THETA FUNCTIONS AND SINGULAR TORSION ON ELLIPTIC CURVES

John BOXALL and David GRANT

§1. Introduction

In this paper we report on various relationships between theta functions and torsion points on certain commutative algebraic groups. In §3 we consider products G_n of degree 2 theta constants, which are Siegel modular forms that vanish at the period matrix of a curve of genus two precisely when the curve has a point of exact order n in its hyperelliptic torsion packet. The proof of the Manin-Mumford conjecture then shows that at any given period matrix, there are only finitely many n for which G_n vanishes.

In §4, by studying the asymptotic behavior of G_n near the points of the Siegel upper half space of degree 2 which correspond to the product of elliptic curves, we are led to consider elliptic modular functions f_n and g_n, which are the products of derivatives at 0 of elliptic theta functions with rational characteristics. On the one hand, as we discuss in §5, an understanding of the zeros of f_n and g_n is a generalization of Jacobi's derivative formula; but on the other hand, we show in §6 that f_n and g_n vanish at the period matrix of an elliptic curve E precisely when E contains a special point of order n we call a "singular torsion point."

In §7 we show that the singular torsion points of E are in 1-1 correspondence with the torsion points on the image of E embedded into the generalized Jacobian of E with modulus twice the origin [S]. Hindry's proof of the generalization of the Manin-Mumford conjecture to commutative algebraic groups then shows that every complex elliptic curve has only finitely many singular torsion points. Finally, for E defined over a number field, we

---

1991 *Mathematics Subject Classification.* 11G05, 11G30, 14K25.

Key words and phrases. Theta functions, curves of genus two, elliptic curves, torsion points.

briefly describe in §8 a procedure which in principle allows one to compute all the singular torsion points on E.

In §2 we recall what information we need from the theory of theta functions. The proofs of the results in §8 will appear in [BG1].

§2. Preliminaries

Let $g \geq 1$ be an integer and let $\mathcal{H}^g$ denote the Siegel space of complex $g \times g$ symmetric matrices with positive definite imaginary part. We write Γ_g for the group $\mathrm{Sp}_{2g}(\mathbb{Z})$ of symplectic $2g \times 2g$ matrices with integral coefficients. We let $\mathcal{A}_g$ be the (uncompactified) moduli space of principally polarized complex abelian varieties of dimension g, so that with the standard action of Γ_g on $\mathcal{H}^g$, we can identify $\mathcal{A}_g(\mathbb{C})$ with $\Gamma_g \backslash \mathcal{H}^g$. If $\mathcal{M}_g$ denotes the moduli space of smooth proper complex curves of genus g, then Torelli's theorem allows us to view $\mathcal{M}_g$ as a subvariety of $\mathcal{A}_g$. In particular, $\mathcal{M}_2$ is a dense open subvariety of $\mathcal{A}_2$ whose complement is a divisor which we denote by $\mathcal{D}$. Then $\mathcal{D}$ is isomorphic to $\mathcal{A}_1 \times \mathcal{A}_1$ and $\mathcal{D}(\mathbb{C})$ corresponds to the points of $\Gamma_2 \backslash \mathcal{H}^2$ that are images under the canonical projection $\mathcal{H}^2 \to \Gamma_2 \backslash \mathcal{H}^2$ of the diagonal matrices in $\mathcal{H}^2$ [I1]. We denote by $\mathcal{D}^*(\mathbb{C})$ and $\mathcal{M}_2^*(\mathbb{C})$ the pullbacks of $\mathcal{D}(\mathbb{C})$ and $\mathcal{M}_2(\mathbb{C})$ under this projection.

The Riemann theta function of genus g with characteristic $\alpha = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$, $a, b \in \mathbb{Q}^g$, is defined on $\mathbb{C}^g \times \mathcal{H}^g$ by

$$\theta_g[\alpha](z, \tau) = \sum_{n \in \mathbb{Z}^g} e^{\pi i^t(n+a)\tau(n+a) + 2\pi i^t(n+a)(z+b)},$$

where $(z, \tau) \in \mathbb{C}^g \times \mathcal{H}^g$ and ${}^tX$ denotes the transpose of the vector X. We often identify the characteristics modulo 1. Since $\theta_g[\alpha + \epsilon](z, \tau) = e^{2\pi i^t aq}\theta_g[\alpha](z, \tau)$ for all $\alpha = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$, $\epsilon = \left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$, $p, q \in \mathbb{Z}^g$, this means that some of our functions will only be defined up to a multiplication by a root of unity. If $\alpha = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$, then the order of α is defined to be the order of (a, b) in $\mathbb{Q}^{2g}/\mathbb{Z}^{2g}$. If $n \in \mathbb{N}^*$, we denote by $\Phi_g(n)$ the set of all characteristics in $\mathbb{Q}^{2g}/\mathbb{Z}^{2g}$ of order n and by $\phi_g(n)$ the cardinality of $\Phi_g(n)$. Thus

$$\phi_g(n) = n^{2g} \prod_{p \mid n} \left(1 - \frac{1}{p^{2g}}\right), \tag{1}$$

the product running over all primes p dividing n.

For every characteristic $\delta = \left[\begin{smallmatrix} \delta_1 \\ \delta_2 \end{smallmatrix}\right] \in \Phi_g(2) \cup \{\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]\}$, $\theta_2[\delta](z, \tau)$ is an even or odd function of z, and δ is called an even or odd characteristic accordingly.

We write $\Phi_g(2)^+$ and $\Phi_g(2)^-$ respectively for the set of even and odd characteristics.

We shall be almost exclusively interested in values of $\theta_g[\alpha](z,\tau)$ with $z=0$, the so-called theta constants or thetanullwerte. We denote $\tau \mapsto \theta_g[\alpha](0,\tau)$ by $\theta_g[\alpha](\tau)$, and when $g=1$, $\tau \mapsto \frac{\partial}{\partial z}\big(\theta_1[\alpha](z,\tau)\big)_{z=0}$ by $\theta_1'[\alpha](\tau)$. We record the following standard results for future reference.

