J. reine angew. Math. **392** (1988), 125—136

Journal für die reine und angewandte Mathematik

A generalization of Jacobi's derivative formula to dimension two

By *David Grant* at Ann Arbor

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0. Introduction

For any $g \geqq 1$, let $\tau \in \mathfrak{H}_g$, the Siegel upper-half space of dimension g, z a column vector in $\mathbb{C}^g$, and a and b column vectors in $\frac{1}{2}\mathbb{Z}^g$. The g-dimensional theta function with theta characteristic $\begin{bmatrix} a \\ b \end{bmatrix}$ is given by

$$(0.\ 1) \qquad \theta\begin{bmatrix} a \\ b \end{bmatrix}(z, \tau) = \sum_{n \in \mathbb{Z}^g} e^{\pi i\, {}^t(n+a)\tau(n+a) + 2\pi i\, {}^t(n+a)(z+b)}.$$

It follows from the definition that $\theta\begin{bmatrix} a \\ b \end{bmatrix}(-z, \tau) = e^{4\pi i\, {}^t ab}\, \theta\begin{bmatrix} a \\ b \end{bmatrix}(z, \tau)$, and $\begin{bmatrix} a \\ b \end{bmatrix}$ is called an even or odd theta characteristic depending on whether $\theta\begin{bmatrix} a \\ b \end{bmatrix}(z, \tau)$ is an even or odd function.

One of the central results in the theory of one-dimensional theta functions is Jacobi's derivative formula, which states that for any τ in the upper-half plane,

$$(0.\ 2) \qquad \theta'\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}(0, \tau) = -\pi\,\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(0, \tau)\, \theta\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}(0, \tau)\, \theta\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}(0, \tau).$$

Various 19th-century authors provided generalizations of (0. 2) to g-dimensional theta functions, expressing the Jacobian of g distinct odd theta functions at zero explicitly as a rational function in the Thetanullwerte. In recent years there has been renewed interest in the problem, and Igusa has succeeded in determining for any g which sets of odd theta functions have a jacobian which at $z=0$ is a polynomial in the Thetanullwerte [6].

When $g=2$, all the formulas have the following form: If α_1 and α_2 are any two distinct odd theta characteristics, then

$$\begin{vmatrix} \frac{\partial}{\partial z_1}\theta[\alpha_1](0,\tau) & \frac{\partial}{\partial z_1}\theta[\alpha_2](0,\tau) \\ \frac{\partial}{\partial z_2}\theta[\alpha_1](0,\tau) & \frac{\partial}{\partial z_2}\theta[\alpha_2](0,\tau) \end{vmatrix} = \pm\pi^2 \prod_{i=1}^{4} \theta[\beta_i](0,\tau),$$

where the β_i are even theta characteristics depending on α_1 and α_2. This was stated by Rosenhain (and apparently also by Riemann), and proved by Weber, Thomae, and Frobenius. We refer the reader to [2], [3], [5], and [6] for the statements and history of the generalizations to other dimensions.

In this paper we offer a different generalization of (0. 2) for $g=2$, relating the derivatives of just one odd theta function to the product of all ten even theta functions (Thm. 2. 11).

Formula (0. 2) has an interesting interpretation as a relation between functions defined on the moduli space of elliptic curves: raising $\theta'\begin{bmatrix}1/2\\1/2\end{bmatrix}(0,\tau)$ to the eighth power yields a cusp form of weight 12; hence it differs only by a scaling factor from the classical Δ-function which assigns to τ the discriminant of the elliptic curve corresponding to τ.

Theorem 2. 11 will hold only for $\tau\in\mathfrak{H}_2$ which are period matrices of curves of genus two, and it is the analogue of (0. 2) for the moduli space of curves of genus two. Indeed, for such $\tau\in\mathfrak{H}_2$, the product of the squares of all the even theta functions differs only by a scaling factor from the discriminant of the curve corresponding to τ (Prop. 2. 5).

The proofs of the formulas will depend on the function theory of curves of genus two and their Jacobian varieties, and on the Taylor expansion for the 2-dimensional σ-function carried out (up to a constant) in [1]. For the convenience of the reader, we will rederive the Taylor expansion in section one before proving the main formula in section two.

