrow \quad \text{f-number} \downarrow \quad \text{and} \quad \text{shutter speed} \uparrow$$

Using Transformations to Correct Lens Distortions



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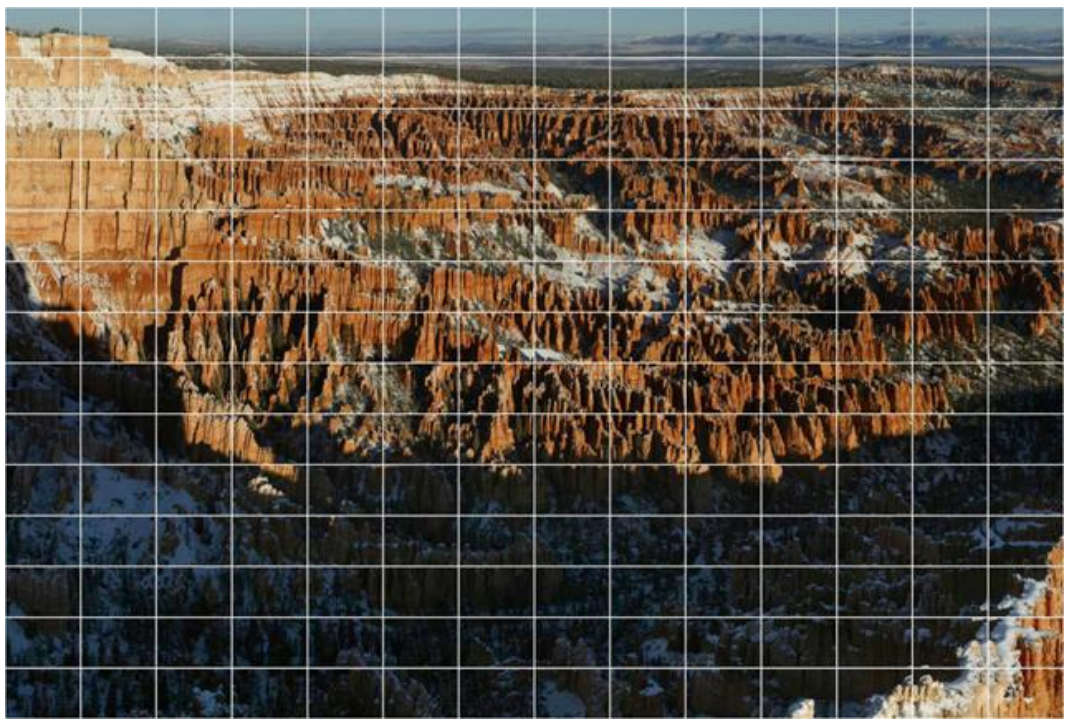
- Helmut Dersch (furtwangen.de) found otherwise.

Using Transformations to Correct Lens Distortions

- Dersch started using barrel distortion corrections in an attempt to make the world's first "gigapixel" image.



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- See examples.