ACTA ARITHMETICA
Online First version

A generalization of Jacobi's derivative formula to dimension two, II

by

DAVID GRANT (Boulder, CO)

Introduction. One of the central results for elliptic theta functions is Jacobi's derivative formula. Recall that for $t \in \mathfrak{h} = \{x + iy \mid y > 0\}$ and $\alpha, \beta \in \frac{1}{2}\mathbb{Z}$, we define the *theta function* in one variable $w \in \mathbb{C}$ with theta characteristic $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$ by

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(w,t) = \sum_{n\in\mathbb{Z}} e^{\pi i(n+\alpha)^2 t + 2\pi i(n+\alpha)(w+\beta)}. \tag{1}$$

Jacobi's derivative formula states that

$$\frac{d\vartheta\left[\begin{smallmatrix}1/2\\ 1/2\end{smallmatrix}\right](0,t)}{dw} = -\pi\vartheta\begin{bmatrix}0\\ 0\end{bmatrix}(0,t)\vartheta\begin{bmatrix}1/2\\ 0\end{bmatrix}(0,t)\vartheta\begin{bmatrix}0\\ 1/2\end{bmatrix}(0,t). \tag{2}$$

Perhaps the easiest proof of (2) comes from noting that the transformation formula for theta functions shows that the eighth powers of both sides of (2) are cusp forms of weight 12 on $\mathrm{SL}_2(\mathbb{Z})$, so their ratio is a constant. The constant is determined by the Fourier expansions given in (1). The same argument shows that both sides of (2) are equal to $-2\pi\eta(t)^3$, where for $q = e^{2\pi i t}$,

$$\eta(t) = q^{1/24}\prod_{n\geq 1}(1-q^n)$$

is Dedekind's eta function.

The first dimension-2 analogue of (2) was stated by Rosenhain [R]. Let $\mathfrak{h}_2$ denote the Siegel upper half-space of 2×2 symmetric complex matrices with positive-definite imaginary part. Writing $\mathbb{C}^2$ and $\mathbb{Z}^2$ as column vectors, and letting ^{t} denote taking the transpose, for any $a, b \in \frac{1}{2}\mathbb{Z}^2$ and $\tau \in \mathfrak{h}_2$ we

2010 *Mathematics Subject Classification*: Primary 14K25; Secondary 11F46.
Key words and phrases: theta functions, Siegel modular forms.
Received 6 December 2018; revised 9 January 2019.
Published online *.

let

$$\theta\begin{bmatrix}a\\b\end{bmatrix}(z,\tau) = \sum_{n\in\mathbb{Z}^2} e^{\pi i\,{}^t(n+a)\tau(n+a)+2\pi i\,{}^t(n+a)(z+b)} \tag{3}$$

denote the theta function in two variables $z = \binom{z_1}{z_2} \in \mathbb{C}^2$ with theta characteristic $\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]$. We call a theta characteristic *even* or *odd* depending respectively on whether $\theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](z,\tau)$ is an even or odd function of z. Since $\theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](-z,\tau) = \theta\left[\begin{smallmatrix}-a\\-b\end{smallmatrix}\right](z,\tau) = (-1)^{4\,{}^tab}\theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](z,\tau)$, whether the theta characteristic $\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]$ is even or odd is determined by the parity of $4\,{}^tab$.

Rosenhain's formula states that if δ_i, $i = 1, 2$, are odd theta characteristics that are distinct modulo 1, then there are even theta characteristics ϵ_k, $1 \le k \le 4$, depending on the δ_i, such that

$$\det_{1\le i,j\le 2}\left[\frac{\partial\theta[\delta_i](0,\tau)}{\partial z_j}\right] = \pm\pi^2 \prod_{k=1}^{4} \theta[\epsilon_k](0,\tau). \tag{4}$$

Proofs of (4) were supplied by Weber and Thomae; the latter for any $g \ge 1$ expressed the jacobian at 0 of g odd theta functions in g variables evaluated at the period matrix τ of a genus g hyperelliptic curve as a constant times a product of thetanullwerte [T]. See [E], [I2], and [GM2] for additional history and references to what is known for such jacobians evaluated at all points τ in the Siegel upper half-space of degree g.

A different kind of generalization for $g = 2$, involving a single derivative of a single theta function, was given in part I of this paper [G1]. The goal of this part II is to provide a new generalization of (2) to theta functions in two variables—one that has geometric meaning and arithmetic applications. If there is any flaw in the beautiful and venerable formula (4), it is that the eighth power of each side is a Siegel modular form of level 2, and not of level 1. This is rectified in Theorem 1 below.

Recall that modulo 1, there are ten even theta characteristics for theta functions in two variables. In the next section (see (13)) we will choose a set $\mathcal{E}$ of representatives for these characteristics and define

$$\varDelta(\tau) = \prod_{\epsilon\in\mathcal{E}} \theta[\epsilon](0,\tau).$$

Then $\varDelta(\tau)$ is up to a multiplicative constant the unique Siegel modular form (with character) of level 1 and weight 5 (see Section 1 and [K, Section 9.2]). For any odd theta characteristic δ, define

$$X[\delta](z,\tau) = \theta[\delta](z,\tau)^3 \det_{1\le i,j\le 2}\left[\frac{\partial^2 \log\theta[\delta](z,\tau)}{\partial z_i\partial z_j}\right],$$

which a computation with partial derivatives shows is analytic.

THEOREM 1. *For any odd theta characteristic* $\delta = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$, $a = \left(\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right)$, $b = \left(\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right)$,

$$R[\delta](\tau) := \det\begin{pmatrix} \frac{\partial\theta[\delta](0,\tau)}{\partial z_1} & \frac{\partial\theta[\delta](0,\tau)}{\partial z_2} \\ \frac{\partial X[\delta](0,\tau)}{\partial z_1} & \frac{\partial X[\delta](0,\tau)}{\partial z_2} \end{pmatrix} = (-1)^{2(a_1+a_2+b_1+b_2)+4a_1b_1} 2\pi^6 \Delta(\tau).$$

Expanding the determinant gives

$$R[\delta](\tau) = \sum_{i=0}^{3} \binom{3}{i} \frac{\partial^3\theta[\delta](0,\tau)}{\partial z_1^{3-i}\partial z_2^{i}} \left(-\frac{\partial\theta[\delta](0,\tau)}{\partial z_1}\right)^{i} \left(\frac{\partial\theta[\delta](0,\tau)}{\partial z_2}\right)^{3-i}.$$

So if we define the differential operator

$$\mathcal{D}_\delta = \frac{\partial\theta[\delta](0,\tau)}{\partial z_2}\frac{\partial}{\partial z_1} - \frac{\partial\theta[\delta](0,\tau)}{\partial z_1}\frac{\partial}{\partial z_2},$$

then Theorem 1 is equivalent to

$$\mathcal{D}_\delta^3(\theta[\delta](0,\tau)) = (-1)^{2(a_1+a_2+b_1+b_2)+4a_1b_1} 2\pi^6 \Delta(\tau), \tag{5}$$

and we will prove it in this form. Note that $R[\delta](\tau)$ is independent of the choice of δ modulo 1.

Theorem 1 is still not the best level-1 generalization of Rosenhain's formula, which expresses the wedge of two vector-valued modular forms in terms of thetanullwerte. For although the gradient at $z = 0$ of $\theta[\delta](z,\tau)$ is a vector-valued modular form, the gradient at $z = 0$ of $X[\delta](z,\tau)$ is not quite so. We remedy this as follows:

THEOREM 2. *For any odd theta characteristic* $\delta = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$, $a = \left(\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right)$, $b = \left(\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right)$, *let*

$$Y[\delta](z,\tau) = X[\delta](z,\tau) + \frac{1}{10}\frac{\mathcal{D}_\delta^5(\theta[\delta](0,\tau))}{\mathcal{D}_\delta^3(\theta[\delta](0,\tau))}\theta[\delta](z,\tau),$$

which is analytic. Then the gradient of $Y[\delta](z,\tau)$ *is a vector-valued modular form, and*

$$\det\begin{pmatrix} \frac{\partial\theta[\delta](0,\tau)}{\partial z_1} & \frac{\partial\theta[\delta](0,\tau)}{\partial z_2} \\ \frac{\partial Y[\delta](0,\tau)}{\partial z_1} & \frac{\partial Y[\delta](0,\tau)}{\partial z_2} \end{pmatrix} = (-1)^{2(a_1+a_2+b_1+b_2)+4a_1b_1} 2\pi^6 \Delta(\tau).$$

We will give explicit expressions for what type of vector-valued modular forms the gradients of $\theta[\delta](z,\tau)$ and $Y[\delta](z,\tau)$ are in Sections 2 and 3. We will also show that $\mathcal{D}_\delta^5(\theta[\delta](0,\tau))/\mathcal{D}_\delta^3(\theta[\delta](0,\tau))$ is a Siegel quasimodular form.