1. Prerequisites

Let $\mathscr{C}$ be a curve of genus 2 defined over $\mathbb{C}$ by the equation

$$(1.\ 1)\qquad y^2 = x^5 + b_1x^4 + b_2x^3 + b_3x^2 + b_4x + b_5 = \prod_{i=1}^{5}(x-a_i).$$

The function x on $\mathscr{C}$ defines a 2 to 1 cover of $\mathbb{P}^1$, branched over ∞ and a_i $(1\leq i\leq 5)$. We think of points on $\mathscr{C}$ as pairs (x, y) with the hyperelliptic involution on $\mathscr{C}$ mapping a point $P=(x,y)$ to $\bar{P}=(x,-y)$.

A basis for the differentials of the first kind is given by $\mu_1 = \frac{dx}{y}$ and $\mu_2 = \frac{x\,dx}{y}$. Given an ordering of the six branch points, there is a standard and canonical way to pick a symplectic basis for $H_1(\mathscr{C}, \mathbb{Z})$; that is, a basis $\{A_1, A_2, B_1, B_2\}$, such that $A_1 \cdot A_2 = B_1 \cdot B_2 = 0$, and $A_i \cdot B_j = \delta_{ij}$, (see [9], pp. 3. 75—3. 77).

The period matrices ω, ω', and τ are defined by

$$\omega_{ij} = \int_{A_j} \mu_i, \quad \omega'_{ij} = \int_{B_j} \mu_i \quad (i, j = 1, 2), \quad \text{and} \quad \tau = \omega^{-1}\omega'.$$

Standard calculations show that $\det \omega \neq 0$ and that $\tau \in \mathfrak{H}_2$.

Let L denote the lattice generated in $\mathbb{C}^2$ by the columns of ω and ω'. Algebraically, the Jacobian J of $\mathscr{C}$ can be described as the symmetric product $\mathscr{C}^{(2)}$ with the locus of unordered pairs $\{(P, \bar{P}) \mid P \in \mathscr{C}\}$ blown down to the origin [8]. The Jacobian J will be identified with $\mathbb{C}^2/L$ via the map

$$(P_1, P_2) \xrightarrow{\Phi} \int_\infty^{P_1} + \int_\infty^{P_2} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \text{ modulo } L.$$

The curve $\mathscr{C}$ can be embedded into J by

$$P \mapsto \Phi(P, \infty).$$

Its image will be denoted by Θ, the theta divisor of J.

A fundamental theorem of Riemann states that there is an odd theta characteristic δ such that $\theta[\delta](z, \tau)$ has a zero of order one precisely along the pullback of Θ to $\mathbb{C}^2$. A calculation [9], pp. 3. 80—3. 85, shows that with the standard choice of symplectic basis for $H_1(\mathscr{C}, \mathbb{Z})$, $\delta = \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \\ 1/2 \end{bmatrix}$.

For any theta characteristic $\begin{bmatrix} a \\ b \end{bmatrix}$, $\theta\begin{bmatrix} a \\ b \end{bmatrix}(z, \tau)$ is analytic. It follows directly from (0. 1) that it satisfies the following:

Translational Formula 1. 2. *Let p and q be column vectors in $\mathbb{Z}^2$. Then*

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z + \tau p + q, \tau) = e^{-\pi i\, {}^t p \tau p - 2\pi i\, {}^t p(z+b) + 2\pi i\, {}^t a q}\, \theta\begin{bmatrix} a \\ b \end{bmatrix}(z, \tau). \tag{1. 2}$$

The differentials of the second kind on $\mathscr{C}$,

$$\zeta_1 = \frac{(3x^3 + 2b_1 x^2 + b_2 x)\,dx}{4y}, \quad \text{and} \quad \zeta_2 = \frac{x^2\,dx}{4y},$$

are used to form a matrix η of quasiperiods

$$\eta_{ij} = \int_{A_j} \zeta_i \qquad (i, j = 1, 2).$$

Let $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ be in $\mathbb{C}^2$. The 2-dimensional σ-function is defined by

$$(1.3) \qquad \sigma(z) = e^{-1/2 {}^t z \eta \omega^{-1} z}\, \theta[\delta](\omega^{-1} z, \tau).$$

