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Some product formulas for theta functions in one and two variables

by

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Dedicated to Harold Stark on the occasion of his 60th birthday

Introduction. For $\tau \in \mathfrak{h} = \{x + iy \mid y > 0\}$, $a, b \in \mathbb{R}$, we define the theta function in one variable $z \in \mathbb{C}$ with characteristic vector $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ by

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z, \tau) = \sum_{n\in\mathbb{Z}} e^{\pi i(n+a)^2\tau + 2\pi i(n+a)(z+b)}.$$

Let m be a positive integer. The following equation between modular forms is crucial to the construction of elliptic units:

$$\prod_{\substack{0\le u,v<m \\ (u,v)\ne(0,0)}} \theta\begin{bmatrix} 1/2 + u/m \\ 1/2 + v/m \end{bmatrix}(0, \tau) = (-1)^{m-1} m\eta(\tau)^{m^2-1}, \tag{1}$$

where $\eta(\tau) = e^{\pi i\tau/12}\prod_{n>0}(1 - e^{2\pi in\tau})$. For example, Stark (see [S, p. 353]) proved (1) (in a disguised form) using Kronecker's second limit formula, a tool which is not available for the study of theta functions in more than 1 variable.

Note that $\eta(\tau)^{24} = \Delta(\tau)$, a cusp form of weight 12 related to the discriminant of the elliptic curve which has τ as a period. No formula analogous to (1) holds for all m relating theta functions in two variables to the "discriminant modular form" attached to τ in the Siegel upper half-space of degree 2 (see §3). One main purpose of this paper is to prove Theorem 2 of Section 3, which shows that an analogous formula does hold when $m = 3$ and $m = 4$.

That such an equation holds with an undetermined constant was shown for $m = 3$ in [Gr4], and independently for $m = 3$ and $m = 4$ by Goren [Go]. Our approach is different from that in [Go], using facts about Siegel modular forms, rather than considering the moduli of genus two curves. At

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the end of Section 3 we briefly discuss how the two approaches compare, relating the product formulas of Theorem 2 to the fact that primitive 3- and 4-torsion points on the Jacobian of a curve of genus 2 do not lie on the embedded image of the curve under the Albanese map using a Weierstrass point as base point. This fact is also central to the arguments in [BoBa], [BaBo], and [Gr1], which for certain genus 2 curves defined over number fields, build units in number fields attached to torsion points. See [A] for similar results for genus 3 curves, and [dSG], [Gr2], [FK] and [Lec] for more on units attached to genus 2 curves. In particular, for one curve, [FK] builds units from 6-torsion points on the Jacobian from the point of view of theta functions. I hope that the type of product formulas given here will lead to a better understanding of the sort of norm computations done in [FK]. In some sense, Theorem 2 says that the arithmetic properties enjoyed by 3- and 4-torsion points on Jacobians of curves of genus 2 defined over number fields are reflected in the geometry of generic curves of genus 2.

Another product formula for theta functions in two variables is given in [Gr3] (see also [C]), and a seemingly unrelated product formula is given in [Bor].

The other main purpose of this paper is to derive generalizations of Jacobi's derivative formula for theta functions in one variable, relating in Theorem 1 the products of derivatives at zero of theta functions with different rational characteristics to powers of $\eta(\tau)$. This is necessary for determining the constants in Theorem 2. For more on this theme, see [BG].

For the convenience of the reader, in Section 1 we recall certain properties of theta functions in several variables. Since we need it in what follows, in Section 2 we give a quick proof of (1) and a bevy of allied formulas. We also state and prove Theorem 1. Theorem 2 is stated and proved in Section 3.

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Finally, the germ of this paper was in some work done under the supervision of Harold Stark [Gr4]. I would like to take this opportunity to thank him for his continued support and friendship, and it is a pleasure to dedicate this paper to him.

1. Properties of theta functions. Let $\mathfrak{h}_g$ denote the Siegel upper half-space of degree g; that is, $g \times g$ symmetric complex matrices with positive-definite imaginary part. We let $\mathrm{Sp}_{2g}(\mathbb{Z})$ denote the integral symplectic group of degree g; i.e., block matrices $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ such that

$$ {}^{\mathrm{t}}\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, $$

where A, B, C, and D are integral $g \times g$ matrices, I is the $g \times g$ identity, and ${}^{\mathrm{t}}$ denotes the transpose. For $N > 0$, we let $\Gamma(N)$ denote the subgroup of matrices congruent to the identity mod N. Elements $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma = \Gamma(1)$ act on $\mathfrak{h}_g$ via $\gamma \circ \tau = (A\tau + B)(C\tau + D)^{-1}$. Let k be a non-negative integer. Recall that $M_k(\Gamma(N))$, the space of Siegel modular forms of degree g, level N, and weight k, consists of holomorphic functions f on $\mathfrak{h}_g$ satisfying

$$ f(\gamma \circ \tau) = j_\gamma(\tau)^k f(\tau), $$

for any $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma(N)$, where $j_\gamma(\tau) = \det(C\tau + D)$. When $g = 1$, we also require that f be analytic on the compactification of $\Gamma(N)\backslash\mathfrak{h}_1$ gotten by adjoining points at the cusps.