Since we need it in the sequel [G3] and it takes little additional work, in Theorem 3 of Section 4 we will also give a quick proof of Rosenhain's formula that provides a compact explanation for the pattern of signs that appear in the formula (cf. [FM] and see also [F, §6.2]).

The structure of the proof of Theorem 1 is relatively straightforward. After reviewing the requisite background on theta functions and modular forms in Section 1, we will show in Section 2 that $R[\delta](\tau)$ is a Siegel modular form with character of weight 5 and level 2 which vanishes wherever $\Delta(\tau)$ does. It is well known that $\Delta(\tau)$ has a simple zero along its zero locus, hence $R[\delta](\tau)/\Delta(\tau)$ is a modular form of weight 0, so a constant. The constant is determined by comparing the limiting behavior of $R[\delta](\tau)$ and $\Delta(\tau)$ as τ approaches the locus of diagonal matrices in $\mathfrak{h}_2$, and then using (2).

More interesting than the proof of Theorem 1 is the geometric interpretation of the formula, which is why it was considered in the first place.

Let us recall the setup of the precursor [G1]. Let $b_i \in \mathbb{C}$, $1 \le i \le 5$, be such that

$$f(x) = x^5 + b_1x^4 + b_2x^3 + b_3x^2 + b_4x + b_5$$

has no multiple roots, so

$$y^2 = f(x) \tag{6}$$

defines an affine model of a smooth, projective, complex curve C of genus 2, which has a single point ∞ at infinity on the normalization of this model. Let $\{A_1, A_2, B_1, B_2\}$ be a chosen symplectic basis for $H_1(C, \mathbb{Z})$ and let $\zeta_1 = \frac{dx}{y}$, $\zeta_2 = \frac{xdx}{y}$ be a basis for the holomorphic differentials on C. Set $\omega = [\int_{A_j} \omega_i]_{1\le i,j\le 2}$, $\omega' = [\int_{B_j} \omega_i]_{1\le i,j\le 2}$, and $\tau = \omega^{-1}\omega' \in \mathfrak{h}_2$. Let $L = \mathbb{Z}^2 \oplus \tau\mathbb{Z}^2$, $A_\tau = \mathbb{C}^2/L$, and Θ be the image of C in A_τ under the Abel–Jacobi map $\phi(P) := \int_\infty^P \omega^{-1}\binom{\zeta_1}{\zeta_2} \mod L$. Then Riemann's Vanishing Theorem says there is an odd theta characteristic δ such that $\theta[\delta](z)$ vanishes precisely on the pullback of Θ to $\mathbb{C}^2$.

The main theorem of [G1] is

Theorem 0. *For $i = 1, 2$, let $c_i[\delta](\tau) = \left(\frac{d}{dz_i}\theta[\delta](\omega^{-1}z, \tau)\right)\Big|_{z=0}$. Then*

$$c_1[\delta](\tau)^8 = \pm 16\pi^{12}\det(\omega)^{-6}\Delta(\tau)^2, \qquad c_2[\delta](\tau) = 0.$$

Given these choices, attached to C is a sigma function $\sigma[\delta](z, \tau)$, which differs from $\theta[\delta](\omega^{-1}z, \tau)$ by a trivial theta function involving the quasi-periods of C (see e.g. [G1]). The content of Theorem 0 is that $\sigma[\delta](z, \tau)$ has a Taylor expansion in z of the form

$$\sigma[\delta](z, \tau) = c_1(\tau)\left(z_1 + \frac{1}{24}b_3z_1^3 - \frac{1}{12}z_2^3 + \cdots\right).$$

From this it is just a calculation to show that

$$d_2[\delta](\tau) := \left(\frac{d}{dz_2}X[\delta](\omega^{-1}z, \tau)\right)\Bigg|_{z=0} = \frac{1}{2}c_1[\delta](\tau)^3,$$

and then standard properties of jacobians and hessians (see Lemma 1(i, ii)) show that

$$c_1[\delta](\tau)d_2[\delta](\tau) = \det(\omega)^{-3}R[\delta](\tau).$$

Hence the methods of [G1] provide another proof of Theorem 1 (though only up to a fourth root of unity), and Theorem 1 in turn gives a proof of Theorem 0 with far less work than in [G1]. It also shows that the plus sign always holds in Theorem 0, and

$$c_1[\delta](\tau)^4 = (-1)^{2(a_1+a_2+b_1+b_2)+4a_1b_1} 4\pi^6 \det(\omega)^{-3} \Delta(\tau).$$

The difference is that Theorem 0 is a statement about theta functions attached to marked Riemann surfaces of genus 2, whereas Theorem 1 is really a statement about theta functions on the Siegel upper half-space. Indeed, the function theory of C and A_τ will not appear in the proof of Theorem 1, but just in its motivation, which we now provide.

The divisor of zeroes of $X[\delta](z,\tau)$ on $\mathbb{C}^2$ is L-periodic so descends to a divisor $(X)_0$ on A_τ, and in [G2] it was shown that there are parameters t_1, t_2 at the origin of A_τ whose zeroes are Θ and $(X)_0$. Theorem 1 then expresses the fact that Θ and $(X)_0$ meet transversally at the origin O of A_τ. By the same token, Rosenhain's formula says that for any Weierstrass point W on the affine model (6) of C, the set Θ and its translate by $\phi(W)$ meet transversally at O.

There are arithmetic applications of the above theorems as well. In a subsequent paper [G3] we will use Theorem 2 to recast the analytic theory of jacobians of genus-2 curves in such a way that A_τ has "modular parameters" on its tangent space. This will provide us with functions on A_τ whose expansions in terms of these parameters have coefficients which are Siegel modular forms. This was already used by Alexander in his thesis [A] to count points over finite fields on CM jacobians of curves of genus 2.

There have been a variety of other types of generalizations of Jacobi's derivative formulas to higher dimensions in recent years. Coogan in her thesis [C] generalized Theorem 0 to hyperelliptic curves of genus 3, finding a second derivative of an even theta function related to the discriminant of the curve. In an unpublished note, Onishi generalized this to all hyperelliptic curves. See also [BEL], where the lead terms of the Taylor expansions of sigma functions are described for all hyperelliptic curves. Similar results were obtained in [N] for superelliptic (n,s)-curves.

It was shown in [GM2] that for any g, the nullwerte of a hessian determinant of a ratio of certain theta functions of even characteristic was a modular form of degree g and the authors found its value in terms of theta-nullwerte. They generalized this to higher order theta functions in [GM3]. In [dJ] de Jong defined a function on the theta divisor of any abelian variety in terms of first and second derivatives of theta functions, that in the case of A_τ reduces to $X[\delta](z,\tau)$ restricted to Θ, which is essentially the numerator of the y-coordinate on C embedded via ϕ in A_τ. His Theorem 6.1 was proved using Theorem 0, and one could in turn prove Theorem 1 from his result by looking at the cubic terms in the Taylor expansions of both sides of his formula. Cléry, van der Geer, and Grushevsky have proved analogues of Jacobi's derivative formula for genus-2 vector-valued modular forms [CGG, Section 17]. Sasaki proved an analogue of Rosenhain's formula for hermitian theta functions [S].