Let $\mathfrak{p}_{ij,\ldots,k}$ denote $-\frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j}\cdots\frac{\partial}{\partial z_k}\log\sigma(z)$, and suppose $z = \Phi((x_1, y_1), (x_2, y_2))$. Baker [1], p. 38, shows that:

$$(1.4) \qquad \mathfrak{p}_{11}(z) =$$

$$\frac{(x_1+x_2)(x_1x_2)^2 + 2b_1(x_1x_2)^2 + b_2(x_1+x_2)x_1x_2 + 2b_3x_1x_2 + b_4(x_1+x_2) + 2b_5 - 2y_1y_2}{4(x_1-x_2)^2},$$

$$\mathfrak{p}_{12}(z) = -\frac{1}{4}x_1x_2,$$

$$\mathfrak{p}_{22}(z) = \frac{1}{4}(x_1+x_2),$$

$$\mathfrak{p}_{111}(z) = (y_2\psi(x_1, x_2) - y_1\psi(x_2, x_1))/(4(x_1-x_2)^3),$$

where

$$\psi(m, n) = 4b_5 + b_4(3m+n) + 2b_3m(m+n) + b_2m^2(m+3n) + 4b_1m^3n + m^3n(3m+n),$$

$$\mathfrak{p}_{112}(z) = (y_1x_2^2 - y_2x_1^2)/(4(x_1-x_2)),$$

$$\mathfrak{p}_{122}(z) = (y_2x_1 - y_1x_2)/(4(x_1-x_2)),$$

and

$$\mathfrak{p}_{222}(z) = (y_1 - y_2)/(4(x_1-x_2)).$$

For a divisor D on J let $l(D)$ be the dimension of the vector space $\mathcal{L}(D)$ of functions f on J such that $(f) + D$ is an effective divisor.

The translational formula (1.2) guarantees that $\mathfrak{p}_{ij}(z) \in \mathcal{L}(2\Theta)$. It follows directly from (1.4) that 1, $\mathfrak{p}_{11}(z)$, $\mathfrak{p}_{12}(z)$, and $\mathfrak{p}_{22}(z)$ are linearly independent. Since Θ is an ample divisor with a self-intersection number of 2, the Riemann-Roch Theorem implies $l(n\Theta) = n^2$ for $n \geqq 1$. Therefore

$$(1.5) \qquad \mathcal{L}(2\Theta) = \mathbb{C} \oplus \mathbb{C}\mathfrak{p}_{11} \oplus \mathbb{C}\mathfrak{p}_{12} \oplus \mathbb{C}\mathfrak{p}_{22}.$$

Baker computed the Taylor expansion for $\sigma(z)$ [1], p. 96[1]),

$$(1.6) \qquad \sigma(z)=c\left(z_1+\frac{1}{24}b_3 z_1^3-\frac{1}{12}z_2^3+(d^\circ \geqq 5)\right),$$

where c is the nonzero constant $\frac{\partial}{\partial z_1}\theta[\delta](\omega^{-1}z,\tau)|_{z=0}$, and $(d^\circ \geqq n)$ denotes a power series all of whose terms have total degree at least n.

The main theorem is derived by calculating c via the function theory on J, so we will now sketch a proof of (1.6).

Let $\sigma_{ij\cdots k}(z)$ denote $\frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j}\cdots\frac{\partial}{\partial z_k}\sigma(z)$. Directly from (1.4) we find that

$$(\mathfrak{p}_{11}(z)\,\mathfrak{p}_{22}(z)-\mathfrak{p}_{12}^2(z))^2+\mathfrak{p}_{11}^2(z)\,\mathfrak{p}_{12}(z)-b_1\mathfrak{p}_{11}^2(z)\,\mathfrak{p}_{22}^2(z)+b_2\mathfrak{p}_{11}(z)\,\mathfrak{p}_{12}(z)\,\mathfrak{p}_{22}(z)$$
$$-b_3\mathfrak{p}_{11}(z)\,\mathfrak{p}_{22}(z)^2+b_4\mathfrak{p}_{12}(z)\,\mathfrak{p}_{22}(z)^2-b_5\mathfrak{p}_{22}(z)^3\in\mathscr{L}(5\Theta).$$