Writing $\mathbb{R}^g$ and $\mathbb{Z}^g$ as column vectors, for any $a, b \in \mathbb{R}^g$, $\tau \in \mathfrak{h}_g$, we let

$$ \theta\begin{bmatrix} a \\ b \end{bmatrix}(z, \tau) = \sum_{n \in \mathbb{Z}^g} e^{\pi i\, {}^{\mathrm{t}}(n+a)\tau(n+a) + 2\pi i\, {}^{\mathrm{t}}(n+a)(z+b)} \tag{2} $$

denote the theta function in g variables $z \in \mathbb{C}^g$ with *characteristic vector* $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$. In particular, if $a, b \in \frac{1}{2}\mathbb{Z}^g$, we call $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ a *theta characteristic*. We call a theta characteristic *even* or *odd* depending respectively on whether $\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z, \tau)$ is an even or odd function of z, i.e., whether $e^{4\pi i\, {}^{\mathrm{t}}ab}$ is 1 or -1. We identify theta characteristics mod 1. It follows immediately from (2) that

$$ \theta\begin{bmatrix} a+p \\ b+q \end{bmatrix}(z, \tau) = e^{2\pi i\, {}^{\mathrm{t}}aq}\theta\begin{bmatrix} a \\ b \end{bmatrix}(z, \tau), \tag{3} $$

for $p, q \in \mathbb{Z}^g$. Hence if $a, b \in \frac{1}{m}\mathbb{Z}^g$, then $\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z, \tau)^m$ depends only on $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ mod 1. Therefore we lose at most a sign when we identify theta characteristics mod 1.

For any $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$, theta functions transform as ([I2, pp. 85, 176, 182])

$$ \theta\begin{bmatrix} a \\ b \end{bmatrix}^\gamma ({}^{\mathrm{t}}(C\tau + D)^{-1}z, \gamma \circ \tau) = \zeta(\gamma) j_\gamma(\tau)^{1/2} e^{\pi i\, {}^{\mathrm{t}}z(C\tau+D)^{-1}Cz}\theta\begin{bmatrix} a \\ b \end{bmatrix}(z, \tau), \tag{4} $$

where

$$ \begin{aligned} &\zeta(\gamma) = \varrho(\gamma) e^{-\pi i[{}^{\mathrm{t}}a\, {}^{\mathrm{t}}BDa - 2\,{}^{\mathrm{t}}a\, {}^{\mathrm{t}}BCb + {}^{\mathrm{t}}b\, {}^{\mathrm{t}}ACb - ({}^{\mathrm{t}}a\, {}^{\mathrm{t}}D - {}^{\mathrm{t}}b\, {}^{\mathrm{t}}C)(A\, {}^{\mathrm{t}}B)_0]}, \\ &\varrho(\gamma) = \text{an eighth root of 1 } (= \text{a fourth root of 1 for } \gamma \in \Gamma(2)), \\ &\begin{bmatrix} a \\ b \end{bmatrix}^\gamma = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix}\begin{bmatrix} a \\ b \end{bmatrix} + \frac{1}{2}\begin{bmatrix} (C\, {}^{\mathrm{t}}D)_0 \\ (A\, {}^{\mathrm{t}}B)_0 \end{bmatrix}, \end{aligned} \tag{5} $$

and where for a matrix M, $(M)_0$ denotes the column vector consisting of the diagonal entries of M, and $j_\gamma(\tau)^{1/2}$ is a choice of branch of square root of $j_\gamma(\tau)$.

For $\gamma \in \Gamma$, the map $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]^\gamma$ mod 1 gives an action on characteristic vectors mod 1 we call the *symplectic action*. It is clear that the theta characteristics are stable under the symplectic action, but it can be shown ([I2, p. 213]) that the subsets of even and odd theta characteristics are also left stable by the symplectic action.

For any positive integer m, and theta characteristic δ, we let $\mathrm{prim}(m)$ be the set of characteristic vectors mod 1 defined by

$$\mathrm{prim}(m) = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \bmod 1 \;\middle|\; ma, mb \in \mathbb{Z}^g,\ (ma, mb, m) = 1 \right\},$$

and set

$$\mathrm{char}_\delta(m) = \delta + \mathrm{prim}(m) \bmod 1.$$

Note that if $\sigma_g(m)$ is the cardinality of $\mathrm{prim}(m)$, then

$$\sigma_g(m) = m^{2g} \prod_{p \mid m} \left(1 - \frac{1}{p^{2g}}\right),$$

the product being over all primes dividing m.

Let α be the involution on characteristic vectors mod 1 that sends $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} -a \\ -b \end{smallmatrix}\right]$, and for any set S of characteristic vectors mod 1 upon which α acts, let S/α denote the quotient set of S modulo the action of α.

LEMMA 1. (i) *For m odd, δ a theta characteristic, and $\gamma \in \Gamma(2)$, the symplectic action of γ on characteristic vectors mod* 1 *leaves* $\mathrm{char}_\delta(m)$ *stable. This induces an action of Γ on the sets*

$$\mathrm{even}(m) = \bigcup_{\delta\ \mathit{even}} \mathrm{char}_\delta(m), \qquad \mathrm{odd}(m) = \bigcup_{\delta\ \mathit{odd}} \mathrm{char}_\delta(m).$$

(ii) *For m a multiple of* 4, $\mathrm{char}_\delta(m) = \mathrm{prim}(m)$ *for all theta characteristics δ. Furthermore, the symplectic action on characteristic vectors mod* 1 *gives an action of Γ on* $\mathrm{prim}(m)$.

(iii) *The symplectic action on characteristic vectors mod* 1 *induces an action of $\Gamma(2)$ on* $\mathrm{char}_\delta(m)/\alpha$. *This induces actions of Γ on* $\mathrm{even}(m)/\alpha$ *and* $\mathrm{odd}(m)/\alpha$ *when m is odd, and on* $\mathrm{prim}(m)$ *when m is a multiple of* 4.

REMARK. For $m = 2m'$, m' odd, we have $\mathrm{prim}(m) = \bigcup_{\delta \neq \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]} \mathrm{char}_\delta(m')$, the union being over all non-zero theta characteristics.