1. Preliminaries. We let $\Gamma = \mathrm{Sp}_4(\mathbb{Z})$ denote the integral symplectic group of degree 2, i.e., 4×4 integral matrices $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ such that

$$ {}^t\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \tag{7} $$

where A, B, C, D are 2×2 matrices and I is the 2×2 identity. Hence ${}^tAC = {}^tCA$ and ${}^tBD = {}^tDB$. Taking inverse transposes in (7) shows that Γ is closed under transposition, so we also have

$$ A\,{}^tB = B\,{}^tA, \qquad D\,{}^tC = C\,{}^tD. \tag{8} $$

Elements $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$ act on $\mathfrak{h}_2$ via $\gamma(\tau) = (A\tau + B)(C\tau + D)^{-1}$. For $N > 0$, we let $\Gamma(N)$ denote the subgroup of matrices congruent to the identity modulo N.

Let $\Gamma' \subseteq \Gamma$ be of finite index, and let $m_\gamma(\tau)$ be a *multiplier system* (factors of automorphy) on Γ', i.e., holomorphic functions that satisfy the cocycle relation $m_{\gamma'\gamma}(\tau) = m_{\gamma'}(\gamma(\tau))m_\gamma(\tau)$. Let ρ be any n-dimensional complex representation of $\mathrm{GL}_2(\mathbb{C})$. We then define a vector-valued modular form of type (m, ρ) on Γ' to be a holomorphic map $f : \mathfrak{h}_2 \to \mathbb{C}^n$ such that

$$ f(\gamma(\tau)) = m_\gamma(\tau)\rho(C\tau + D)f(\tau) $$

for all $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma'$ (see, e.g., [FM]).

In particular, when $n = 1$, we get the scalar Siegel modular forms. Specifically, if k is a non-negative integer, and χ a character on Γ', a Siegel modular form of degree 2 on Γ' with character χ and weight k is a holomorphic function f on $\mathfrak{h}_2$ satisfying

$$ f(\gamma(\tau)) = \chi(\gamma)\det(C\tau + D)^k f(\tau) $$

for any $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma'$. If N is minimal such that $\Gamma(N) \subseteq \Gamma'$, we say f has *level* N. The only modular forms of any level and character which have weight 0 are constants.

For any $\gamma \in \Gamma$ we write

$$ (z, \tau)^\gamma = ({}^t(C\tau + D)^{-1}z, \gamma(\tau)), $$

which defines an action of Γ on $\mathbb{C}^2 \times \mathfrak{h}_2$.

For any $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$, and theta characteristic $\lambda = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$, theta functions transform as [BL, p. 227]

$$\theta[\lambda^\gamma](z,\gamma)^\gamma = m[\lambda]_\gamma(\tau) e^{{}^t z \mu_\gamma(\tau) z} \theta[\lambda](z,\tau), \tag{9}$$

where $m[\lambda]_\gamma(\tau) = \zeta[\lambda]_\gamma \det(C\tau + D)^{1/2}$, for $\det(C\tau+D)^{1/2}$ some choice of square root of $\det(C\tau + D)$ and $\zeta[\lambda]_\gamma$ an eighth root of unity that depends on that choice, where $\mu_\gamma(\tau) = \pi i (C\tau + D)^{-1} C$,

$$\lambda^\gamma = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \frac{1}{2} \begin{bmatrix} (C\,{}^t D)_0 \\ (A\,{}^t B)_0 \end{bmatrix},$$

and where for a matrix M, $(M)_0$ denotes the column vector consisting of the diagonal entries of M.

It follows immediately from (3) that

$$\theta \begin{bmatrix} a+p \\ b+q \end{bmatrix} (z,\tau) = e^{2\pi i\, {}^t a q} \theta \begin{bmatrix} a \\ b \end{bmatrix} (z,\tau)$$

for $p, q \in \mathbb{Z}^2$. Therefore we lose at most a sign when we identify theta characteristics modulo 1. However, since we will be concerned with signs, for any theta characteristic $[\lambda]$ with $\lambda \in \frac{1}{2}\mathbb{Z}^4$, we will let $\{\lambda\}$ denote the theta characteristic with entries in $\{0, 1/2\}$ such that $\{\lambda\} \equiv [\lambda] \bmod 1$. Absorbing the sign into the multiplier, we will define $\ell\{\lambda\}_\gamma(\tau) = \pm m[\lambda]_\gamma(\tau)$, so that

$$\theta\left\{\lambda^\gamma\right\}(z,\tau)^\gamma = \ell\{\lambda\}_\gamma(\tau) e^{{}^t z \mu_\gamma(\tau) z} \theta\{\lambda\}(z,\tau). \tag{10}$$

For $\gamma \in \Gamma$, and λ a theta characteristic, the map $\{\lambda\} \mapsto \{\lambda^\gamma\}$ gives an action on characteristic vectors modulo 1 we call the *symplectic action*. It is clear that the theta characteristics are stable under the symplectic action, and (9) shows that the subsets of even and odd theta characteristics are also left stable by the symplectic action. The actions on the even and odd characteristics are transitive. Moreover, as noted in [I1, p. 398], the action gives a homomorphism of Γ into the group S_6 of symmetries of the six odd theta characteristics modulo 1, which is surjective with kernel precisely $\Gamma(2)$. So if for any odd theta characteristic δ, we let Γ_δ denote the stabilizer of $\{\delta\}$ under the symplectic action, then Γ_δ has index 6 in Γ, $\Gamma_\delta \supset \Gamma(2)$, and $\Gamma_\delta/\Gamma(2)$ is isomorphic to S_5 via its action on the other five odd theta characteristics modulo 1. A reasonable name for Γ_δ is an *odd theta subgroup* of Γ.

The six odd theta characteristics modulo 1 are represented by

$$\mathcal{O} := \tag{11}$$
$$\left\{ \delta_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 0 \end{bmatrix}, \delta_2 = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix}, \delta_3 = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 1/2 \end{bmatrix}, \delta_4 = \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \delta_5 = \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ 1/2 \end{bmatrix}, \delta_6 = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \\ 1/2 \end{bmatrix} \right\}.$$

Let $\eta_i = \{\delta_i + \delta_6\}$, $1 \le i \le 6$, and write $\eta_i = \left[\begin{smallmatrix} \eta_i' \\ \eta_i'' \end{smallmatrix}\right]$, $\eta_i' = \left(\begin{smallmatrix} \eta_{i1}' \\ \eta_{i2}' \end{smallmatrix}\right)$, $\eta_i'' = \left(\begin{smallmatrix} \eta_{i1}'' \\ \eta_{i2}'' \end{smallmatrix}\right)$. Then as noted in [G1], for $\{i, j\} \subset \{1, 2, 3, 4, 5\}$,

$$i < j \iff (-1)^{4\,{}^t\eta_i'\eta_j''} = 1.$$

This can be extended to an ordering of $\{1, 2, 3, 4, 5, 6\}$ by checking that for $\{i, j\} \subset \{1, 2, 3, 4, 5, 6\}$,

$$i < j \iff (-1)^{4\,{}^t\eta_i'\eta_j'' + (1-2\eta_{i1}')(1-2\eta_{i1}'')} = 1. \tag{12}$$

More properties of the numbering of the δ_i will be given in Section 4.

Representatives for the ten even theta characteristics modulo 1 are

$$\mathcal{E} := \tag{13}$$
$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \right\}.$$

Note that for each even characteristic $\epsilon \in \mathcal{E}$, there is a unique partition of $\{1, 2, 3, 4, 5, 6\}$ into sets $\{i, j, k\} \cup \{l, m, n\}$ such that $\epsilon = \{\delta_i + \delta_j + \delta_k\} = \{\delta_l + \delta_m + \delta_n\}$. We will let $\{i, j, k\}^\sim$ denote the complement of $\{i, j, k\}$ in $\{1, 2, 3, 4, 5, 6\}$, so in this way even characteristics correspond to subsets of $\{1, 2, 3, 4, 5, 6\}$ of size 3 modulo the action of $\sim$.

We write $\tau = \left(\begin{smallmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{smallmatrix}\right)$ for $\tau \in \mathfrak{h}_2$. Let Z denote the image of the analytic subvariety $\tau_{12} = 0$ of $\mathfrak{h}_2$ under the action by Γ.