Multiplying by $\sigma(z)^6$ and replacing $\mathfrak{p}_{ij}$ $(i,j=1,2)$ by its definition in terms of partial derivatives yields:

$$\sigma_1(0)^5\sigma_2(0)-b_1\sigma_1(0)^4\sigma_2(0)^2+b_2\sigma_1(0)^3\sigma_2(0)^3-b_3\sigma_1(0)^2\sigma_2(0)^4$$
$$+b_4\sigma_1(0)\,\sigma_2(0)^5-b_5\sigma_2(0)^6=0.$$

Hence either $\sigma_2(0)=0$, or $\sigma_1(0)/\sigma_2(0)$ is the negative of one of the roots in (1.1). These six choices correspond to the choice of odd theta characteristic in the definition of $\sigma(z)$. The fact that $\sigma(z)$ vanishes precisely on Θ will determine which of these holds.

Since $\mathscr{L}(\Theta)=\mathbb{C}$, $\mathfrak{p}_{22}(z)/\mathfrak{p}_{12}(z)$ restricts to a regular function on Θ. Suppose that $z=\Phi((x,y),\infty)$. Then $\sigma(z)=0$, so by continuity and (1.4) we have

$$\frac{\mathfrak{p}_{22}(z)}{\mathfrak{p}_{12}(z)}=\frac{\sigma_2(z)}{\sigma_1(z)}=\frac{-1}{x}.$$

Taking the limit as $x\to\infty$ yields $\sigma_2(0)=0$. Since $\mathscr{C}$ is smooth, $\sigma_1(0)\neq 0$.

We can now find the cubic terms in (1.6). It follows directly from (1.4) that:

$$(1.7) \qquad 16\mathfrak{p}_{222}^2(z)=64\mathfrak{p}_{22}^3(z)+16b_1\mathfrak{p}_{22}^2(z)+(16\mathfrak{p}_{12}(z)+4b_2)\,\mathfrak{p}_{22}(z)+4\mathfrak{p}_{11}(z)+b_3,$$
$$\mathfrak{p}_{112}(z)=4\mathfrak{p}_{222}(z)\,\mathfrak{p}_{12}(z)-4\mathfrak{p}_{122}(z)\,\mathfrak{p}_{22}(z),$$

and

$$\mathfrak{p}_{111}(z)=-b_2\mathfrak{p}_{122}(z)+2b_1\mathfrak{p}_{112}(z)+8\mathfrak{p}_{112}(z)\,\mathfrak{p}_{22}(z)-4\mathfrak{p}_{222}(z)\,\mathfrak{p}_{11}(z)-4\mathfrak{p}_{122}(z)\,\mathfrak{p}_{12}(z).$$

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[1]) Our formulas differ slightly from Baker's [1] because he assumes that the curve $\mathscr{C}$ is defined by $y^2=4x^5+\cdots$.

By differentiating (1. 7) and substituting values from (1. 4) we obtain:

(1. 8) $$\wp_{1222}(z)=6\wp_{12}(z)\wp_{22}(z)+b_1\wp_{12}(z)-\frac{1}{2}\wp_{11}(z),$$

$$\wp_{1122}(z)=2\wp_{11}(z)\wp_{22}(z)+4\wp_{12}^2(z)+\frac{1}{2}b_2\wp_{12}(z),$$

$$\wp_{1112}(z)=6\wp_{11}(z)\wp_{12}(z)+b_3\wp_{12}(z)-\frac{1}{2}b_4\wp_{22}(z)-\frac{1}{4}b_5,$$

$$\wp_{1111}(z)=6\wp_{11}^2(z)+b_4\wp_{12}(z)+b_3\wp_{11}(z)-b_5\wp_{22}(z)-\frac{1}{2}b_1b_5+\frac{1}{8}b_4.$$