Proof. (i) Suppose m is odd, $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in \mathrm{char}_\delta(m)$, so

$$\begin{bmatrix} a \\ b \end{bmatrix} = \delta + \begin{bmatrix} c \\ d \end{bmatrix} \bmod 1,$$

for some $\left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right] \in \mathrm{prim}(m)$. The symplectic action of $\gamma \in \Gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ on $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ from (5) is such that

$$\begin{bmatrix} a \\ b \end{bmatrix}^{\gamma} = \delta^{\gamma} + \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \bmod 1.$$

It is easy to check that the symplectic action of $\Gamma(2)$ fixes δ mod 1, and that the *multiplicative action* on characteristic vectors mod 1 defined by

$$\begin{bmatrix} c \\ d \end{bmatrix} \to \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \bmod 1$$

is an action of Γ on $\mathrm{prim}(m)$. Therefore the symplectic action of $\Gamma(2)$ on characteristic vectors mod 1 defines an action on $\mathrm{char}_{\delta}(m)$. Hence the symplectic action of Γ on characteristic vectors mod 1 defines actions on the sets $\mathrm{even}(m)$ and $\mathrm{odd}(m)$.

(ii) For m a multiple of 4, and δ a theta characteristic, it is clear that $\mathrm{char}_{\delta}(m) = \mathrm{prim}(m)$. For $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$, the multiplicative action on characteristic vectors mod 1 permutes $\mathrm{prim}(m)$, and differs from the symplectic action by the addition of an element in $\frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g$. But $\mathrm{prim}(m)$ is invariant under the addition of such elements.

(iii) We have $\alpha(\delta) = \delta$ for any theta characteristic δ, so α acts on $\mathrm{char}_{\delta}(m)$. It follows as in the proof of (i) that the symplectic action of $\Gamma(2)$ commutes with α, so gives an action on $\mathrm{char}_{\delta}(m)/\alpha$. The rest follows as in the proofs of (i) and (ii).

Let $\mathrm{Prim}(m)$ and $\mathrm{Char}_{\delta}(m)$ denote respectively sets of representatives for the classes of characteristic vectors mod 1 in $\mathrm{prim}(m)$ and $\mathrm{char}_{\delta}(m)$. Let $\mathrm{Prim}(m)/\alpha$ and $\mathrm{Char}_{\delta}(m)/\alpha$ denote respectively sets of representatives for the classes of characteristic vectors mod 1 and modulo α in $\mathrm{prim}(m)/\alpha$ and $\mathrm{char}_{\delta}(m)/\alpha$.

PROPOSITION 1. *Let m be any positive integer.*

(i) *For any theta characteristic δ, and any $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in \mathrm{Char}_{\delta}(m)$,*

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (0,\tau)^{4m} \in M_{2m}(\Gamma(2m)),$$

and is independent of the choice of $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ mod 1.

(ii) *For any theta characteristic δ, and m odd, if*

$$\phi_{\delta,m}(\tau) = \prod_{\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in \mathrm{Char}_{\delta}(m)} \theta \begin{bmatrix} a \\ b \end{bmatrix} (0,\tau),$$

then $(\phi_{\delta,m}(\tau))^{4m}$ *is a modular form of level* 2. *For* m *a multiple of* 4, *if*

$$\phi_m(\tau) = \prod_{\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in \mathrm{Prim}(m)} \theta \begin{bmatrix} a \\ b \end{bmatrix} (0,\tau),$$

then $(\phi_m(\tau))^{4m}$ *is a modular form of level* 1.

(iii) *For* m *odd, if*

$$\phi_{even,m}(\tau) = \prod_{\delta \text{ } even} \phi_{\delta,m}(\tau) \quad and \quad \phi_{odd,m}(\tau) = \prod_{\delta \text{ } odd} \phi_{\delta,m}(\tau),$$

then $(\phi_{odd,m}(\tau))^{4m}$ *and* $(\phi_{even,m}(\tau))^{4m}$ *are modular forms of level* 1.

(iv) *If* m *is odd, take*

$$f(\tau) = \psi_{even,m}(\tau) = \prod_{\delta \text{ } even} \prod_{\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in \mathrm{Char}_\delta(m)/\alpha} \theta \begin{bmatrix} a \\ b \end{bmatrix} (0,\tau), \quad or$$

$$f(\tau) = \psi_{odd,m}(\tau) = \prod_{\delta \text{ } odd} \prod_{\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in \mathrm{Char}_\delta(m)/\alpha} \theta \begin{bmatrix} a \\ b \end{bmatrix} (0,\tau).$$

If m *is a multiple of* 4 *take*

$$f(\tau) = \psi_m(\tau) = \prod_{\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in \mathrm{Prim}(m)/\alpha} \theta \begin{bmatrix} a \\ b \end{bmatrix} (0,\tau).$$

- *If* $g = 1$ *and* $m \geq 3$, *then* $f(\tau)^{\gcd(8m,12)}$ *is a modular form of level* 1.
- *If* $g = m = 1$, *then* $f(\tau)^8$ *is a modular form of level* 1.
- *If* $g = 2$, *then* $f(\tau)^2$ *is a modular form of level* 1.
- *If* $g \geq 3$, *then* $f(\tau)$ *is a modular form of level* 1.

Proof. (i) Since $\Gamma(2m) \subset \Gamma(2)$, for $\gamma \in \Gamma(2m)$, $\varrho(\gamma)^4 = 1$. It is easy to verify then that $\zeta(\gamma)^{4m} = 1$. Further, $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]^\gamma \equiv \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ mod 1, so by (3) and (4),

$$j_\gamma(\tau)^{2m} \theta \begin{bmatrix} a \\ b \end{bmatrix} (0,\tau)^{4m} = \left(\theta \begin{bmatrix} a \\ b \end{bmatrix}^\gamma (0, \gamma \circ \tau) \right)^{4m} = \left(\theta \begin{bmatrix} a \\ b \end{bmatrix} (0, \gamma \circ \tau) \right)^{4m},$$

and by (3), $\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](0,\tau)^{4m}$ depends only on $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ mod 1.