As in the Introduction, we define $\Delta(\tau) = \prod_{\epsilon \in \mathcal{E}} \theta\{\epsilon\}(0, \tau)$. Then $\Delta(\tau)$ is up to a constant multiple the unique Siegel modular form of level 1 and weight 5 for some character χ. It is well known that $\Delta(\tau)$ has a zero of order 1 along Z and no other zeroes, and that χ is a quadratic character on Γ [K, p. 115]. Moreover, χ is the only non-trivial character on Γ [K, p. 44], so must restrict to the sign character on $\Gamma/\Gamma(2) \simeq S_6$. Hence it restricts to the sign character on $\Gamma_\delta/\Gamma(2) \simeq S_5$, so is a quadratic character on Γ_δ. We employ the letter Δ because (see e.g. [G1]) it can be shown using Thomae's formula that for $\tau \notin Z$, $\Delta(\tau)^2$ differs only by a multiplicative constant from the discriminant of the curve of genus 2 given in (6) whose period matrix is τ.

It follows from the definition (3) that if $\tau = \left(\begin{smallmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{smallmatrix}\right) \in \mathfrak{h}_2$, then for $z = \left(\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right)$, $a = \left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right]$, $b = \left[\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right]$, $a_i, b_i \in \frac{1}{2}\mathbb{Z}$, $i = 1, 2$, we have

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \tau) \bigg|_{\tau_{12}=0} = \vartheta \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} (z_1, \tau_{11}) \vartheta \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} (z_2, \tau_{22}). \tag{14}$$

If $\lambda = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$, we will write

$$\mathrm{Top}(\lambda) := \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \quad \text{and} \quad \mathrm{Bottom}(\lambda) = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix},$$

and say that λ *decomposes* into a "top" elliptic theta characteristic $\mathrm{Top}(\lambda)$ and a "bottom" elliptic theta characteristic $\mathrm{Bottom}(\lambda)$. Factorizations analogous to (14) occur for the derivatives of $\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z,\tau)$ with respect to z_1 and z_2 when $\tau_{12} = 0$. We will say that $\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z,\tau)$ *decomposes* as in (14) when $\tau_{12} = 0$.

For every $\delta \in \mathcal{O}$, precisely one of $\mathrm{Top}(\delta)$ and $\mathrm{Bottom}(\delta)$ is the odd elliptic theta characteristic $\left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right]$. So we will let $\mathcal{O}^T = \{\delta_1, \delta_2, \delta_3\}$ and $\mathcal{O}_B = \{\delta_4, \delta_5, \delta_6\}$ denote the subsets of $\mathcal{O}$ with "odd tops" and "odd bottoms", respectively. We have numbered the elements in $\mathcal{O}$ so that if $\delta_i \in \mathcal{O}^T$, then δ_{7-i} is the element in $\mathcal{O}_B$ such that $\mathrm{Top}(\delta_i) = \mathrm{Bottom}(\delta_{7-i})$ for $1 \le i \le 6$. Hence, if we let τ' denote the matrix τ with τ_{11} and τ_{22} reversed, then it follows directly from (3) that for $1 \le i \le 6$,

$$\theta\{\delta_i\}\left(\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \tau\right) = \theta\{\delta_{7-i}\}\left(\begin{pmatrix} z_2 \\ z_1 \end{pmatrix}, \tau'\right), \tag{15}$$

which will simplify several proofs. Note that this is just (9) with $A = D = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $B = C = 0$.

2. Proof of Theorem 1. Let Φ_1 and Φ_2 be elements in some commutative $\mathbb{C}$-algebra, and $\Phi = \left(\begin{smallmatrix} \Phi_1 \\ \Phi_2 \end{smallmatrix}\right)$. For $n \ge 1$, we let $\Phi^{(n)}$ denote the column vector of length $n+1$ whose ith entry is $\Phi_1^{n-i}\Phi_2^i$ for $0 \le i \le n$.

Let z_1, z_2 be the coordinate functions on $\mathbb{C}^2$, $z = \left(\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right)$, and let ∇_n be the row vector of differential operators on the space of functions on $\mathbb{C}^2$ analytic at the origin whose ith entry is $\frac{1}{i!(n-i)!}\left(\frac{\partial}{\partial z_1}\right)^{n-i}\left(\frac{\partial}{\partial z_2}\right)^i$ for $0 \le i \le n$. We let $\nabla_n(f(0))$ denote $\nabla_n(f(z))|_{z_1=z_2=0}$. Then with our normalizations, for any function f on $\mathbb{C}^2$ analytic at the origin, the homogeneous degree-n part of the Taylor expansion at the origin of f in the variables z_1 and z_2 is given by $\nabla_n(f(0))z^{(n)}$. And if $\Psi = \left(\begin{smallmatrix} \Psi_1 \\ \Psi_2 \end{smallmatrix}\right) \in \mathbb{C}^2$, then

$$\nabla_n(f(0))\Psi^{(n)} = \frac{1}{n!}\left(\Psi_1\frac{\partial}{\partial z_1} + \Psi_2\frac{\partial}{\partial z_2}\right)^n f(0). \tag{16}$$

To simplify notation, we let subscripts $i, \ldots, j$ denote partial derivatives with respect to the correspondingly indexed variables $z_i, \ldots, z_j$.

As in the Introduction, let $\mathcal{D}_\delta = \theta\{\delta\}_2(0,\tau)\frac{\partial}{\partial z_1} - \theta\{\delta\}_1(0,\tau)\frac{\partial}{\partial z_2}$.

Proposition 1. *For any odd theta characteristic δ, let*

$$R\{\delta\}(\tau) = \mathcal{D}_\delta^3\theta\{\delta\}(0,\tau).$$

Then $R\{\delta\}(\tau)$ is a modular form of weight 5 and some character ψ_δ on Γ_δ.

Proof. Let $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_\delta$. Taking linear terms of (10) gives

$$\nabla_1\big(\theta\{\delta^\gamma\}(0,\gamma(\tau))\big)\,{}^t(C\tau + D)^{-1}z = \ell\{\delta\}_\gamma(\tau)\nabla_1(\theta\{\delta\}(0,\tau))z,$$

so by taking transposes,

$$\begin{pmatrix} \theta_1\{\delta^\gamma\}(0,\gamma(\tau)) \\ \theta_2\{\delta^\gamma\}(0,\gamma(\tau)) \end{pmatrix} = \ell\{\delta\}_\gamma(\tau)(C\tau+D)\begin{pmatrix} \theta_1\{\delta\}(0,\tau) \\ \theta_2\{\delta\}(0,\tau) \end{pmatrix}. \tag{17}$$

If we let ρ denote the standard two-dimensional representation of GL_2 on $\mathbb{C}^2$, this means that ${}^t\nabla_1(\theta\{\delta\}(0,\tau))$ is a vector-valued Siegel modular form of type $(\ell\{\delta\}_\gamma(\tau),\rho)$ on Γ_δ (see [GM1] or [SM] where this is shown for the gradient of any odd theta function).

Let $K = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Multiplying (17) on the left by ${}^t(C\tau+D)K$ gives what we will call the *linear relation*:

(18)

$${}^t(C\tau+D)\begin{pmatrix} \theta_2\{\delta^\gamma\}(0,\gamma(\tau)) \\ -\theta_1\{\delta^\gamma\}(0,\gamma(\tau)) \end{pmatrix} = \ell\{\delta\}_\gamma(\tau)\det(C\tau+D)\begin{pmatrix} \theta_2\{\delta\}(0,\tau) \\ -\theta_1\{\delta\}(0,\tau) \end{pmatrix},$$

since for any 2×2 matrix M, ${}^tMKM = (\det M)K$.