From their definitions in terms of partial derivatives, it follows readily that for any $i, j, k, l \in \{1, 2\}$,

$$(\sigma(z)^2(\wp_{ijkl}(z)-2\wp_{ij}(z)\wp_{kl}(z)-2\wp_{ik}(z)\wp_{jl}(z)-2\wp_{il}(z)\wp_{jk}(z))|_{z=0}$$
$$=\sigma_i(0)\sigma_{jkl}(0)+\sigma_j(0)\sigma_{ikl}(0)+\sigma_k(0)\sigma_{ijl}(0)+\sigma_l(0)\sigma_{ijk}(0).$$

Alternately taking i, j, k, and l to be 1 or 2, and combining with (1. 8) yields

$$\sigma_{111}(0)=\frac{b_3}{4}\sigma_1(0), \quad \sigma_{112}(0)=\sigma_{122}(0)=0, \quad \text{and} \quad \sigma_{222}(0)=\frac{-1}{2}\sigma_1(0),$$

which establishes (1. 6).

2. Derivation of the main formula

It follows immediately from (1. 2) and (1. 3) that

(2. 1) $$g(u, v)=-c^2\frac{\sigma(u+v)\sigma(u-v)}{\sigma^2(u)\sigma^2(v)}$$

is a function on $J \times J$. Using (1. 5) and (1. 6) it is easy to derive the following formula of Baker [1], p. 100:

(2. 2) $$g(u, v)=\wp_{11}(u)-\wp_{11}(v)+4\wp_{12}(u)\wp_{22}(v)-4\wp_{12}(v)\wp_{22}(u).$$

Our goal is to calculate the eighth power of $c=\frac{\partial}{\partial z_1}\theta[\delta](\omega^{-1}z, \tau)|_{z=0}$. The chief technical tool is a classical formula of Thomae.

For any $1 \leqq i \leqq 5$, $\Phi((a_i, 0), \infty)$ is a 2-torsion point on J, hence can be written as

$$\Phi((a_i, 0), \infty)=\omega'\eta_i'+\omega\eta_i'' \text{ modulo } L,$$

where η_i', η_i'' are column vectors with entries of 0 or 1/2. For notational convenience, we write $\eta_i = \begin{bmatrix} \eta_i' \\ \eta_i'' \end{bmatrix}$, and $\delta = \begin{bmatrix} \delta' \\ \delta'' \end{bmatrix}$, where $\delta' = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$, and $\delta'' = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$. A calculation similar to that for δ [9], pp. 3.86—3.88, shows that

$$(2.3) \qquad \eta_1 = \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \eta_2 = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix}, \quad \eta_3 = \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \\ 0 \end{bmatrix}, \quad \eta_4 = \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad \eta_5 = \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 1/2 \end{bmatrix}.$$

Set $S = \{1, 2, 3, 4, 5\}$, and $U = \{1, 3, 5\}$. For a subset V of S, let $\eta_v = \sum_{i \in V} \eta_i$ and let $U \circ V$ denote the symmetric difference of U and V.

It follows from (0.1) that if η and $\tilde{\eta}$ are theta characteristics, and $\eta \equiv \tilde{\eta} \bmod 1$, then $\theta[\eta]^2(z, \tau) = \theta[\tilde{\eta}]^2(z, \tau)$. Hereafter we will identify two theta characteristics if they are congruent mod 1. We will also make repeated use of the observation that $\delta \equiv \eta_1 + \eta_3 + \eta_5 \equiv \eta_2 + \eta_4 \bmod 1$. The following is due to Thomae (see [9], p. 3.121):

Lemma 2.4. *Let V be a subset of S satisfying $|U \circ V| = 3$. Then*

$$\theta[\eta_V]^4(0, \tau) = d \prod_{\substack{i, j \in U \circ V \\ i<j}} (a_i - a_j) \prod_{\substack{i, j \notin U \circ V \\ i<j}} (a_i - a_j),$$

where $d = \pm \left(\frac{\det \omega}{4\pi^2} \right)^2$, *and the sign of d is independent of V.*