(ii) By Lemma 1, and part (i), this is just the product of modular forms which are permuted under the action of $\Gamma(2)/\Gamma(2m)$ (or $\Gamma/\Gamma(2m)$ for m a multiple of 4), where the action is $f(\tau) \mapsto f(\gamma \circ \tau)/j_\gamma(\tau)^{2m}$. This drops the level to 2 for m odd, and to 1 for m a multiple of 4.

(iii) For m odd, as in (ii), this is just the product under the action of $\Gamma/\Gamma(2)$ of modular forms on $\Gamma(2)$, which drops the level to 1.

(iv) If $g(\tau)^n$ is a modular form of weight nk, k an integer, then the map $\gamma \mapsto g(\gamma \circ \tau)/(g(\tau) j_\gamma(\tau)^k)$ is a character on Γ. Let m be odd or a multiple of 4.

Since $\theta\left[\begin{smallmatrix}-a\\-b\end{smallmatrix}\right](0,\tau) = \theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](0,\tau)$ for $m \geq 3$, $f(\tau)^2$ differs by at most a multiplicative constant from $\phi_{\mathrm{odd},m}(\tau)$, $\phi_{\mathrm{even},m}(\tau)$, or $\phi_m(\tau)$. So for $m \geq 3$, since it is easy to check that $\sigma_{2g}(m)$ is a multiple of 4, we see from (4) that $f(\tau)^{8m}$ is a modular form whose weight is divisible by $8m$. It is known ([M, p. 169]) that the number of even and odd theta characteristics is $2^{g-1}(2^g+1)$ and $2^{g-1}(2^g-1)$, respectively. Hence when $m = 1$, since α pointwise fixes theta characteristics, $f(\tau)$ differs by at most a multiplicative constant from $\phi_{\mathrm{odd},1}(\tau)$ or $\phi_{\mathrm{even},1}(\tau)$, so if $g > 1$, $f(\tau)^4$ is a modular form whose weight is divisible by 4.

So in any case, unless $g = m = 1$, we find that $f(\tau)$ is a modular form with character on Γ. For $g = 2$ every character of Γ is of order dividing 2; and for $g \geq 3$ there are no non-trivial characters of Γ ([K, pp. 43–44]). For $g = 1$, [Leh, p. 349] shows that for $m \geq 3$, $f(\tau)$ times some power of $\eta(\tau)^2$ is a modular form, so $f(\tau)^{12}$ is a modular form.

Finally, if $g = m = 1$, $\psi_{\mathrm{odd},1}(\tau) = 0$ and Lemma 2(i) below shows that $\psi_{\mathrm{even},1}(\tau)$ is a constant multiple of $\eta(\tau)^3$, so $\psi_{\mathrm{even},1}(\tau)^8$ is a modular form.

Remark. We will see in Proposition 2 that if $g = 1$, and $c(f)$ is the number of theta functions in the product defining $f(\tau)$ in Proposition 1(iv), then $f(\tau)$ is a constant times $\eta(\tau)^{c(f)}$.

2. Theta functions in one variable. Here $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. The only odd theta characteristic is represented by $\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right]$. We take $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]$, and $\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right]$ as representatives for the three even theta characteristics. We recall some classic facts about modular forms of degree 1 (see, e.g. [M]). For $\tau \in \mathfrak{h} = \mathfrak{h}_1$, set $q = e^{2\pi i\tau}$. For any modular form, its "q-expansion" is its Fourier series at "$i\infty$" in q. If $f(\tau)$ is holomorphic on $\mathfrak{h}$ and $f(\tau)^n$ is a modular form, then $f(\tau)$ has a q-expansion in $q^{1/n} = e^{2\pi i\tau/n}$. The exponent of q in the lead term of the q-expansion is the order of zero of a form at $i\infty$. Recall we define

$$\eta(\tau) = q^{1/24}\prod_{n>0}(1-q^n) \quad \text{and} \quad \Delta(\tau) = \eta(\tau)^{24}.$$

We let $\theta'\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](z,\tau)$ denote $\frac{d}{dz}\theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](z,\tau)$.

Lemma 2. (i) *We have*

$$\frac{1}{2\pi i}\theta'\begin{bmatrix}1/2\\1/2\end{bmatrix}(0,\tau) = \frac{i}{2}\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}1/2\\0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\1/2\end{bmatrix}(0,\tau) = i\eta^3(\tau).$$

(ii) $\Delta(\tau)$ *is a modular form of level* 1 *and weight* 12. *It has no zeros on* $\mathfrak{h}$ *and a simple zero at* $i\infty$.

(iii) *The only modular forms of any level which have weight* 0 *are constants.*

Proof. See [M, p. 42 and pp. 64–72]. (i) is Jacobi's derivative formula.