Lemma 1. *Let* $a = \left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right]$, $b = \left[\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right]$, *with* $a_i, b_i \in \mathbb{Q}$. *Then, for all* $(\tau_1,\tau_2) \in \mathcal{H}^1 \times \mathcal{H}^1$ *we have*

(i) $\theta_2\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} = \theta_1\left[\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}\right](\tau_1)\theta_1\left[\begin{smallmatrix} a_2 \\ b_2 \end{smallmatrix}\right](\tau_2)$,

(ii) $2\pi i \,\frac{\partial}{\partial \sigma}\left(\theta_2\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]\left(\begin{smallmatrix} \tau_1 & \sigma \\ \sigma & \tau_2 \end{smallmatrix}\right)\right)_{\sigma=0} = \theta_1'\left[\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}\right](\tau_1)\theta_1'\left[\begin{smallmatrix} a_2 \\ b_2 \end{smallmatrix}\right](\tau_2)$.

Lemma 2. *Let* $\alpha = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$, $a, b \in \mathbb{Q}$.

(i) *We have* $\theta_1[\alpha](\tau) = 0$ *precisely when* $(a,b) \equiv (\frac{1}{2}, \frac{1}{2}) \pmod{\mathbb{Z}\times\mathbb{Z}}$.

(ii) $\theta_1'[\alpha](\tau)$ *is a modular form of weight* $\frac{3}{2}$ *on an appropriate subgroup of* Γ_1. *It vanishes identically if and only if* $\alpha \in \Phi_1(2)^+$.

(iii) (Jacobi's derivative formula.) *For all* $\tau \in \mathcal{H}^1$, *we have*

$$\theta_1'\left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right](\tau) = -\pi\,\theta_1\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right](\tau)\theta_1\left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right](\tau)\theta_1\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right](\tau) = -2\pi\eta(\tau)^3,$$

where for $q = e^{2\pi i\tau}$, $\eta(\tau) = e^{\frac{\pi i \tau}{12}}\prod_{n\geq 1}(1-q^n)$.

Lemma 1 is proved via easy calculations using the definitions. Lemma 2 (i) is in [M], and (ii) follows from the functional equation of the theta function and by looking at q-expansions. There are many proofs of Lemma 2 (iii) in the literature, including one in [M].

§3. The genus 2 case

Let X be a smooth proper curve of genus two over $\mathbb{C}$, and let J be the Jacobian of X. Let (A_1, A_2, B_1, B_2) be a symplectic basis of $H_1(X(\mathbb{C}),\mathbb{Z})$. Then there exists a unique basis (ω_1,ω_2) of the holomorphic differentials on X such that $\int_{A_i}\omega_j = \delta_{ij}$ for i, $j \in \{1,2\}$. Then the matrix

$$\tau_X = \begin{pmatrix} \int_{B_1}\omega_1 & \int_{B_2}\omega_1 \\ \int_{B_1}\omega_2 & \int_{B_2}\omega_2 \end{pmatrix}$$

lies in $\mathcal{H}^2$ and the orbit of τ_X under Γ_2 represents the isomorphism class of X in $\mathcal{M}_2$. If $\tau \in \mathcal{H}^2$, denote by Λ_τ the lattice $\{\tau m + n \mid m, n \in \mathbb{Z}^2\}$ of $\mathbb{C}^2$.

Let $X^{(2)}$ denote the symmetric square of X, whose points we identify with positive divisors on X of degree 2. Fix a Weierstrass point W on X and define a morphism of complex analytic varieties $\epsilon_W : X^{(2)}(\mathbb{C}) \to \mathbb{C}^2/\Lambda_{\tau_X}$ by

$$\epsilon_W(P+Q) = \int_W^P \begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} + \int_W^Q \begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} \pmod{\Lambda_{\tau_X}},$$

where the integrals are taken over any paths joining W to P and to Q. Embed X in $X^{(2)}$ by sending $P \in X(\mathbb{C})$ to the divisor $P+W$. Then Riemann's vanishing theorem then shows that there exists a $\delta = \delta_W \in \Phi_2(2)^-$ such that

$$\epsilon_W(X(\mathbb{C})) = \{z \in \mathbb{C}^2/\Lambda_{\tau_X} \mid \theta_2[\delta](z, \tau_X) = 0\}. \tag{2}$$

In fact, given a choice of symplectic basis for $H_1(X(\mathbb{C}), \mathbb{Z})$, the map $W \to \delta_W$ is a bijection between the set of Weierstrass points on X and $\Phi_2(2)^-$ [M]. Note that it follows from the definitions that $\theta_2[\delta](z+u,\tau)$ is equal to an exponential factor times $\theta_2[\delta](z,\tau)$ for all $(u,z,\tau) \in (\Lambda_\tau, \mathbb{C}^2, \mathcal{H}^2)$, so that the vanishing of $\theta_2[\delta](z,\tau)$ depends only on $z \pmod{\Lambda_\tau}$. We write Θ_W for $\epsilon_W(X(\mathbb{C}))$. Let $\mu : X^{(2)} \to J$ be the map defined by sending $P+Q$ to the class of $P+Q-2W$ in the Picard group of C. Then ϵ_W and μ induce an isomorphism $J(\mathbb{C}) \simeq \mathbb{C}^2/\Lambda_{\tau_X}$. Using this isomorphism, we often identify Θ_W with its image in J. If $n \in \mathbb{N}^*$, and A is a commutative algebraic group defined over a field k with algebraic closure $\bar{k}$, we denote by $A[n]$ the group of points in $A(\bar{k})$ of order dividing n and by $A[n]^*$ the subset of $A[n]$ consisting of the points of (exact) order n. Let $A_{\mathrm{tors}} = \cup_{n=1}^{\infty} A[n]$ be the torsion subgroup of $A(\bar{k})$.