Looking now at the cubic terms in (10), we have

$$\begin{aligned} &\nabla_3\big(\theta\{\delta^\gamma\}(0,\gamma(\tau))\big)\big({}^t(C\tau+D)^{-1}z\big)^{(3)} \\ &\quad = \ell\{\delta\}_\gamma(\tau)\big(\nabla_3(\theta\{\delta\}(0,\tau))z^{(3)} + ({}^tz\mu_\gamma(\tau)z)\nabla_1(\theta\{\delta\}(0,\tau))z\big). \end{aligned} \tag{19}$$

Since setting $z = \ell\{\delta\}_\gamma(\tau)\det(C\tau+D)\,{}^t(\theta_2\{\delta\}(0,\tau), -\theta_1\{\delta\}(0,\tau))$ makes the last term on the right side of (19) vanish, plugging the linear relation (18) into (19) yields

$$\begin{aligned} &\nabla_3(\theta\{\delta^\gamma\}(0,\gamma(\tau))\begin{pmatrix} \theta_2\{\delta^\gamma\}(0,\gamma(\tau)) \\ -\theta_1\{\delta^\gamma\}(0,\gamma(\tau)) \end{pmatrix}^{(3)} \\ &\quad = \ell\{\delta\}_\gamma(\tau)^4\det(C\tau+D)^3\nabla_3(\theta\{\delta\}(0,\tau))\begin{pmatrix} \theta_2\{\delta\}(0,\tau) \\ -\theta_1\{\delta\}(0,\tau) \end{pmatrix}^{(3)}. \end{aligned} \tag{20}$$

Since $\ell\{\delta\}_\gamma(\tau)^4 = \pm\det(C\tau+D)^2$, we can set $\psi_\delta(\gamma) = \frac{\ell\{\delta\}_\gamma(\tau)^4}{\det(C\tau+D)^2} = \pm 1$, which is independent of τ and so is a character on Γ_δ. Thus

$$\nabla_3(\theta\{\delta\}(0,\tau))\begin{pmatrix} \theta_2\{\delta\}(0,\tau) \\ -\theta_1\{\delta\}(0,\tau) \end{pmatrix}^{(3)}$$

is a modular form of weight 5 on Γ_δ with character ψ_δ. By (16) this is precisely $1/3!$ times $R\{\delta\}(\tau)$. ■

For f_1 and f_2 differentiable functions of $z = \left(\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right)$, let $j(f_1,f_2)$ represent the jacobian determinant with respect to z of f_1 and f_2.

For any odd theta characteristic δ we now set

$$\xi\{\delta\}(z,\tau) = j(\theta\{\delta\}(z,\tau), X\{\delta\}(z,\tau)).$$

PROPOSITION 2. *For any odd theta characteristic δ, $R\{\delta\}(\tau) = \xi\{\delta\}(0,\tau)$ vanishes along Z.*

Proof. Since the symplectic action takes odd theta characteristics to odd theta characteristics, it suffices to show, for every odd δ, that $R\{\delta\}(\tau)$ vanishes when $\tau_{12}=0$. Indeed, for any odd natural number i, $\mathcal{D}_\delta^i\theta\{\delta\}(0,\tau)$ vanishes when $\tau_{12}=0$. To see this, assume without loss of generality by (15) that $\delta\in\mathcal{O}^T$. Then when $\tau_{12}=0$, $\mathcal{D}_\delta=-\theta\{\delta\}_1(0,\tau)\frac{\partial}{\partial z_2}$ and $\big(\frac{\partial}{\partial z_2}\big)^i\theta\{\delta\}(0,\tau)$ vanishes. ∎

It follows from Propositions 1 and 2 that $R\{\delta\}(\tau)/\Delta(\tau)$ is a Siegel modular form of weight 0 on Γ_δ with character ψ_δ, and hence is a constant c_δ. Determining the constant will then complete the proof of Theorem 1. (This also shows that χ restricted to Γ_δ is ψ_δ.)

PROPOSITION 3. *Let $\delta=\left[\begin{smallmatrix}a\\ b\end{smallmatrix}\right]$ be an odd theta characteristic, and $a=\left(\begin{smallmatrix}a_1\\ a_2\end{smallmatrix}\right)$, $b=\left(\begin{smallmatrix}b_1\\ b_2\end{smallmatrix}\right)$. Then*

$$c_\delta=(-1)^{2(a_1+a_2+b_1+b_2)+4a_1b_1}2\pi^6.$$

Proof. Since theta functions are solutions of the heat equation, we see that for any theta characteristic λ,

$$\frac{\partial\theta[\lambda](z,\tau)}{\partial\tau_{12}}=\frac{1}{2\pi i}\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2}\theta[\lambda](z,\tau). \tag{21}$$

Applying $\frac{1}{2\pi i}\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2}$ to $\prod_{\epsilon\in\mathcal{E}}\theta\{\epsilon\}(z,\tau)$ shows that

$$\frac{\partial}{\partial\tau_{12}}\Delta(\tau)\bigg|_{\tau_{12}=0}=-2^7\pi i\eta(\tau_{11})^{12}\eta(\tau_{22})^{12},$$

which follows from applying (2) multiple times.

Now to determine c_δ, we want to differentiate $R\{\delta\}(\tau)$ with respect to τ_{12} and set $\tau_{12}=0$. By (21), $\frac{\partial}{\partial\tau_{12}}$ acts on the derivatives of $\theta\{\delta\}(z,\tau)$ as $\frac{1}{2\pi i}\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2}$, so we will differentiate

$$\sum_{i=0}^{3}\binom{3}{i}\frac{\partial^3\theta\{\delta\}(z,\tau)}{\partial z_1^{3-i}\partial z_2^i}\Big(-\frac{\partial\theta\{\delta\}(z,\tau)}{\partial z_1}\Big)^i\Big(\frac{\partial\theta\{\delta\}(z,\tau)}{\partial z_2}\Big)^{3-i}, \tag{22}$$

with this operator when $\tau_{12}=0$ and then set $z=0$.

First we assume that $\delta\in\mathcal{O}^T$. Let $\alpha=\mathrm{Top}(\delta)$ and $\beta=\mathrm{Bottom}(\delta)$, which are respectively odd and even elliptic theta characteristics. In this case, the only terms in (22) which vanish to only order 1 when $\tau_{12}=0$ are when $i=3$ and $i=2$. We will consider these in turn.

The term where $i=3$ contributes

$$-\frac{1}{2\pi i}\vartheta[\alpha]_1(0,\tau_{11})^4\vartheta[\beta](0,\tau_{22})^3\vartheta[\beta]_{2222}(0,\tau_{22}),$$

while the term where $i=2$ contributes

$$\frac{3}{2\pi i}\vartheta[\alpha]_1(0,\tau_{11})^4\vartheta[\beta](0,\tau_{22})^2\vartheta[\beta]_{22}(0,\tau_{22})^2.$$

Hence

$$\frac{\partial}{\partial\tau_{12}} R\{\delta\}(\tau)\Big|_{\tau_{12}=0} = \frac{1}{2\pi i}\vartheta[\alpha]_1(0,\tau_{11})^4 r[\beta](\tau_{22}) = -c_\delta 2^7\pi i\eta(\tau_{11})^{12}\eta(\tau_{22})^{12}, \tag{23}$$

where

$$r[\beta](t) := -\vartheta[\beta](0,t)^3\left(\frac{\partial}{\partial w}\right)^4\vartheta[\beta](0,t) + 3\vartheta[\beta](0,t)^2\left(\frac{\partial}{\partial w}\right)^2\vartheta[\beta](0,t)^2.$$

Now (23) implies that $r[\beta](t)$ is a constant κ_β times $\eta(t)^{12}$, and comparing q-expansions using (1) gives $\kappa_\beta = (-1)^{1+2a_2+2b_2}32\pi^4$.

Then applying Jacobi's derivative formula (2) to (23) gives

$$c_\delta = \frac{\frac{1}{2\pi i}(-2\pi)^4\kappa_\beta}{-2^7\pi i} = (-1)^{1+2a_2+2b_2}2\pi^6$$

when $\delta\in\mathcal{O}^T$. Since the roles of z_1 and z_2 get reversed on the two sides of (15), one deduces that for $\delta\in\mathcal{O}_B$,

$$c_\delta = (-1)^{2a_1+2b_1}2\pi^6.$$

The proposition and Theorem 1 now follow from checking that $2a_1+2b_1+2a_2+2b_2+4a_1b_1$ agrees with $1+2a_2+2b_2$ modulo 2 for $\delta\in\mathcal{O}^T$ and with $2a_1+2b_1$ modulo 2 for $\delta\in\mathcal{O}_B$. ∎

Remark. One can give a direct proof that $r[\beta](t)$ is a multiple of $\eta(t)^{12}$ by recognizing it as $-\vartheta[\beta](w,t)^4\left(\frac{\partial}{\partial w}\right)^4\log(\vartheta[\beta](w,t))\big|_{w=0}$, so via the functional equation for ϑ, it is an elliptic modular form with character of weight 6 and level 2. Then one checks that the q-expansions of $r[\beta](t)$ for all choices of even elliptic characteristics $[\beta]$ have lead term a multiple of $q^{1/2}=e^{\pi i t}$, so $r[\beta](t)/\eta(t)^{12}$ is a modular form of level 2 and weight 0, hence a constant.