For any positive integer m, we define

$$\operatorname{prod}\begin{bmatrix}1/2\\1/2\end{bmatrix}(m)(\tau)=\prod_{\substack{0\le u,v<m\\(u,v)\neq(0,0)}}\theta\begin{bmatrix}1/2+u/m\\1/2+v/m\end{bmatrix}(0,\tau),$$

$$\operatorname{prod}\begin{bmatrix}0\\0\end{bmatrix}(m)(\tau)=\prod_{\substack{0\le u,v<m\\(u,v)\neq(m/2,m/2),\,m\text{ even}\\(u,v)\neq(0,0),\,m\text{ odd}}}\theta\begin{bmatrix}u/m\\v/m\end{bmatrix}(0,\tau),$$

$$\operatorname{prod}\begin{bmatrix}1/2\\0\end{bmatrix}(m)(\tau)=\prod_{\substack{0\le u,v<m\\(u,v)\neq(0,m/2),\,m\text{ even}\\(u,v)\neq(0,0),\,m\text{ odd}}}\theta\begin{bmatrix}1/2+u/m\\v/m\end{bmatrix}(0,\tau),$$

$$\operatorname{prod}\begin{bmatrix}0\\1/2\end{bmatrix}(m)(\tau)=\prod_{\substack{0\le u,v<m\\(u,v)\neq(m/2,0),\,m\text{ even}\\(u,v)\neq(0,0),\,m\text{ odd}}}\theta\begin{bmatrix}u/m\\1/2+v/m\end{bmatrix}(0,\tau).$$

LEMMA 3. *The following are lead terms of q-expansions*:

	m odd	*m even*
$\operatorname{prod}\begin{bmatrix}1/2\\1/2\end{bmatrix}(m)(\tau)$:	$mq^{(m^2-1)/24}$	$-mq^{(m^2-1)/24}$
$\operatorname{prod}\begin{bmatrix}0\\0\end{bmatrix}(m)(\tau)$:	$q^{(m^2-1)/24}$	$(-1)^{(m-2)/2}mq^{(m^2-1)/24}$
$\operatorname{prod}\begin{bmatrix}1/2\\0\end{bmatrix}(m)(\tau)$:	$(-1)^{(m-1)/2}q^{(m^2-1)/24}$	$(-1)^{(m-2)/2}mq^{(m^2-1)/24}$
$\operatorname{prod}\begin{bmatrix}0\\1/2\end{bmatrix}(m)(\tau)$:	$q^{(m^2-1)/24}$	$-mq^{(m^2-1)/24}$

Proof. For any theta function $\theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](0,\tau)$ with $a,b\in\mathbb{Q}$, we can compute its q-expansion directly from its definition (2). Alternatively, one can use the product expansion for theta functions (see [M, p. 69]). We leave the verification of the lemma to the reader.

PROPOSITION 2.

	m odd	*m even*
$\operatorname{prod}\begin{bmatrix}1/2\\1/2\end{bmatrix}(m)(\tau)=$	$m\eta(\tau)^{m^2-1}$	$-m\eta(\tau)^{m^2-1}$
$\operatorname{prod}\begin{bmatrix}0\\0\end{bmatrix}(m)(\tau)=$	$\eta(\tau)^{m^2-1}$	$(-1)^{(m-2)/2}m\eta(\tau)^{m^2-1}$

$$\text{prod}\begin{bmatrix}1/2\\0\end{bmatrix}(m)(\tau) = \quad (-1)^{(m-1)/2}\eta(\tau)^{m^2-1} \quad (-1)^{(m-2)/2}m\eta(\tau)^{m^2-1}$$

$$\text{prod}\begin{bmatrix}0\\1/2\end{bmatrix}(m)(\tau) = \quad \eta(\tau)^{m^2-1} \quad -m\eta(\tau)^{m^2-1}$$

Proof. First let us consider prod $\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right](m)(\tau)$. Since $\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right]$ represents the only odd theta characteristic when $g = 1$, applying Proposition 1 to all the factors on the right hand side of

$$\left(\text{prod}\begin{bmatrix}1/2\\1/2\end{bmatrix}(m)(\tau)\right)^{4m} = \Bigl(\prod_{d|m,\,4|d}\phi_d(\tau)\cdot\prod_{2d'|m,\,d'\text{ odd}}\phi_{\text{even},d'}(\tau)\cdot\prod_{d|m,\,d>1\text{ odd}}\phi_{\text{odd},d}(\tau)\Bigr)^{4m},$$

we see that $\left(\text{prod}\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right](m)(\tau)\right)^{4m}$ is a modular form of level 1, and weight $2m(m^2-1)$. By Lemma 3, the lead term of its q-expansion is a constant times $q^{m(m^2-1)/6}$. Therefore, by Lemma 2,

$$\frac{\left(\text{prod}\begin{bmatrix}1/2\\1/2\end{bmatrix}(m)(\tau)\right)^{4m}}{\varDelta(\tau)^{m(m^2-1)/6}}$$

is a modular form of level 1 and weight 0, and hence a constant. Since $\mathfrak{h}$ is connected, prod $\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right](m)(\tau)$ and $\eta(\tau)^{m^2-1}$ differ only by a constant. The constant is determined by the q-expansion in Lemma 3.

For any $\delta = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right], \left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right]$, all of which represent even theta characteristics, the same argument only shows, a priori, that $(\text{prod}[\delta](m)(\tau))^{4m}$ is of level 2. But since each image under the action of $\varGamma$ has a q-expansion whose lead term is a constant times $q^{m(m^2-1)/6}$, we deduce again that $(\text{prod}[\delta](m)(\tau))^{4m}/\varDelta(\tau)^{m(m^2-1)/6}$ is a modular form of level 2 and weight 0, and hence a constant. Therefore, again $\text{prod}[\delta](m)(\tau)$ and $\eta(\tau)^{m^2-1}$ differ only by a constant, determined by the q-expansions in Lemma 3.

Proposition 3. *Let m be any positive integer.*

(i) *For any theta characteristic δ, and any $\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right] \in \text{Char}_\delta(m)$,*

$$\theta'\begin{bmatrix}a\\b\end{bmatrix}(0,\tau)^{4m} \in M_{6m}(\varGamma(2m)),$$

and is independent of the choice of $\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]$ mod 1.