Let $n \in \mathbb{N}^*$ and suppose $z \in \Theta_W$. Then z lies in $J[n]^*$ if and only z is of order n as an element of $\mathbb{C}^2/\Lambda_{\tau_X}$, that is to say, if and only if z is of the form $(\tau_X p + q)/n$ for some $p, q \in \mathbb{Z}^2$ with the greatest common divisor of n and the coefficients of p and q equal to one. This is the same thing as saying that the characteristic $\begin{bmatrix} p/n \\ q/n \end{bmatrix}$ belongs to $\Phi_2(n)$. Again, an easy calculation shows that $\theta_2[\delta]((\tau p + q)/n, \tau)$ is the same as $\theta_2[\delta + \begin{bmatrix} p/n \\ q/n \end{bmatrix}](\tau)$ up to an exponential factor. Using (2) we conclude that Θ_W contains an point of order n if and only if $\theta_2[\delta+\alpha](\tau)$ vanishes at $\tau = \tau_X$ for some $\alpha \in \Phi_2(n)$. If $\delta \in \Phi_2(2)$ and $n \in \mathbb{N}^*$, we define the functions $F_{\delta,n}$ and G_n on $\mathcal{H}^2$ by

$$F_{\delta,n}(\tau) = \prod_{\alpha \in \Phi_2(n)} \theta[\delta+\alpha](\tau), \qquad G_n = \prod_{\delta \in \Phi_2(2)^-} F_{\delta,n}. \tag{3}$$

We have proved (see also [Go1]):

Proposition 3. *Let $n \in \mathbb{N}^*$.*

(i) *Let W be a Weierstrass point on X. A necessary and sufficient condition for Θ_W to contain a point of $J[n]^*$ is that $F_{\delta_W,n}$ vanishes at τ_X.*

(ii) *A necessary and sufficient condition for X to have a Weierstrass point W such that Θ_W contains a point of $J[n]^*$ is that G_n vanishes at τ_X.*

Remarks. 1) When $n = 2m$, m odd, up to constant multiples we have

$$F_{\delta,n} = \prod_{\substack{\epsilon \in \Phi_2(2) \\ \epsilon \neq \delta}} F_{\epsilon,m}.$$

So in what follows we will only consider $F_{\delta,n}$ when n is odd or a multiple of 4.

2) When n is a multiple of 4, $[\alpha] \mapsto [\delta + \alpha]$ is just a permutation of $\Phi_2(n)$, so up to constant multiples, $F_{\delta,n}$ does not depend on δ in this case, and we often denote it by F_n.

3) When $\delta \in \Phi_2(2)^-$, we have $\theta_2[\delta](\tau) = 0$. It follows that $F_{\delta,1} = 0$. It is shown in [Gr1] that for $n \geq 3$, G_n is a modular form on Γ_2. With more work, using Proposition 5 one can show that G_n is the square of a modular form on Γ_2.

Recall that the Manin-Mumford conjecture in its original form asserts that if X is a smooth projective curve of genus ≥ 2 over $\mathbb{C}$ embedded in its Jacobian, then $X(\mathbb{C})$ contains only finitely many torsion points of the Jacobian. This is of course a theorem, first proved by Raynaud [R] and several other proofs have appeared since, see for example [Col] and [H]. In particular, $\Theta_W \cap J_{\text{tors}}$ is finite for all Weierstrass points W (these intersections comprise the image under ϵ_W of the so-called hyperelliptic torsion packet on X [Col]). So Proposition 3 implies that given $\tau \in \mathcal{H}^2$ whose orbit under Γ_2 represents a point of $\mathcal{M}_2(\mathbb{C})$, only finitely many of the functions G_n vanish at τ. Equivalently, only finitely many of the functions $\theta_2[\alpha](\tau)$, $\alpha = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right], a, b \in \mathbb{Q}^2/\mathbb{Z}^2$, can vanish at any given $\tau \in \mathcal{M}_2^*(\mathbb{C})$.

§4. Asymptotic behavior

Given the results of the last section, it is natural to study the asymptotic behavior of $F_{\delta,n}$ and G_n on the points of $\mathcal{A}_2$ near $\mathcal{D} \cong \mathcal{A}^1 \times \mathcal{A}^1$, to see what their limiting behavior tells us about torsion points on elliptic curves. To do so, we first need to recall some of the properties of the genus 2 discriminant function.

We define $\Delta_2 : \mathcal{H}^2 \to \mathbb{C}$ by $\Delta_2(\tau) = 2^{-12} \prod_{\delta \in \Phi_2(2)^+} \theta[\delta](\tau)^2$. This does not depend on the choice of representatives of the δ's. We call Δ_2 the genus two discriminant function because when $\tau \notin \mathcal{D}^*(\mathbb{C})$, $\Delta_2(\tau)$ is related to the discriminant of a quintic or sextic polynomial P such that $y^2 = P(x)$ defines a Weierstrass model of a genus 2 curve with period matrix τ (see [Gr2] and [L]).

Proposition 4. (*i*) *Δ_2 is a modular form of weight 10 on Γ_2.*
(*ii*) *Δ_2 has a zero of order 2 along $\mathcal{D}^*$ and no other zeros.*
(*iii*) *We have*

$$\Delta_2 \left(\begin{smallmatrix} \tau_1 & \sigma \\ \sigma & \tau_2 \end{smallmatrix}\right) = (2\pi i)^2 \Delta(\tau_1)\Delta(\tau_2)\sigma^2 + O(\sigma^3),$$

where $\Delta(\tau) = \eta(\tau)^{24}$ is the usual genus one Δ-function.

For a proof of (*i*) and (*ii*), see [K], page 118. Part (*iii*) is an easy application of Lemmas 1 and 2. We can now study the behavior of $F_{\delta,n}$ near $\mathcal{H}^1 \times \mathcal{H}^1$.

The elements $\delta \in \Phi_2(2)^-$ are $\delta = \left[\begin{smallmatrix} \delta' \\ \delta'' \end{smallmatrix}\right]$ with $\delta' = (\delta_1', \delta_2')$, $\delta'' = (\delta_1'', \delta_2'')$ and $\left(\begin{smallmatrix} \delta_1' & \delta_2' \\ \delta_1'' & \delta_2'' \end{smallmatrix}\right)$ equal to one of $\left(\begin{smallmatrix} \frac12 & 0 \\ \frac12 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & \frac12 \\ 0 & \frac12 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} \frac12 & 0 \\ \frac12 & \frac12 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} \frac12 & \frac12 \\ \frac12 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & \frac12 \\ \frac12 & \frac12 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} \frac12 & \frac12 \\ 0 & \frac12 \end{smallmatrix}\right)$.

In each case, there is exactly one index $i = i_\delta \in \{1, 2\}$ such that $\left[\begin{smallmatrix} \delta_i' \\ \delta_i'' \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right]$. Let $j = j_\delta = 3 - i_\delta$ be the other index. We denote $\left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right]$ by δ_0 and set $\delta^+ = \left[\begin{smallmatrix} \delta_j' \\ \delta_j'' \end{smallmatrix}\right]$. We see that in each case $\delta^+ \in \Phi_1(2)^+$.