Let $\mathscr{C}$ be given by (1.1). Then we define the discriminant $\Delta(\mathscr{C})$ by

$$\Delta(\mathscr{C}) = \prod_{1 \leq i < j \leq 5} (a_i - a_j)^2.$$

Proposition 2.5. *Let $\mathscr{C}$ be a curve of genus two defined over $\mathbb{C}$ by*

$$y^2 = x^5 + b_1 x^4 + b_2 x^3 + b_3 x^2 + b_4 x + b_5 = \prod_{i=1}^{5} (x - a_i).$$

If τ is the period matrix of $\mathscr{C}$, then

$$\Delta(\mathscr{C}) = d^{-5} \prod_{\eta \text{ even}} \theta[\eta]^2(0, \tau).$$

Remarks. 1) That the squares of the two sides of the above equation are equal follows directly from 2.4. All the work in the proof is to determine a sign.

2) As a function of the period matrix τ, the product of the squares of the ten even theta functions at zero is a Siegel modular form of dimension two and weight ten [4].

Proof. For $1 \leqq i < j \leqq 5$, let the symbol $(a_i - a_j)^{1/2}$ be a chosen square root of $(a_i - a_j)$. For $V \subseteq S$, $|U \circ V| = 3$, we define $e(V) = \pm 1$, by

$$\theta[\eta_V]^2(0, \tau) = e(V) d^{1/2} \prod_{\substack{i, j \in U \circ V \\ i<j}} (a_i - a_j)^{1/2} \prod_{\substack{i, j \notin U \circ V \\ i<j}} (a_i - a_j)^{1/2}, \tag{2. 6}$$

where $d^{1/2}$ is the square root of d such that $e(\emptyset) = 1$. The 10 choices of V for which η_V is an even theta characteristic are

$$\emptyset, (\{i, j\} \mid i \in U, j \in S - U), \text{ and } (\{i, j, k, l\} \mid i, k \in U, j, l \in S - U, i \neq k, j \neq l).$$

Therefore, to prove the theorem it suffices to show that

$$e(\{i, j, k, l\}) = e(\{i, j\}) \, e(\{k, l\}), \tag{2. 7}$$

for $i, k \in U$, $j, l \in S - U$, $i \neq k$, $j \neq l$.

To show that the signs have this coherence, we need one more tool. When $z = \Phi((x_1, y_1), (x_2, y_2))$, Mumford [9], p. 3.113, shows that for any $V = \{i, j, k\}$,

$$(a_k - x_1)(a_k - x_2) = \varepsilon \prod_{\substack{n \in V \\ n \neq k}} (a_k - a_n) \left(\frac{\theta[\eta_{U \circ V} + \eta_k](0, \tau) \, \theta[\delta + \eta_k](\omega^{-1} z, \tau)}{\theta[\eta_{U \circ V}](0, \tau) \, \theta[\delta](\omega^{-1} z, \tau)} \right)^2,$$

where

$$\varepsilon = (-1)^{4 {}^t\delta' \eta_k'' + 4 {}^t\eta_{U \circ V}'' \eta_k'}.$$

Plugging in $z = \omega'(\eta_l' + \eta_m') + \omega(\eta_l'' + \eta_m'')$, it follows from (0. 1) that

(2. 8)

$$\frac{(a_k - a_l)(a_k - a_m)}{(a_k - a_i)(a_k - a_j)} = \varepsilon \frac{\theta[\delta + \eta_i + \eta_j]^2(0, \tau) \, \theta[\delta + \eta_k + \eta_l + \eta_m]^2(0, \tau)}{\theta[\delta + \eta_i + \eta_j + \eta_k]^2(0, \tau) \, \theta[\delta + \eta_l + \eta_m]^2(0, \tau)} (-1)^{4 {}^t(\eta_l' + \eta_m')\eta_k''}.$$

Let

$$\langle a_i - a_j \rangle = \begin{cases} a_i - a_k & \text{if } i < k, \\ a_k - a_i & \text{if } k < i. \end{cases}$$

Then we observe from (2. 3) that

$$(a_i - a_j) = (-1)^{4 {}^t\eta_i'' \eta_j'} \langle a_i - a_j \rangle. \tag{2. 9}$$