(ii) *For any theta characteristic* δ, *and* m *odd, if*

$$\Phi_{\delta,m}(\tau) = \prod_{\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in \mathrm{Char}_\delta(m)} \theta' \begin{bmatrix} a \\ b \end{bmatrix} (0, \tau),$$

then $(\Phi_{\delta,m}(\tau))^{4m}$ *is a modular form of level* 2. *For* m *a multiple of* 4, *if*

$$\Phi_m(\tau) = \prod_{\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in \mathrm{Prim}(m)} \theta' \begin{bmatrix} a \\ b \end{bmatrix} (0, \tau),$$

then $(\Phi_m(\tau))^{4m}$ *is a modular form of level* 1.

(iii) *For* m *odd, if*

$$\Phi_{even,m}(\tau) = \prod_{\delta\ even} \Phi_{\delta,m}(\tau) \quad \mathit{and} \quad \Phi_{odd,m}(\tau) = \Phi_{\delta,m}(\tau)$$

for $\delta = \left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right] \bmod 1$, *then* $(\Phi_{odd,m}(\tau))^{4m}$ *and* $(\Phi_{even,m}(\tau))^{4m}$ *are modular forms of level* 1.

(iv) *If* m *is odd, take*

$$F(\tau) = \Psi_{even,m}(\tau) = \prod_{\delta\ even} \prod_{\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in \mathrm{Char}_\delta(m)/\alpha} \theta' \begin{bmatrix} a \\ b \end{bmatrix} (0, \tau), \quad \mathit{or}$$

$$F(\tau) = \Psi_{odd,m}(\tau) = \prod_{\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in \mathrm{Char}_\delta(m)/\alpha} \theta' \begin{bmatrix} a \\ b \end{bmatrix} (0, \tau)$$

for $\delta = \left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right] \bmod 1$. *If* m *is a multiple of* 4, *take*

$$F(\tau) = \Psi_m(\tau) = \prod_{\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in \mathrm{Prim}(m)/\alpha} \theta' \begin{bmatrix} a \\ b \end{bmatrix} (0, \tau).$$

Then if $m \geq 3$, $F(\tau)$ *times some power of* $\eta(\tau)^2$ *is a modular form of level* 1, *so* $F(\tau)^{\gcd(8m,12)}$ *is a modular form of level* 1. *If* $m = 1$, $F(\tau)^8$ *is a modular form of level* 1.

Proof. These follow just as Proposition 1 by differentiating (4) and using the resulting formula at $z = 0$.

Unlike the products in Proposition 2, the products $F(\tau)$ in Proposition 3(iv) are not necessarily constants times a power of $\eta(\tau)$. However, we will show in Theorem 1 that this is true for $m = 3$ and $m = 4$. For an analysis of these products for all m, see [BG].

For any representative $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ of a theta characteristic, we let

$$\mathrm{derivprod} \begin{bmatrix} a \\ b \end{bmatrix} (3)(\tau) = \prod_{\substack{0 \leq u,v < 3 \\ (u,v) \neq (0,0)}} \theta' \begin{bmatrix} a + u/3 \\ b + v/3 \end{bmatrix} (0, \tau),$$

and set

$$\operatorname{derivprod}(4)(\tau) = \prod_{\substack{0\le u,v<4 \\ (u,v)\ne(0,0),(0,2),(2,0),(2,2)}} \theta'\begin{bmatrix} u/4 \\ v/4 \end{bmatrix}(0,\tau).$$

Part (ii) of the following theorem can be considered a generalization of Jacobi's derivative formula (Lemma 2(i)).

THEOREM 1. *For* $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right]$:

(i) *The lead term of the q-expansions of* $\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](0,\tau)^8 \operatorname{derivprod}\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](3)(\tau)$ *is*

$$\frac{(-1)^{2a+1} 2^8 \pi^8}{3^5} \cdot q^{4/3}.$$

The lead term of the q-expansion of derivprod$(4)(\tau)$ *is* $(-\pi^{12}/2^3) q^{3/2}$.

(ii)

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(0,\tau)^8 \operatorname{derivprod}\begin{bmatrix} a \\ b \end{bmatrix}(3)(\tau) = \frac{(-1)^{2a+1} 2^8 \pi^8}{3^5}\, \eta(\tau)^{32}$$

and

$$\operatorname{derivprod}(4)(\tau) = \frac{-\pi^{12}}{2^3} \eta(\tau)^{36}.$$

Proof. (i) This is a computation whose verification we leave to the reader. (ii) This is entirely similar to the proof of Proposition 2.

3. Theta functions in two variables. Here $\Gamma = \mathrm{Sp}_4(\mathbb{Z})$. The structure of the ring $\bigcup_{k\ge 0} M_k(\Gamma)$ was determined by Igusa [I1] and subsequently by Hammond [H], and Freitag [F].

There are six odd theta characteristics, represented by

$$\begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}.$$

Representatives for the 10 even theta characteristics are

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{bmatrix},$$

$$\begin{bmatrix} 0 \\ 1/2 \\ 1/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}.$$

We write

$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix} \qquad \text{for } \tau \in \mathfrak{h}_2.$$

Let Z denote the image of the subvariety $\tau_{12} = 0$ of $\mathfrak{h}_2$ under the action by Γ.

We define

$$\Delta_2(\tau) = 2^{-12} \prod_{\delta \text{ even}} \theta[\delta]^2(0,\tau).$$

We need to accumulate some facts.

LEMMA 4. (i) *$\Delta_2(\tau)$ is a modular form of level 1 and weight 10, which has zeros of order 2 along Z and no other zeros.*

(ii) *A modular form of any level which is of weight 0 is a constant.*

Proof. These can be found in [K, pp. 115, 119].