If $n \in \mathbb{N}^*$ and $\delta \in \Phi_1(2) \cup \{\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]\}$ we define the function f_n on $\mathcal{H}^1$ by

$$f_{\delta,n}(\tau) = \prod_{\alpha \in \Phi_1(n)} \theta_1'[\delta + \alpha](\tau).$$

In particular, we set $f_n = f_{\delta_0,n}$. When n is a multiple of 4, up to constant multiples, $f_{\delta,n}$ is independent of δ and so it is just f_n in this case.

Proposition 5. *Let $n \geq 3$ and let $\tau = \left(\begin{smallmatrix} \tau_1 & \sigma \\ \sigma & \tau_2 \end{smallmatrix}\right) \in \mathcal{H}^2$.*
(*i*) *Suppose n is odd, $\delta \in \Phi_2(2)^-$. Then for some constant $c \neq 0$ (depending on n and δ) we have*

$$F_{\delta,n}(\tau) = c f_{\delta^+,n}(\tau_{j_\delta}) \eta(\tau_{i_\delta})^{3\phi_1(n)} \big(\eta(\tau_1)\eta(\tau_2)\big)^{\phi_2(n) - \phi_1(n)} \sigma^{\phi_1(n)} \big(1 + O(\sigma)\big)$$

as $\sigma \to 0$;
(*ii*) *Suppose n is a multiple of 4. Then there is a constant $c \neq 0$ (depending on n) such that*

$$F_n(\tau) = c f_n(\tau_1) f_n(\tau_2) \big(\eta(\tau_1)\eta(\tau_2)\big)^{\phi_2(n) + \phi_1(n)} \sigma^{2\phi_1(n)} \big(1 + O(\sigma)\big)$$

as $\sigma \to 0$.

Proof: We use Lemma 1 which gives

$$\theta_2[\delta+\alpha](\tau) = \theta_1\left[\begin{smallmatrix}a_1\\b_1\end{smallmatrix}\right](\tau_1)\theta_1\left[\begin{smallmatrix}a_2\\b_2\end{smallmatrix}\right](\tau_2) + \tfrac{1}{2\pi i}\theta_1'\left[\begin{smallmatrix}a_1\\b_1\end{smallmatrix}\right](\tau_1)\theta_1'\left[\begin{smallmatrix}a_2\\b_2\end{smallmatrix}\right](\tau_2)\sigma + O(\sigma^2)$$

whenever $\delta+\alpha = \left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]$, with $a=(a_1,a_2)$ and $b=(b_1,b_2)$. We apply this to each term in the product (3) defining $F_{\delta,n}$. By Lemma 2(i), the terms for which $\theta_1\left[\begin{smallmatrix}a_1\\b_1\end{smallmatrix}\right](\tau_1)\theta_1\left[\begin{smallmatrix}a_2\\b_2\end{smallmatrix}\right](\tau_2)$ vanishes are just those for which either $\left[\begin{smallmatrix}a_1\\b_1\end{smallmatrix}\right]=\delta_0$ or $\left[\begin{smallmatrix}a_2\\b_2\end{smallmatrix}\right]=\delta_0$. When n is odd, this can only happen for one of the indices, which is then equal to i_δ, in which case $\left[\begin{smallmatrix}a_{j_\delta}\\b_{j_\delta}\end{smallmatrix}\right]+\delta_+ \in \Phi_1(n)$. When n is a multiple of 4, this occurs with either index, and we must again have $\left[\begin{smallmatrix}a_{j_\delta}\\b_{j_\delta}\end{smallmatrix}\right]+\delta_+ \in \Phi_1(n)$. Thus, among the expressions $\theta_1\left[\begin{smallmatrix}a_1\\b_1\end{smallmatrix}\right](\tau_1)\theta_1\left[\begin{smallmatrix}a_2\\b_2\end{smallmatrix}\right](\tau_2)$, $\phi_1(n)$ vanish when n is odd and $2\phi_1(n)$ vanish when n is a multiple of 4. This accounts for the powers of σ and the term $f_{\delta^+,n}(\tau_{j_\delta})$ when n is odd and $f_n(\tau_1)f_n(\tau_2)$ when n is a multiple of 4. We use Lemma 2(iii) to transform the powers of $\theta_1'[\delta_0](\tau_{i_\delta})$ into powers of $\eta^3(\tau_{i_\delta})$. To deal with the terms where $\theta_1\left[\begin{smallmatrix}a_1\\b_1\end{smallmatrix}\right](\tau_1)\theta_1\left[\begin{smallmatrix}a_2\\b_2\end{smallmatrix}\right](\tau_2)$ does not vanish, we need the following lemma, whose use will complete the proof of the proposition.

Lemma 6. *Let* $n \geq 3$ *and let* $\delta \in \Phi_1(2) \cup \{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\}$. *Then there is a constant* $c_{n,\delta} \neq 0$ *such that* $\prod_{\alpha\in\Phi_1(n)} \theta_1[\delta+\alpha](\tau) = c_{n,\delta}\eta(\tau)^{\phi_1(n)}$.

For a proof of similar results, see [KL] or [Gr1].

Remarks. 1) If $n \geq 3$ and $\alpha \in \Phi_1(n)$, by Lemma 2, $\theta_1'[\alpha](\tau)$ cannot vanish identically. Hence $f_{\delta^+,n}$ cannot vanish identically when $n \geq 3$ and so Proposition 5 gives the exact order of vanishing of $F_{\delta,n}$ along $\mathcal{D}^*(\mathbb{C})$.

2) It is shown in [Gr1] and [Go2] that $F_{\delta,3}$ and F_4 are respectively constant multiples of Δ_2^4 and Δ_2^{12}, and hence do not vanish on $\mathcal{M}_2^*(\mathbb{C})$. By Proposition 3, this means that Θ_W cannot contain any points of J of order 3 or 4. This can of course also be verified directly using the Riemann-Roch theorem, see for example [BG1]

Now suppose $n \geq 5$. Then the quotient $r_n = \frac{G_n^{10}}{\Delta_2^{3\phi_2(n)}}$ is modular of weight 0 with respect to Γ_2, and using Propositions 4 and 5 and (1) one sees that it actually has a pole along $\mathcal{D}$ and thus cannot be a constant. Since a non-constant modular function necessarily vanishes somewhere, and cannot vanish only at infinity when the genus is ≥ 2, and r_n has a pole along $\mathcal{D}^*$, we deduce that G_n vanishes somewhere on $\mathcal{M}_2^*(\mathbb{C})$. From this one can deduce that for all $\alpha \in \Phi_2(n)$, $\theta_2[\alpha](\tau)$ vanishes somewhere on $\mathcal{M}_2^*(\mathbb{C})$.