Likewise we observe that for any k,

$$(-1)^{4 {}^t\delta' \eta_k'' + 4 {}^t(\delta'' + \eta_k'')\eta_k'} = 1.$$

Hence,

$$\varepsilon = (-1)^{4\,{}^t\delta' \eta_k'' + 4\,{}^t(\delta'' + \eta_i'' + \eta_j'' + \eta_k'')\eta_k'} = (-1)^{4\,{}^t(\eta_i'' + \eta_j'')\eta_k'},$$

and so by (2. 8) and (2. 9),

$$\frac{\langle a_k - a_l \rangle \langle a_k - a_m \rangle}{\langle a_k - a_i \rangle \langle a_k - a_j \rangle} = \frac{\theta[\delta + \eta_i + \eta_j]^2 (0, \tau)\, \theta[\delta + \eta_k + \eta_l + \eta_m]^2 (0, \tau)}{\theta[\delta + \eta_i + \eta_j + \eta_k]^2 (0, \tau)\, \theta[\delta + \eta_l + \eta_m]^2 (0, \tau)}.$$

In particular when $l = j$,

$$\frac{\langle a_k - a_m \rangle}{\langle a_k - a_i \rangle} = \frac{\theta[\delta + \eta_i + \eta_j]^2 (0, \tau)\, \theta[\delta + \eta_k + \eta_j + \eta_m]^2 (0, \tau)}{\theta[\delta + \eta_i + \eta_j + \eta_k]^2 (0, \tau)\, \theta[\delta + \eta_j + \eta_m]^2 (0, \tau)}. \tag{2. 10}$$

Finally, to show (2. 7), take $i, k \in U$, $j, l \in S - U$, $i \neq k$, $j \neq l$, and let $m = U - i - k$. Then from (2. 6),

$$\frac{\theta[\eta_i + \eta_j]^2 (0, \tau)\, \theta[\eta_k + \eta_l]^2 (0, \tau)}{\theta[0]^2 (0, \tau)\, \theta[\eta_i + \eta_j + \eta_k + \eta_l]^2 (0, \tau)} = \frac{e(\{i, j\})\, e(\{k, l\}) \langle a_k - a_j \rangle \langle a_l - a_i \rangle}{e(\{i, j, k, l\}) \langle a_k - a_i \rangle \langle a_l - a_j \rangle}$$

$$= \frac{e(\{i, j\})\, e(\{k, l\})\, \theta[\delta + \eta_i + \eta_l]^2 (0, \tau)\, \theta[\delta + \eta_j + \eta_k]^2 (0, \tau)}{e(\{i, j, k, l\})\, \theta[\delta + \eta_j + \eta_l]^2 (0, \tau)\, \theta[\delta + \eta_i + \eta_k]^2 (0, \tau)},$$

after applying (2. 10) twice. Since $\delta \equiv \eta_i + \eta_k + \eta_m \equiv \eta_j + \eta_l \bmod 1$, we conclude that

$$e(\{i, j, k, l\}) = e(\{i, j\})\, e(\{k, l\}).$$

Theorem 2. 11. *Let $\mathscr{C}$ be a curve of genus two defined over $\mathbb{C}$ by*

$$y^2 = f(x) = x^5 + b_1 x^4 + b_2 x^3 + b_3 x^2 + b_4 x + b_5 = \prod_{i=1}^{5} (x - a_i),$$

with ω and δ as in section one. Then

$$\left(\frac{\partial}{\partial z_1} \theta[\delta] (\omega^{-1} z, \tau)|_{z=0} \right)^8 = 2^{-16} \pi^{-8} (\det \omega)^4 \Delta(\mathscr{C})$$

$$= \pm 16 \pi^{12} (\det \omega)^{-6} \prod_{\eta \text{ even}} \theta[\eta]^2 (0, \tau).$$

Proof. It follows directly from (1. 3) and (2. 1) that

$$g(u, v) = -c^2 \frac{\sigma(u+v)\, \sigma(u-v)}{\sigma^2(u)\, \sigma^2(v)} = -c^2 \frac{\theta[\delta] (\omega^{-1}(u+v))\, \theta[\delta] (\omega^{-1}(u-v))}{\theta[\delta]^2 (\omega^{-1} u)\, \theta[\delta]^2 (\omega^{-1} v)}.$$