3. Proof of Theorem 2. The following is elementary.

Lemma 1. *Let $z=\binom{z_1}{z_2}$ be complex variables, M be any 2×2 complex matrix, and f, f_i, $1\le i\le 4$, be meromorphic functions of z. Let $j(f_1,f_2)$ denote the jacobian determinant of f_1 and f_2 with respect to z, and $h(f)$ the determinant of the hessian matrix $\mathcal{H}(f)$ of f with respect to z. Then*

(i) $j(f_1(Mz),f_2(Mz)) = (\det M)j(f_1,f_2)(Mz)$,
(ii) $h(f(Mz)) = (\det M)^2 h(f)(Mz)$,
(iii) $\mathcal{H}({}^t zMz) = M + {}^tM$.

For any odd theta characteristic δ we have defined

$$X\{\delta\}(z,\tau) = \theta\{\delta\}(z,\tau)^3 h(\log\theta\{\delta\}(z,\tau)).$$

We need to determine how its gradient at the origin transforms under Γ.

PROPOSITION 4. *Let* $K = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. *For any* $\delta \in \mathcal{O}$, $\gamma \in \Gamma$, *define*

$$U\{\delta\}_\gamma(\tau) = -2\big(\nabla_1(\theta\{\delta\}(0,\tau))K\big)\mu_\gamma(\tau)\,{}^t\big(\nabla_1(\theta\{\delta\}(0,\tau))K\big).$$

Then

$$\begin{aligned}\nabla_1(X\{\delta^\gamma\}\big(0,\gamma(\tau))\big) &= \ell\{\delta\}_\gamma(\tau)^3 \det(C\tau+D)^2 \\ &\quad \times \big(\nabla_1(X\{\delta\}(0,\tau)) + U\{\delta\}_\gamma(\tau)\nabla_1(\theta\{\delta\}(0,\tau))\big)\,{}^t(C\tau+D).\end{aligned}$$

Proof. Taking logs and then hessian determinants of (10) and using Lemma 1(ii, iii) gives

$$h(\log\theta\{\delta^\gamma\}(z,\tau)^\gamma) = \det(C\tau+D)^2 \det\big(2\mu_\gamma(\tau) + H(\log\theta\{\delta\}(z,\tau))\big),$$

since by (8), $\mu_\gamma(\tau) = \pi i(C\tau+D)^{-1}C$ is symmetric. Hence by (10),

$$\begin{aligned}X\{\delta^\gamma\}(z,\tau)^\gamma &= \ell\{\delta\}_\gamma(\tau)^3 \det(C\tau+D)^2 \\ &\quad \times e^{3\,{}^tz\mu_\gamma(\tau)z}\theta\{\delta\}(z,\tau)^3 \det\big(2\mu_\gamma(\tau) + H(\log\theta\{\delta\}(z,\tau))\big).\end{aligned} \tag{24}$$

Now the definition of $X\{\delta\}(z,\tau)$ in terms of the partial derivatives of $\theta\{\delta\}(z,\tau)$ shows that

$$\begin{aligned}\theta\{\delta\}(z,\tau)^3 &\det\big(2\mu_\gamma(\tau) + H(\log\theta\{\delta\}(z,\tau))\big) \\ &= X\{\delta\}(z,\tau) + \theta\{\delta\}(z,\tau)V\{\delta\}_\gamma(z,\tau),\end{aligned} \tag{25}$$

where $V\{\delta\}_\gamma(z,\tau)$ is analytic.

Hence taking gradients at the origin of (24), using (25) and the fact that δ is odd, we get

$$\begin{aligned}&\nabla_1(X\{\delta^\gamma\}(0,\gamma(\tau)))\,{}^t(C\tau+D)^{-1} \\ &\quad = \ell\{\delta\}_\gamma(\tau)^3 \det(C\tau+D)^2\big(\nabla_1(X\{\delta\}(0,\tau)) + \nabla_1(\theta\{\delta\}(0,\tau))V\{\delta\}_\gamma(0,\tau)\big).\end{aligned}$$

If we let $U\{\delta\}_\gamma(\tau) = V\{\delta\}_\gamma(0,\tau)$, and write $\mu_\gamma(\tau)$ as $[\mu_{i,j}]_{1\le i,j\le 2}$, then working out the determinant in (25) we get

$$\begin{aligned}U\{\delta\}_\gamma(\tau) &= 2(-\theta_2[\delta](0,\tau)^2\mu_{11} + 2\theta_1[\delta](0,\tau)\theta_2[\delta](0,\tau)\mu_{12} - \theta_1[\delta](0,\tau)^2\mu_{22}) \\ &= -2(\nabla_1(\theta\{\delta\}(0,\tau))K)\mu_\gamma(\tau)\,{}^t\big(\nabla_1(\theta\{\delta\}(0,\tau))K\big),\end{aligned}$$

which gives the proposition. ∎

Proposition 4 is a genus-2 version of a classic genus-1 situation: the negative of the second logarithmic derivative of $\vartheta\left\{\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right\}(w,t)$ is not quite modular. One has to subtract $\frac{1}{3}\vartheta\left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right]'''(0,t)/\vartheta\left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right]'(0,t)$ (a multiple of the Eisenstein series $E_2(t)$, a quasimodular form [BGHZ, I, §1.8]) in order for it to be the Weierstrass $\wp$-function $\wp(w,t)$, which is a Jacobi form [EZ]. In genus 2, derivatives of theta functions again provide the required Siegel quasimodular form.

PROPOSITION 5. *For $\delta \in \mathcal{O}$, let*

$$E\{\delta\}(\tau) = \frac{1}{10}\mathcal{D}_\delta^5(\theta\{\delta\}(0,\tau))/\mathcal{D}_\delta^3(\theta\{\delta\}(0,\tau)).$$

Then for every $\gamma \in \Gamma$,

$$E\{\delta^\gamma\}(\gamma(\tau)) = \ell\{\delta\}_\gamma(\tau)^2 \det(C\tau+D)^2(E\{\delta\}(\tau) - U\{\delta\}_\gamma(\tau)).$$

Proof. As in the proof of Proposition 1, looking now at quintic terms in (10), we have

$$\begin{aligned}(26)\qquad \nabla_5(\theta\{\delta^\gamma\}(0,\gamma(\tau)))({}^t(C\tau+D)^{-1}z)^{(5)} &= \ell\{\delta\}_\gamma(\tau)\big(\nabla_5(\theta\{\delta\}(0,\tau))z^{(5)} \\ &+ ({}^tz\mu_\gamma(\tau)z)\nabla_3(\theta\{\delta\}(0,\tau))z^{(3)} + \tfrac{1}{2}({}^tz\mu_\gamma(\tau)z)^2\nabla_1(\theta\{\delta\}(0,\tau))z\big).\end{aligned}$$

Since setting $z = \ell\{\delta\}_\gamma(\tau)\det(C\tau+D)\,{}^t(\theta_2\{\delta\}(0,\tau), -\theta_1\{\delta\}(0,\tau))$ makes the last term on the right of (26) vanish, plugging the linear relation (18) into (26) yields

$$\frac{1}{5!}\mathcal{D}_\delta^5(\theta\{\delta^\gamma\}(0,\gamma(\tau))) = \nabla_5\big(\theta\{\delta^\gamma\}(0,\gamma(\tau))\big)\begin{pmatrix}\theta_2\{\delta^\gamma\}(0,\gamma(\tau)) \\ -\theta_1\{\delta^\gamma\}(0,\gamma(\tau))\end{pmatrix}^{(5)} =$$

$$\ell\{\delta\}_\gamma(\tau)^6\det(C\tau+D)^5\left(\frac{1}{5!}\mathcal{D}_\delta^5(\theta\{\delta\}(0,\tau)) - \frac{1}{2}U\{\delta\}_\gamma(\tau)\frac{1}{3!}\mathcal{D}_\delta^3(\theta\{\delta\}(0,\tau))\right).$$