Put more prosaically, this shows that if $n \geq 5$, then there actually exist genus two curves X with Weierstrass points W such that Θ_W contains points of $J[n]^*$. In fact, the family of such curves sweeps out a finite union of surfaces in the three-dimensional variety $\mathcal{M}_2$.

§5. The genus 1 case

We now return our attention to questions about elliptic curves that are suggested by the behavior of the functions $F_{\delta,n}$ near $\mathcal{H}^1 \times \mathcal{H}^1$.

Since η does not vanish on $\mathcal{H}^1$, Proposition 5 suggests that one study the zero set of the functions $f_{\delta^+,n}$.

If $n \in \mathbb{N}^*$, we write $g_n = \prod_{\delta \in \Phi_1(2)^+} f_{\delta,n}$, and recall we set $f_n = f_{\delta_0,n}$. If f, g are two non-zero meromorphic functions on $\mathcal{H}^1$, we write $f \sim g$ to mean that $\frac{f}{g}$ is constant.

Note that the f_n's and g_n's are related: when n is odd $f_{2n} \sim g_n$, $g_{2n} \sim {f_n}^3 {g_n}^2$, and when n is even, $g_{2n} \sim f_{2n}^3$, so one need only calculate f_n when n is odd or a multiple of 4, and g_n for n odd.

By Jacobi's derivative formula (Lemma 2 (*iii*)), one knows that $f_1 \sim \eta^3$ and it also follows from Lemma 2 that $g_1 = 0$. On the other hand, it was shown in [Gr1] that

$$g_3 \sim \Delta^3, \;\; f_4 \sim \eta^{36},$$

which can be regarded as generalizations of Jacobi's derivative formula.

More generally we are led to ask the shape of further generalizations: What are the zeros of f_n and g_n?

Much of the remainder of the paper is devoted to showing that a solution to this problem is related to a “Manin-Mumford problem”of a certain type. It will follow from Proposition 8 that for any given $\tau \in \mathcal{H}^1$, there are only finitely many n such that $f_n(\tau)$ or $g_n(\tau)$ vanish.

As for individual f_n and g_n, it is shown in [Gr1] that if f is f_3^3 or f_n or g_n for any $n > 4$, then f is a power of η^2 times the square of a modular form on Γ_1. However, computations with the q-expansions of these f shows that none (except $f_6 \sim g_3$) is a constant times a power of η.

For example, letting j denote Klein's j-function $j = q^{-1} + 744 + \cdots$, a computer calculation shows that ${f_3}^3\Delta(\tau)^{-3} \sim j^2$, $f_5\Delta(\tau)^{-3} \sim (-20480 + 243j)^2$, and that $g_5(\tau)\Delta(\tau)^{-9} \sim (19465109 + 248832j)^2$.

These are not as pretty as the expressions for f_1, g_3, and f_4, but are in some sense the generalizations of Jacobi's derivative formula that nature provides.

For generalizations of Jacobi's derivative formula in different directions, see [I2], [Gr1], and [Coo].

§6. Singular torsion

In the previous section we were studying the zero set of the $\theta_1'[\alpha](\tau)$ as a function of τ. In this section we will fix τ and study the zeros of $\theta_1'[\alpha](\tau)$ as a function of α. Since we shall only be concerned with genus one theta constants, we simply write $\theta[\alpha](\tau)$ for $\theta_1[\alpha](\tau)$ and $\theta'[\alpha](\tau)$ for $\theta_1'[\alpha](\tau)$ from now on.

If $\tau \in \mathcal{H}^1$ we write Λ_τ for the lattice $\{m\tau + n \mid m, n \in \mathbb{Z}\}$ and O for the origin of the complex torus $\mathbb{C}/\Lambda_\tau$. Recall that $\delta_0 = \left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right]$.

Lemma 7. *Let $n \geq 2$. Let $\tau \in \mathcal{H}^1$, and let $\alpha = \left[\begin{smallmatrix}a\\b\end{smallmatrix}\right] \in \Phi_1(n)$. Let P_α be the point $a\tau + b \pmod{\Lambda_\tau}$ on $\mathbb{C}/\Lambda_\tau$, and let e_α be an elliptic function on $\mathbb{C}/\Lambda_\tau$ with divisor $n(P_\alpha - O)$. Let*

$$e_\alpha(z) = \frac{a}{z^n} + \frac{b}{z^{n-1}} + O\Big(\frac{1}{z^{n-2}}\Big)$$

be the Laurent expansion of e_α at O. Then $\theta'[\delta_0 + \alpha](\tau)$ vanishes if and only if $b = 0$.

To see this, one notes that, up to a multiplicative constant,

$$e_\alpha(z) = \Big(\frac{\theta[\delta_0 + \alpha](z, \tau)}{\theta[\delta_0](z, \tau)}\Big)^n,$$

which follows from standard transformation formulas for theta functions. Since $\theta[\delta_0](z, \tau)$ is an odd function of z, it is equal to $\theta'[\delta_0](\tau)z + O(z^3)$. Since $\theta[\delta_0+\alpha](z, \tau) = \theta[\delta_0+\alpha](\tau)+\theta'[\delta_0+\alpha](\tau)z+O(z^2)$, and $\theta'[\delta_0](\tau)\theta[\delta_0+\alpha](\tau) \neq 0$ by Lemma 2, the result follows at once.

The interest of Lemma 7 is that it suggests the following purely algebraic definition. For the rest of the paper, let k be an algebraically closed field of characteristic not 2.

Definition. Let E be an elliptic curve over k. We say that $P \in E[n]$ is a *singular n-torsion point* if, letting t be an odd uniformizer at the origin O of E, a function $e_P \in k(E)$ with divisor $n(P - O)$ has an expansion in powers of t of the form

$$e_P = \frac{a}{t^n} + O\Big(\frac{1}{t^{n-2}}\Big)$$

i.e., with no term in $\frac{1}{t^{n-1}}$.