We call $\Delta_2(\tau)$ the *discriminant modular form* (of degree 2). The reason for the name is that via Thomae's formula, it can be shown for $\tau \notin Z$ that $\Delta_2(\tau)$ differs only by a multiplicative constant from the discriminant of the curve of genus 2 whose period matrix is τ (see [Gr3]).

It follows from the definition (2) that if $\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix} \in \mathfrak{h}_2$, then for $a = \left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right]$, $b = \left[\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right]$, $a_i, b_i \in \mathbb{R}$, $i = 1, 2$, we have

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(0,\tau)\bigg|_{\tau_{12}=0} = \theta\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}(0,\tau_{11})\,\theta\begin{bmatrix} a_2 \\ b_2 \end{bmatrix}(0,\tau_{22}). \tag{6}$$

Recall that $\theta\left[\begin{smallmatrix} a_i \\ b_i \end{smallmatrix}\right](0,\tau_{ii}) = 0$ if and only if $\left[\begin{smallmatrix} a_i \\ b_i \end{smallmatrix}\right] \equiv \left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right] \bmod 1$ ([M, p. 11]).

THEOREM 2. *For any odd theta characteristic $\delta = \left[\begin{smallmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{smallmatrix}\right]$,*

$$f_\delta(\tau) := \prod_{\substack{0 \le u_i, v_i < 3 \\ (u_1,u_2,v_1,v_2) \ne (0,0,0,0)}} \theta\begin{bmatrix} \delta + \begin{matrix} u_1/3 \\ u_2/3 \\ v_1/3 \\ v_2/3 \end{matrix} \end{bmatrix}(0,\tau) = c_3(\delta)\Delta_2(\tau)^4,$$

where $c_3(\delta) = (-1)^{2a_1+2a_2}3^4$, and

$$g(\tau) := \prod_{\substack{0 \le u_i, v_i < 4 \\ (u_1,u_2,v_1,v_2) \text{ not all even}}} \theta\begin{bmatrix} u_1/4 \\ u_2/4 \\ v_1/4 \\ v_2/4 \end{bmatrix}(0,\tau) = c_4 \cdot \Delta_2(\tau)^{12},$$

where $c_4 = 2^{24}$.

Proof. By Proposition 1, $f_\delta(\tau)^{12}$ is a modular form of level 2 and weight 480. Note that by (6), for every odd characteristic δ, 8 terms in the product $f_\delta(\tau)$ vanish when $\tau_{12} = 0$. Since the 6 choices of $f_\delta(\tau)^{12}$ are permuted by Γ, for each δ, $f_\delta(\tau)^{12}$ vanishes at least to order 96 on Z. So by Lemma 4, $f_\delta(\tau)^{12}/(\Delta_2(\tau))^{48}$ is a modular form of level 2 and weight 0, which is necessarily a constant. Therefore $f_\delta(\tau)^{12}$ differs by a constant from $\Delta_2(\tau)^{48}$, and since $\mathfrak{h}_2$ is connected, $f_\delta(\tau)$ differs by a constant from $\Delta_2(\tau)^4$. A calculation with (3) shows that the constant is independent of the choice of representative for δ mod 1.

By Proposition 1, $g(\tau)^2$ is a modular form of weight 240 and level 1. Of the 240 terms in the product g, 24 vanish along Z. Therefore $g(\tau)^2/\Delta_2(\tau)^{24}$ is a modular form of weight 0 and level 1, and hence a constant. Therefore $g(\tau)$ and $\Delta_2(\tau)^{12}$ differ by a constant.

It remains to compute $c_3(\delta)$ and c_4. For this we need to take the Taylor expansion of $\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](0,\tau)$ in τ_{12} at 0. If the function does not vanish on $\tau_{12} = 0$, the lead term in the expansion is given by (6). If it does vanish, the lead term is given by τ_{12} times

$$\frac{d}{d\tau_{12}}\,\theta\begin{bmatrix} a \\ b \end{bmatrix}(0,\tau)\bigg|_{\tau_{12}=0} = \frac{1}{2\pi i}\theta'\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}(0,\tau_{11})\theta'\begin{bmatrix} a_2 \\ b_2 \end{bmatrix}(0,\tau_{22}).$$

For starters, we compute the lead term of the Taylor expansion of $\Delta_2(\tau)$ as

$$\begin{aligned} 2^{-12}(2\pi i)^2 &\left(\frac{1}{2\pi i}\,\theta'\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}(0,\tau_{11})\right)^2 \left(\frac{1}{2\pi i}\theta'\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}(0,\tau_{22})\right)^2 \\ &\times \left(\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(0,\tau_{11})\theta\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}(0,\tau_{11})\theta\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}(0,\tau_{11})\right)^6 \\ &\times \left(\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(0,\tau_{22})\theta\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}(0,\tau_{22})\theta\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}(0,\tau_{22})\right)^6 (\tau_{12})^2 \\ = &- 2^2\pi^2\Delta(\tau_{11})\Delta(\tau_{22})(\tau_{12})^2, \end{aligned}$$

by Lemma 2.

The formulas in Section 2 now give enough ammunition to calculate $c_3(\delta)$ and c_4.

We will compute the Taylor expansion of $f_{\delta_0}(\tau)$ when $\delta_0 = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \\ 1/2 \end{bmatrix}$. The other choices for odd theta characteristics are treated similarly. The lead term of the expansion is

$$\left(\frac{1}{2\pi i}\,\theta'\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}(0,\tau_{22})\right)^8 \theta\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}(0,\tau_{11})^8\,\text{derivprod}\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}(3)(\tau_{11})$$

$$\times\left(\operatorname{prod}\begin{bmatrix}1/2\\1/2\end{bmatrix}(3)(\tau_{22})\right)^9\left(\operatorname{prod}\begin{bmatrix}1/2\\0\end{bmatrix}(3)(\tau_{11})\right)^8(\tau_{12})^8$$
$$=3^4 2^8\pi^8\Delta(\tau_{11})^4\Delta(\tau_{22})^4(\tau_{12})^8,$$

so $c_3(\delta_0)=3^4$.