Dividing by (20) and multiplying by 2 gives the result. ∎

Theorem 2 now follows from

PROPOSITION 6. *For any $\delta \in \mathcal{O}$, let*

$$Y\{\delta\}(z,\tau) = X\{\delta\}(z,\tau) + E\{\delta\}(\tau)\theta\{\delta\}(z,\tau).$$

Then ${}^t\nabla_1(Y\{\delta\}(0,\tau))$ is a vector-valued modular form on Γ_δ of type

$$(\ell\{\delta\}_\gamma(\tau)^3\det(C\tau+D)^2, \rho).$$

Proof. We first need to verify that $E\{\delta\}(\tau)$ is analytic, in other words, that $\mathcal{D}_\delta^5(\theta\{\delta\}(0,\tau))$ vanishes on Z. By Proposition 5, this is true for $\mathcal{D}_\delta^5(\theta\{\delta\}(0,\tau))$ if and only if it is true for $\mathcal{D}_{\delta^\gamma}^5(\theta\{\delta^\gamma\}(0,\gamma(\tau)))$ for all $\gamma \in \Gamma$, so it suffices to show, for every odd theta characteristic δ, that $\mathcal{D}_\delta^5(\theta\{\delta\}(0,\tau))$ vanishes when $\tau_{12} = 0$. But this was done in the proof of Proposition 2.

Propositions 4 and 5 now show that for any $\gamma = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in \Gamma_\delta$,

$${}^t\nabla_1\big(Y\{\delta\}(0,\gamma(\tau))\big) = \ell\{\delta\}_\gamma(\tau)^3\det(C\tau+D)^2(C\tau+D)\,{}^t\nabla_1(Y\{\delta\}(0,\tau)). \ ∎$$

4. A quick proof of Rosenhain's formula, with signs. The methods of this paper can give a quick proof of Rosenhain's formula up to a multiplicative constant—and it is not hard to determine the constant—so we will now present both arguments.

Let $\lambda_{1/2} := {}^t[1/2\ 1/2\ 1/2\ 1/2] = \{\delta_1+\delta_2+\delta_3\} = \{\delta_4+\delta_5+\delta_6\}$. One can check easily from the functional equations of theta functions that for any two distinct odd theta characteristics $\delta_i, \delta_j \in \mathcal{O}$, if we let

$$f_{ij} = j(\theta\{\delta_i\}(0,\tau), \theta\{\delta_j\}(0,\tau)) \times \prod_{\substack{\{k,\ell,m\}\subset\{1,2,3,4,5,6\} \\ |\{i,j\}\cap\{k,\ell,m\}|=1 \bmod \sim}} \theta\{\delta_k+\delta_\ell+\delta_m\}(0,\tau),$$

then f_{ij}^2 is a Siegel modular form with character of level 2 and weight 10, and Γ acts on the set of all f_{ij}. Therefore, to see that each f_{ij} is a constant times $\Delta(\tau)$, it suffices to show that each f_{ij}^2 vanishes to order 2 when $\tau_{12} = 0$, i.e., each f_{ij} vanishes when $\tau_{12} = 0$.

If for some $\{k,\ell,m\} \subset \{1,2,3,4,5,6\}$ with $|\{i,j\}\cap\{k,\ell,m\}| = 1$, we have $\delta_k+\delta_\ell+\delta_m = \lambda_{1/2} \mod 1$, then this follows since $\theta[\lambda_{1/2}](0,\tau)$ vanishes when $\tau_{12} = 0$. One checks that this happens when one of δ_i and δ_j is in $\mathcal{O}^T$ and the other in $\mathcal{O}_B$. If that is not the case, then δ_i and δ_j are both in $\mathcal{O}^T$ or both in $\mathcal{O}_B$ and $j(\theta\{\delta_i\}(0,\tau), \theta\{\delta_j\}(0,\tau))$ vanishes when $\tau_{12} = 0$. These two cases correspond respectively to the "Gopel even tetrads" and the "Rosenhain odd tetrads" described in [H, Sections 50, 51] (see also [BL, Section 10.2]).

So we have shown that

$$f_{ij}(\tau) = c_{ij}\Delta(\tau)$$

for some constants c_{ij}. Hence

$$j(\theta\{\delta_i\}(0,\tau), \theta\{\delta_j\}(0,\tau)) = c_{ij} \prod_{k\in\{1,2,3,4,5,6\}-\{i,j\}} \theta\{\delta_i+\delta_j+\delta_k\}(0,\tau). \tag{27}$$

We will now look at the asymptotic behavior of (27) as $\tau_{12} \to 0$ to verify that each c_{ij} is $\pm\pi^2$, and determine whether the plus or minus sign holds.

To explain the result, we need to recall a classical formula about elliptic thetanullwerte, which follows from comparing quadratic terms in [M, formulas (E_1) and (E_2), p. 23] and using Jacobi's derivative formula:

Let $\nu_1 = \left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right], \nu_2 = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], \nu_3 = \left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]$ be representatives for the three even elliptic theta characteristics modulo 1. We have chosen the ordering of the indices so that for any choice of permutation (i,j,k) of $\{1,2,3\}$,

$$\frac{\vartheta[\nu_j]''(0,t)}{\vartheta[\nu_j](0,t)} - \frac{\vartheta[\nu_i]''(0,t)}{\vartheta[\nu_i](0,t)} = \pm\pi^2\vartheta[\nu_k](0,t)^4, \tag{28}$$

with the plus sign holding precisely when $i < j$. (Warning: Our numbering of the ν_i gives an ordering of the even elliptic theta functions that is different from the classical choice of Jacobi. See [WW, p. 487] for a discussion on the various historic notations.)

Using this, we can now explain the numbering of genus-2 odd characteristics we gave in (11). Let $\mu = \left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right]$. If $\delta_i \in \mathcal{O}^T$, then δ_i decomposes into elliptic theta characteristics μ, ν_i, and if $\delta_i \in \mathcal{O}_B$, then δ_i decomposes into $\nu_{7-i}\mu$. This ordering of odd theta characteristics gives a compact description of the sign in Rosenhain's formula (see [E, Appendix A], and compare to [FM, Lemma 5.1] and [F, §6.23]).

THEOREM (Rosenhain, with signs). *Suppose* $1 \le i < j \le 6$. *Then* $c_{ij} = \pi^2$, *that is,*

$$j(\theta\{\delta_i\}(0,\tau), \theta\{\delta_j\}(0,\tau)) = \pi^2 \prod_{k\in\{1,2,3,4,5,6\}-\{i,j\}} \theta\{\delta_i+\delta_j+\delta_k\}(0,\tau).$$

More generally, without any assumption on the ordering of distinct i *and* j, *if* $\eta_i = \{\delta_i + \delta_6\}$ *for all* i, *then from* (12),

$$\begin{aligned} &j(\theta\{\delta_i\}(0,\tau), \theta\{\delta_j\}(0,\tau)) \\ &\quad = (-1)^{4\,{}^t\eta_i'\eta_j'' + (1-2\eta_{i1}')(1-2\eta_{i1}'')}\pi^2 \prod_{k\in\{1,2,3,4,5,6\}-\{i,j\}} \theta\{\delta_i+\delta_j+\delta_k\}(0,\tau). \end{aligned}$$

Proof. We have three cases to check. Suppose $i < j$.