Here the uniformizer t is said to be odd if $[-1]^*t = -t$, where $[-1]$ denotes multiplication by -1 on E. It is clear that the definition is independent of the choice of t.

If $P \in E[n]^*$ and P is a singular n-torsion point, we call P a *singular* torsion point, and write E_{sing} for the subset of singular torsion points of E_{tors}. If k has characteristic 0, it follows easily that for any $m \geq 1$ (and if the characteristic of k is p, for any m prime to p) that $P \in E[n]$ is a singular n-torsion point if and only if P is a singular mn-torsion point. So if the characteristic of k is 0, E_{sing} is just the set of singular n-torsion points for all n. But if the characteristic of k is p, for any $P \in E[n]$, P is automatically a singular np-torsion point, so we need the more restrictive definition of E_{sing} above, and let E'_{sing} denote the singular n-torsion points of E of order prime to p. Note that always $E[2]^* \subseteq E_{\mathrm{sing}}$.

Now the Manin-Mumford conjecture and Proposition 3 suggest

Proposition 8. *Let E be an elliptic curve over a field of characteristic* 0. *Then E_{sing} is a finite set.*

As we shall see in a moment, Proposition 8 is in fact a special case of Theorem 2 of [H].

§7. The generalized Jacobian case

In this section we want to explain how E_{sing} may be viewed as the set of torsion points lying on a curve contained in a certain extension of E by the additive group $\mathbb{G}_a$. Let O denote the origin of E and let G denote the generalized Jacobian of E with modulus $2O$ as defined in [S]. Explicitly, let $\mathrm{Div}_0(E)$ be the group of degree zero divisors on E, $\mathrm{Pr}(E)$ the subgroup of principal divisors, $\mathrm{Div}_0(E)_O$ the subgroup of divisors in $\mathrm{Div}_0(E)$ whose support does not contain O and $\mathrm{Pr}(E)_O = \mathrm{Pr}(E) \cap \mathrm{Div}_0(E)_O$. The inclusion $\mathrm{Div}_0(E)_O \to \mathrm{Div}_0(E)$ induces an isomorphism $\mathrm{Div}_0(E)_O/\mathrm{Pr}(E)_O \simeq \mathrm{Div}_0(E)/\mathrm{Pr}(E)$ and $E(k)$ is isomorphic to this latter group by $P \mapsto P - O \pmod{\mathrm{Pr}(E)}$ as usual. Let $\mathrm{Pr}_{2O}(E)$ denote the subgroup of $\mathrm{Pr}(E)$ consisting of divisors of functions f such that $f-1$ has a zero of order at least 2 at O. Then, with the above identifications, the exact sequence $0 \to \mathbb{G}_a(k) \to G(k) \to E(k) \to 0$ is isomorphic to the exact sequence

(4)
$$0 \to \mathrm{Pr}(E)_O/\mathrm{Pr}_{2O}(E) \to \mathrm{Div}_0(E)_O/\mathrm{Pr}_{2O}(E) \to \mathrm{Div}_0(E)_O/\mathrm{Pr}(E)_O \to 0.$$

Indeed, picking $t \in k(E)$, an odd uniformizer at O, the isomorphism $\mathbb{G}_a(k) \simeq \mathrm{Pr}(E)_O/\mathrm{Pr}_{2O}(E)$ is given by $a \mapsto (1 + at) \pmod{\mathrm{Pr}_{2O}(E)}$. Let $p : G \to E$ be the projection. Then p has a section $s : E \to G$ defined by $s(P) = P - O + (t) \pmod{\mathrm{Pr}_{2O}(E)}$. It is easy to see that s does not depend on the choice of t.

Proposition 9. *Let G'_{tors} denote the elements of G_{tors} of order not divisible by the characteristic of k. Then the projection p induces a bijection from $s(E)(k) \cap G'_{\mathrm{tors}}$ onto E'_{sing}.*

To see this, suppose $P \in E(k)$, $P \neq O$ is such that $s(P) \in G[n]^*$. Then $n(P - O + (t))$ is the divisor of a function $f \in \mathrm{Pr}_{2O}(E)$, so we can write $f = 1 + ut^2$ with u regular at O. But then $n(P - O)$ is the divisor of $\frac{f}{t^n}$, and $\frac{f}{t^n} = \frac{1}{t^n} + O\big(\frac{1}{t^{n-2}}\big)$ at O, so that $P \in E[n]$, and if n is not a multiple of the characteristic of k, then $P \in E'_{\mathrm{sing}}$. Conversely, suppose $P \in E[n]^*$ is a singular torsion point, with n not divisible by the characteristic of k. Then there is a function $e_P \in k(E)$ with divisor $n(P - O)$ such that $e_P = \frac{a}{t^n} + O\big(\frac{1}{t^{n-2}}\big)$, $a \neq 0$. Thus $t^n e_P / e_P(O) = 1 + ut^2$ for u regular at O, and has divisor $n(P - O + (t))$. Hence $s(P) \in G[n]$. Putting these together, we see that p induces a bijection from $s(E)(k) \cap G[n]^*$ onto $E_{\mathrm{sing}} \cap E[n]^*$ for all n not divisible by the characteristic of k, giving us the result.

We can now apply Theorem 2 of [H] to show that $s(E) \cap G_{\mathrm{tors}}$ is finite when the characteristic of k is 0, and thus prove Proposition 8. To do this, we have to check that $s(E)$ does not contain a translate of a nontrivial algebraic subgroup H of G by some point $u \in G(k)$. Assume that $T_u H \subseteq s(E)$, where T_u is the translation-by-u map on G. Then H and $s(E)$ are of dimension 1 and $s(E)$ is irreducible, so $s(E) = T_u H$. Therefore H is an elliptic curve isomorphic to E, and $T_{-u} \circ s : E \to H$ is a surjective morphism between abelian varieties of the same dimension, and hence there exists $v \in G(k)$ such that $T_v \circ s : E \to H$ is an isomorphism. Thus $p \circ T_v \circ s$ is an automorphism of E. Since $p \circ T_v \circ s = T_{p(v)} \circ p \circ s = T_{p(v)}$, this implies $T_{p(v)}$ is the identity, and $v \in \ker p \simeq \mathbb{G}_a(k)$. But on the other hand, $s(E[2]^*) \subseteq G_{\mathrm{tors}}$ by Proposition 9, and, since $v = T_v \circ s(P) - s(P)$ for all $P \in E(k)$ and $T_v \circ s$ is an isomorphism, we deduce that $v \in G_{\mathrm{tors}}$. Hence $v \in \mathbb{G}_a(k) \cap G_{\mathrm{tors}} = \{0\}$ and s is an isomorphism. This implies that G is the trivial extension of E by $\mathbb{G}_a$ which violates the results in [S, VII], and completes the proof of Proposition 8.