Finally, the lead term of the Taylor expansion of $g(\tau)$ is

$$\left(\frac{1}{2\pi i}\theta'\begin{bmatrix}1/2\\1/2\end{bmatrix}(0,\tau_{11})\right)^{12}\left(\frac{1}{2\pi i}\theta'\begin{bmatrix}1/2\\1/2\end{bmatrix}(0,\tau_{22})\right)^{12}$$
$$\times\operatorname{derivprod}(4)(\tau_{11})\operatorname{derivprod}(4)(\tau_{22})$$
$$\times\left(\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau_{11})\theta\begin{bmatrix}0\\1/2\end{bmatrix}(0,\tau_{11})\theta\begin{bmatrix}1/2\\0\end{bmatrix}(0,\tau_{11})\right)^{-3}$$
$$\times\left(\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau_{22})\theta\begin{bmatrix}0\\1/2\end{bmatrix}(0,\tau_{22})\theta\begin{bmatrix}1/2\\0\end{bmatrix}(0,\tau_{22})\right)^{-3}$$
$$\times\left(\operatorname{prod}\begin{bmatrix}0\\0\end{bmatrix}(4)(\tau_{11})\right)^{15}\left(\operatorname{prod}\begin{bmatrix}0\\0\end{bmatrix}(4)(\tau_{22})\right)^{15}(\tau_{12})^{24}$$
$$=2^{48}\pi^{24}\Delta(\tau_{11})^{12}\Delta(\tau_{22})^{12}(\tau_{12})^{24},$$

so $c_4=2^{24}$.

Remark. For any $\tau\in\mathfrak{h}_2$ not in Z, τ is the period matrix of some complex curve $\mathcal{C}$ of genus 2. The curve has six Weierstrass points, w_k, $1\le k\le 6$, and the canonical divisor class is $2w_k$ for any k. Fix one choice of k. We can pick a symplectic basis A_1,A_2,B_1,B_2 for $H_1(\mathcal{C},\mathbb{Z})$ (i.e., such that $A_1\cdot A_2=B_1\cdot B_2=0$, $A_i\cdot B_j=\delta_{ij}$), and a normalized basis μ_1,μ_2 of holomorphic differentials of $\mathcal{C}$ such that

$$\Big[\int_{A_i}\mu_j\Big]_{i,j=1,2}=I,\qquad\Big[\int_{B_i}\mu_j\Big]_{i,j=1,2}=\tau.$$

Then we have an embedding

$$\mathcal{C}\xrightarrow{\Phi_k}\mathbb{C}^2/L$$

given by

$$P\mapsto\int_{w_k}^{P}(\mu_1,\mu_2)\bmod L,$$

where L is the lattice in $\mathbb{C}^2$ generated by the columns of I and τ. (For background and details, see [Gr5].) The map Φ_k extends by linearity to divisors of $\mathcal{C}$, and the Abel–Jacobi Theorem says that if D is a divisor of degree 0, then D is the divisor of a function if and only if $\Phi_k(D)$ is the origin in $\mathbb{C}^2/L$. It follows that $\Phi_k(w_j)$, $j = 1, \ldots, 6$, are precisely the 2-torsion points of $\mathbb{C}^2/L$ which lie on $\Phi_k(\mathcal{C})$.

A fundamental theorem of Riemann says that there is an odd theta characteristic $\delta = \delta(k)$ such that $\theta[\delta](z, \tau)$, $z \in \mathbb{C}^2$, has a zero of order 1 along the pullback of $\Phi_k(\mathcal{C})$ to $\mathbb{C}^2$ and no other zeros. For $a, b \in \mathbb{R}^2$, since $\theta\big[\delta + {a \atop b}\big](0, \tau)$ differs by an exponential from $\theta[\delta](\tau a + b, \tau)$, we see that $\theta\big[\delta + {a \atop b}\big](0, \tau) = 0$ if and only if $\tau a + b \in \Phi_k(\mathcal{C})$. Theorem 2 says $\theta\big[\delta + {c \atop d}\big](0, \tau) \neq 0$ for $\tau \notin Z$, when $3c \equiv 3d \equiv 0 \bmod 1$, and c or $d \not\equiv 0 \bmod 1$, and that $\theta\big[{c \atop d}\big](0, \tau) \neq 0$ for $\tau \notin Z$ when $4c \equiv 4d \equiv 0 \bmod 1$, and $2c$ or $2d \not\equiv 0 \bmod 1$. With this we get

Corollary. *There is no point P on $\mathcal{C}$, $P \neq w_k$, such that $3(P - w_k)$ is the divisor of a function, and there is no point P on $\mathcal{C}$, $P \neq w_j$, $1 \leq j \leq 6$, such that $4(P - w_k)$ is the divisor of a function.*

This corollary can easily be derived from the Riemann–Roch Theorem (see, e.g. [Box]). Having done so directly, Goren gave a moduli-theoretic proof of Theorem 2 up to an unspecified constant [Go]. Likewise, we can see that there is no analogue of Theorem 2 for $m = 5$, because on the curve $\mathcal{C} : y^2 = x^5 + 1$, the divisor of $y - 1$ is $5((0, 1) - \infty)$, where ∞ denotes the Weierstrass point at infinity (see [BG]). See [BGL] for a complete description of the moduli space of curves of genus 2 such that there is a $P \in C$, $P \neq \infty$, such that $5(P - \infty)$ is the divisor of a function.

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