(i) $\delta_i \in \mathcal{O}^T$, $\delta_j \in \mathcal{O}_B$ (Gopel even tetrad case). In this case, when $\tau_{12} = 0$, the left side of (27) is $\vartheta[\mu]'(0,\tau_{11})\vartheta[\mu]'(0,\tau_{22})\vartheta[\nu_i](0,\tau_{22})\vartheta[\nu_{7-j}](0,\tau_{11})$ and the right side is $c_{ij}\vartheta[\nu_i](0,\tau_{22})\vartheta[\nu_{7-j}](0,\tau_{11})\prod_{\ell=1}^3 \vartheta[\nu_\ell](0,\tau_{11})\vartheta[\nu_\ell](0,\tau_{22})$. Applying (2) gives $c_{ij} = \pi^2$.

(ii) $\delta_i, \delta_j \in \mathcal{O}^T$ (a Rosenhain odd tetrad case). In this case both sides of (27) vanish, but we will take the derivative with respect to τ_{12} and then set $\tau_{12} = 0$. We do this via the heat equation, applying $\frac{1}{2\pi i}\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2}$ to $j(\theta\{\delta_i\}(z,\tau), \theta\{\delta_j\}(z,\tau))$ and to

$$c_{ij} \prod_{\{k\}\subset\{1,2,3,4,5,6\}-\{i,j\}} \theta\{\delta_k+\delta_\ell+\delta_m\}(z,\tau),$$

and then setting $z = 0$ and $\tau_{12} = 0$. This yields

$$\begin{aligned} &\frac{1}{2\pi i}\vartheta[\mu]'(0,\tau_{11})^2\big(\vartheta[\nu_i](0,\tau_{22})\vartheta[\nu_j]''(0,\tau_{22}) - \vartheta[\nu_j](0,\tau_{22})\vartheta[\nu_i]''(0,\tau_{22})\big) \\ &\qquad = c_{ij}\frac{1}{2\pi i}\vartheta[\mu]'(0,\tau_{11})\vartheta[\mu]'(0,\tau_{22})\vartheta[\nu_k](0,\tau_{22})^3 \prod_{\ell=1}^3 \vartheta[\nu_\ell](0,\tau_{11}), \end{aligned}$$

where k is the complement of $\{i,j\}$ in $\{1,2,3\}$. It follows from (2) and (28) that $c_{ij} = \pi^2$.

(iii) $\delta_i, \delta_j \in \mathcal{O}_B$ (the other Rosenhain odd tetrad case). The result can be seen from an argument analogous to (ii), or directly from (ii) by using (15). ∎

Acknowledgements. I would like to thank John Boxall for helpful conversations on this work.

References

[A] N. C. Alexander, *Point counting on reductions of CM abelian surfaces*, Ph.D. thesis, Univ. of California, Irvine, 2011.

[BL] C. Birkenhake and H. Lange, *Complex Abelian Varieties*, 2nd ed., Springer, Berlin, 2004.

[BGHZ] J. H. Bruinier, G. van der Geer, G. Harder and D. Zagier, *The 1-2-3 of Modular Forms. Lectures at a Summer School in Nordfjordeid, Norway*, Springer, Berlin, 2008.

[BEL] V. M. Buchstaber, V. Z. Enolskiĭ, and D. V. Leĭkin, *Hyperelliptic kleinian functions and applications*, in: Solitons, Geometry and Topology: On the Crossroad, V. M. Buchstaber and S. P. Novikov (eds.), Amer. Math. Soc. Transl. (2) 179, Amer. Math. Soc., Providence, RI, 1997, 1–33.

[CGG] F. Cléry, G. van der Geer, and S. Grushevsky, *Siegel modular forms of genus 2 and level 2*, Int. J. Math. 26 (2015), no. 5, art. 1550034, 51 pp.

[C] G. G. H. Coogan, *A generalization of Jacobi's derivative formula*, Ph.D. thesis, Univ. of Colorado at Boulder, 1999.

[dJ] R. de Jong, *Theta functions on the theta divisor*, Rocky Mountain J. Math. 40 (2010), 155–176.

[EZ] M. Eichler and D. Zagier, *The Theory of Jacobi Forms*, Progr. Math. 55, Birkhäuser, Boston, 1985.

[E] K. Eilers, *Rosenhain–Thomae formulae for higher genera hyperelliptic curves*, J. Nonlinear Math. Phys. 25 (2018), 86–105.

[F] A. Fiorentino, *On a ring of modular forms related to the theta gradients map in genus 2*, J. Algebra 388 (2013), 81–100.

[FM] E. Freitag and R. Salvati Manni, *Basic vector valued Siegel modular forms of genus two*, Osaka J. Math. 52 (2015), 879–894.

[G1] D. Grant, *A generalization of Jacobi's derivative formula to dimension two*, J. Reine Angew. Math. 392 (1988), 125–136.

[G2] D. Grant, *Formal groups in genus two*, J. Reine Angew. Math. 411 (1990), 96–121.

[G3] D. Grant, *Modular models of curves of genus two and their jacobians*, in preparation.

[GM1] S. Grushevsky and R. Salvati Manni, *Gradients of odd theta functions*, J. Reine Angew. Math. 573 (2004), 45–59.

[GM2] S. Grushevsky and R. Salvati Manni, *Two generalizations of Jacobi's derivative formula*, Math. Res. Lett. 12 (2005), 921–932.

[GM3] S. Grushevsky and R. Salvati Manni, *Theta functions of arbitrary order and their derivatives*, J. Reine Angew. Math. 590 (2006), 31–43.

[H] R. W. H. T. Hudson, *Kummer's Quartic Surface*, Cambridge Univ. Press, Cambridge, 1990.

[I1] J. Igusa, *On Siegel modular forms of genus two. II*, Amer. J. Math. 86 (1964), 392–412.

[I2] J. Igusa, *On Jacobi's derivative formula and its generalizations*, Amer. J. Math. 102 (1980), 409–446.

[K] H. Klingen, *Introductory Lectures on Siegel Modular Forms*, Cambridge Stud. Adv. Math. 20, Cambridge Univ. Press, Cambridge, 1990.

[M] D. Mumford, *Tata Lectures on Theta. I*, Progr. Math. 28, Birkhäuser, Boston, 1983.

[N] A. Nakayashiki, *On algebraic expressions of sigma functions for (n, s) curves*, Asian J. Math. 14 (2010), 175–211.

[R] G. Rosenhain, *Mémoire sur les fonctions de deux variables et à quatre périodes qui sont les inverses des intégrales ultra-elliptiques de la première classe*, in: Mémoires présentés par divers savants à l'Académie des Sciences de l'Institut de France, 2ème série, 11, Imprimerie Royale, Paris, 1851, 361–468.

[SM] R. Salvati Manni, *On the not identically zero Nullwerte of jacobians of theta functions with odd characteristics*, Adv. Math. 47 (1983), 88–104.

[S] R. Sasaki, *Derivative formulas for hermitian theta functions of degree two*, Japan. J. Math. 27 (2001), 1–49.

[T] J. Thomae, *Beitrag zur Bestimmung von* $\vartheta(0, 0, \ldots, 0)$ *durch die Klassenmoduln algebraischer Functionen*, J. Reine Angew. Math. 71 (1870), 201–222.

[WW] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge Univ. Press, Cambridge, 1927.

David Grant
Department of Mathematics
University of Colorado Boulder
Boulder, CO 80309-0395, U.S.A.
E-mail: grant@colorado.edu

Abstract (will appear on the journal's web site only)

We give a new level-1 generalization of Rosenhain's derivative formula for theta functions in two variables. Namely, we show that for τ in the degree-2 Siegel upper half-space, the jacobian at 0 of an odd theta function $\theta[\delta](z,\tau)$ in two variables $z = \binom{z_1}{z_2}$ with the numerator of its logarithmic Hessian, $X[\delta](z,\tau)$, gives a constant times the genus-2 level-1 Siegel modular form (with character) of weight 5. The gradient of $\theta[\delta](z,\tau)$ is a vector-valued modular form and we modify $X[\delta](z,\tau)$ by the addition of a multiple of $\theta[\delta](z,\tau)$ times a Siegel quasimodular form, so that its gradient at 0 is a vector-valued modular form as well. These formulas complement the results in the precursor paper from 1988 and will play a crucial role in an upcoming article on "modular models" for jacobians of curves of genus 2, and we discuss their geometric and arithmetic significance.