We close this section by describing G and s explicitly in the case $k = \mathbb{C}$ when E has a Weierstrass model $y^2 = x^3 + Ax + B$ with origin O. Let $\omega_1 = \frac{dx}{2y}$, $\omega_2 = \frac{x\,dx}{2y}$ be the standard basis of the differential forms on E with pole of order at most 2 at O and holomorphic elsewhere, and let $L = \{\int_\gamma \omega_1 \mid \gamma \in H_1(E(\mathbb{C}), \mathbb{Z})\} = \{\int_\gamma \omega_1 \mid \gamma \in H_1(E(\mathbb{C}) \setminus \{O\}, \mathbb{Z})\}$ be the period lattice of ω_1. We identify $E(\mathbb{C})$ with $\mathbb{C}/L$ via $P \mapsto \int_O^P \omega_1 \pmod{L}$ as usual. If $P \in E(\mathbb{C})$, we denote by z_P any representative of $\int_O^P \omega_1$ in $\mathbb{C}$. Let $\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in L \setminus \{0\}} \left(\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2} \right)$ be the usual Weierstrass elliptic function and let ζ be the unique odd meromorphic function satisfying $\wp = \zeta'$. Recall that there is a unique homomorphism $\eta : L \to \mathbb{C}$ such that $\eta(\lambda) = \zeta(z+\lambda) - \zeta(z)$ for all $z \in \mathbb{C} \setminus L$ and all $\lambda \in L$. Extend η by $\mathbb{R}$-linearity to a map $\mathbb{C} \to \mathbb{C}$, also denoted by η. Using the parameterization $(\mathbb{C} \setminus L)/L \to E(\mathbb{C}) \setminus \{O\}$ given by $(x, y) = (\wp(z), \frac{1}{2}\wp'(z))$, it follows that if $\gamma \in H_1(E(\mathbb{C}) \setminus \{O\}, \mathbb{Z})$ and $\lambda \in L$, then $\int_\gamma \omega_1 = \lambda$ implies $\int_\gamma \omega_2 = \eta(\lambda)$. Furthermore, $\int_Q^P \omega_2 = \zeta(z_P) - \zeta(z_Q) \pmod{\{\eta(\lambda) \mid \lambda \in L\}}$. Hence if we write

$$M = \left\{ \begin{pmatrix} \lambda \\ \eta(\lambda) \end{pmatrix} \;\middle|\; \lambda \in \Lambda \right\},$$

then by [S, V, §19], the exponential map of G can be identified with the canonical map $\mathbb{C}^2 \to \mathbb{C}^2/M$. Furthermore, if we define $u : \mathrm{Div}_0(E)_O \to \mathbb{C}^2/M$ by

$$u\Big(\sum_i P_i - \sum_i Q_i\Big) = \begin{pmatrix} \sum z_{P_i} - \sum z_{Q_i} \\ \sum \zeta(z_{P_i}) - \sum \zeta(z_{Q_i}) \end{pmatrix} \pmod{M}, \tag{5}$$

then u is surjective and $\ker u = \mathrm{Pr}_{2O}(E)$. Thus the exact sequence (4) becomes

$$0 \to \mathbb{C} \to \mathbb{C}^2/M \to \mathbb{C}/L \to 0.$$

Here the map $\mathbb{C}^2/M \to \mathbb{C}/L$ is induced by the projection of $\mathbb{C}^2$ onto its first factor. The map $\mathbb{C} \to \mathbb{C}^2/M$ depends on the choice of the function t, but is necessarily of the form $a \mapsto \binom{0}{va} \pmod{M}$ for some $v \in \mathbb{C}^*$.

Proposition 10. (*i*) *The section $s : E \to G$ induces the map $(\mathbb{C} \setminus L)/L \to \mathbb{C}^2/M$ given by $z \pmod{L} \mapsto \binom{z}{\zeta(z)} \pmod{M}$.*

(*ii*) *Let $z \in \mathbb{C}$. Then $z \pmod{L}$ belongs to E_{sing} if and only if $z \in \mathbb{Q}L$, $z \notin L$ and $\zeta(z) = \eta(z)$.*

Proof. (*i*) If W_1, W_2, W_3 are the three points of order 2 on E, we can take t to be a function with divisor $O + W_1 - W_2 - W_3$. Then if $P \in E(\mathbb{C})$ with $P \neq O$, $s(P)$ is represented by $u(P + W_1 - W_2 - W_3)$. But $z_{W_1} - z_{W_2} - z_{W_3} = \lambda$ is an element of L and the addition formula for ζ shows that $\zeta(z_{W_i}) = \frac{1}{2}\eta(2z_{W_i})$, and hence that $\zeta(z_{W_1}) - \zeta(z_{W_2}) - \zeta(z_{W_3}) = \eta(\lambda)$. The result follows using (5).

(*ii*) Suppose $z \pmod{L} \in E_{\mathrm{sing}}$. Then $z \in \mathbb{Q}L$ and $z \notin L$ since z must represent a non-zero torsion point. Then by (*i*) and Proposition 9, $s(P) = \binom{z}{\zeta(z)} \in \mathbb{Q}M/M$. If $n \in \mathbb{N}^*$ is such that $nz \in L$, then $n\zeta(z) = \eta(nz) = n\eta(z)$ so $\zeta(z) = \eta(z)$. The converse is similar.

Remarks. (*a*) The discussion leading up to Proposition 10 shows that our extension of E by $\mathbb{G}_a$ is just one of a family of such extensions with applications to transcendental number theory (see for example [W], page 64, and [B], [Coh1], [Coh2]).

(*b*) Part (*ii*) shows that E_{sing} is just the set of torsion points on which the real-analytic L-periodic function $z \mapsto \zeta(z) - \eta(z)$, $z \notin L$, vanishes (see [Coh1],[Coh2]). This can of course be proved directly using classical relations between Weierstrass and theta functions. Furthermore, $\zeta(z)-\eta(z)$ is just the non-holomorphic weight one Eisenstein series $G_1(z)$ obtained by analytic continuation to $s=1$ of $\sum_{\lambda \in L} \frac{(\bar{z}+\bar{\lambda})}{|z+\lambda|^{2s}}$ (see for example [GS], §1). So E_{sing} is just the set of points of $(\mathbb{Q}L \setminus L)/L$ on which $G_1(z)$ vanishes. A natural question is thus whether G_1 has only finitely many zeros $\pmod L$ (not necessarily in $(\mathbb{Q}L \setminus L)/L$). A positive answer would obviously imply Proposition 8.

§8. Effective results for singular torsion

Recall that an essential step in Coleman's proof [Col] of the Manin-Mumford conjecture is the following:

Theorem. *Let X be a smooth proper curve of genus $g \geq 2$ defined over a number field K. Suppose X is embedded in its Jacobian J using an Albanese embedding with base point in $X(K)$. Then any torsion point of J that lies on X is defined over an extension of K that is unramified except possibly at places above rational primes $p \leq 2g$, places that are ramified in $K/\mathbb{Q}$ or where X has bad reduction.*

Inspired by this, we have proved

Theorem 11. *Let E be an elliptic curve over a number field K and let $P \in E_{\mathrm{sing}}$. Then the order of P is divisible only by 2, 3, and primes p such that every prime π of K above p is either ramified or a place of bad reduction of E.*

A proof of this, as well as of the next Proposition, will appear in [BG2]. For E an elliptic curve we denote by $E[n^\infty]$ the subgroup of E_{tors} of points of order dividing a power of n.

Proposition 12. *Let k_0 be a finite field of characteristic $p \neq 2$, let $n \in \mathbb{N}^*$ be prime to p, let E be an elliptic curve over k_0 and let $L = \prod_{\ell | n} \ell'$, where $\ell' = \ell$ if ℓ is odd and $\ell' = 4$ if $\ell = 2$. Let $k_1 = k_0(E[L])$ and let M be the smallest integer, divisible only by the primes dividing n and such that*

$E(k_1) \cap E[n^\infty] \subseteq E[M]$ (or $E(k_2) \cap E[n^\infty] \subseteq E[M]$ when n is even, k_2 being the quadratic extension of k_1). Then $E_{\rm sing} \cap E[n^\infty] \subseteq E[M]$.

This leads, in principle, to a method of calculating $E_{\rm sing}$ in numerical examples. Indeed, Theorem 11 gives an explicit $n \in \mathbb{N}^*$ such that $E_{\rm sing} \subseteq E[n^\infty]$ and then Proposition 12 gives a method of determining $E_{\rm sing}$ by reduction at a place not excluded by Theorem 11.

Acknowledgements. We would like to thank Patrice Philippon, Bjorn Poonen and David Roberts for suggesting (in different ways) the relationship between singular torsion and group extensions discussed in §7, as well as Brett Simon for the calculations summarized in §5.

References

[B] A. Baker, *On the quasi-periods of the Weierstrass ζ-function*, Nachr. Akad. Wiss. Göttingen II: Math. Phys. Kl **16**, 145–157.

[BG1] J. Boxall, D. Grant, *Examples of Torsion points on Genus Two Curves*, Trans. Amer. Math. Soc. **352** (2000), no. 10, 4533–4555.

[BG2] J. Boxall, D. Grant, *Effective determination of singular torsion on elliptic curves*, in preparation.

[Coh1] S. P. Cohen, *Heights of torsion points on commutative group varieties*, Proc. London Math. Soc. **52** (1986), 427–444.

[Coh2] P. Cohen, *Heights of torsion points on commutative group varieties II*, Proc. London Math. Soc. **62** (1991), 99–120.

[Col] R. F. Coleman, *Ramified torsion points on curves*, Duke Math. J. **54** (1987), 615–640.

[Coo] Gwynneth Coogan, *Two generalizations of Jacobi's derivative formula*, submitted to Ann. Inst. Fourier.

[GS] C. Goldstein, N. Schappacher, *Séries d'Eisenstein et fonctions L de courbes elliptiques à multiplication complexe*, J. reine u. angewandte Math. **327** (1981), 184–218.

[Go1] E. Z. Goren, *Units in abelian extensions of cm fields of degree four*, Thesis, Hebrew Univerity of Jerusalem (1996).

[Go2] E. Z. Goren, *A note on vanishing properties of certain theta constants of genus 2*, Landau Center, Preprint No. 15 (1995/6) 11 pages.

[Gr1] D. Grant, *Some product formulas for genus 2 theta functions*, submitted to Acta Arithmetica.

[Gr2] D. Grant, *A generalization of Jacobi's derivative formula to dimension two*, J. Reine Angew. Math. **392** (1988), 125–136.

[H] M. Hindry, *Autour d'une conjecture de Serge Lang*, Inventiones Math. **94** (1988), 575–603.

[I1] J. Igusa, *Arithmetic variety of moduli for genus two*, Annals of Math. **72** (1960), 612–649.

[I2] J. Igusa, *On Jacobi's derivative formula and its generalizations*, Amer. J. Math. **102** (1980), 409–446.

[KL] D. Kubert, S. Lang, *Modular Units*, Springer Verlag, 1981.

[L] P. Lockhart, *On the discriminant of a hyperelliptic curve*, Trans. Amer. Math. Soc. **342** (1994), 729–752.
[M] D. Mumford, *Tata Lectures on Theta I, II*, Birkhäuser, 1982, 1984.
[R] M. Raynaud, *Courbes sur une variété abélienne et points de torsion*, Invent. Math. **71** (1983), 207–233.
[S] J.-P. Serre, *Groupes algébriques et corps de classes*, Hermann, 1959.
[W] M. Waldschmidt, *Nombres transcendants et groupes algébriques,* Astérisque, vol. 69–70, 1979.

Département de mathématiques et de mécanique, CNRS Upresa 6081, Université de Caen, Boulevard maréchal Juin, B.P.5186, 14032 Caen cedex, France.
E-mail address: boxall@math.unicaen.fr

Department of Mathematics, University of Colorado at Boulder, Boulder, Colorado 80309-0395 USA.
E-mail address: grant@boulder.colorado.edu