To evaluate c^2, we can plug in

$$u_0 = \Phi((a_1, 0), (a_2, 0)), \quad \text{and} \quad v_0 = \Phi((a_1, 0), (a_3, 0)).$$

It then follows directly from (2. 3) and (0. 1) that

$$g(u_0, v_0)=\frac{-c^2\theta[\delta+\eta_2+\eta_3]^2(0,\tau)}{\theta[\delta+\eta_1+\eta_2]^2(0,\tau)\,\theta[\delta+\eta_1+\eta_3]^2(0,\tau)}=\frac{-c^2\theta\begin{bmatrix}0\\0\\0\\1/2\end{bmatrix}^2(0,\tau)}{\theta\begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}^2(0,\tau)\,\theta\begin{bmatrix}0\\0\\1/2\\1/2\end{bmatrix}^2(0,\tau)}.$$

Now by (2. 6),

$$\theta\begin{bmatrix}0\\0\\0\\1/2\end{bmatrix}^2(0,\tau)=\theta[\eta_3+\eta_4]^2(0,\tau)=\pm(d(a_1-a_4)(a_1-a_5)(a_4-a_5)(a_2-a_3))^{1/2},$$

$$\theta\begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}^2(0,\tau)=\theta[\eta_1+\eta_4]^2(0,\tau)=\pm(d(a_3-a_4)(a_3-a_5)(a_4-a_5)(a_1-a_2))^{1/2},$$

$$\theta\begin{bmatrix}0\\0\\1/2\\1/2\end{bmatrix}^2(0,\tau)=\theta[\eta_1+\eta_2+\eta_3+\eta_4]^2(0,\tau)$$
$$=\pm(d(a_2-a_4)(a_2-a_5)(a_4-a_5)(a_1-a_3))^{1/2},$$

and hence,

$$(2.\,12)\quad g(u_0, v_0)=\pm c^2 d^{-1/2}\times\left(\frac{(a_1-a_4)(a_1-a_5)(a_2-a_3)}{(a_1-a_2)(a_1-a_3)(a_2-a_4)(a_2-a_5)(a_3-a_4)(a_3-a_5)(a_4-a_5)}\right)^{1/2}.$$

We can also calculate $g(u_0, v_0)$ from (2. 2) and (1. 4). If $z=\Phi((x_1, y_1),(x_2, y_2))$, then (1. 4) implies

$$4\mathfrak{p}_{11}(z)=\frac{f(x_1)+f(x_2)-2y_1y_2}{(x_1-x_2)^2}-(x_1+x_2)^3+x_1x_2(x_1+x_2)$$
$$-b_1(x_1+x_2)^2-b_2(x_1+x_2)-b_3.$$

So using $b_1 = -\sum_{1\leqq i\leqq 5} a_i$, and $b_2 = \sum_{1\leqq i<j\leqq 5} a_i a_j$, we get

$$\mathfrak{p}_{11}(u_0) - \mathfrak{p}_{11}(v_0) = \frac{1}{4}(a_2-a_3)(a_1a_4+a_1a_5-a_4a_5).$$

Therefore

$$g(u_0, v_0) = -\frac{1}{4}(a_1-a_4)(a_1-a_5)(a_2-a_3). \tag{2. 13}$$

Equating (2. 12) and (2. 13) and raising to the fourth power gives us:

$$c^8 = 2^{-8}\left(\frac{\det\omega}{4\pi^2}\right)^4 \Delta(\mathscr{C}).$$

Combining this with 2. 5 yields the remainder of the theorem.

Remarks. 1) It follows from (1. 6) that $\frac{\partial}{\partial z_2}\theta[\delta](\omega^{-1}z, \tau)|_{z=0} = 0$.

2) Among the six odd theta characteristics, δ was singled out because of our choice of symplectic basis for $H_1(\mathscr{C}, \mathbb{Z})$. Suppose that $\{\mathscr{A}_1, \mathscr{A}_2, \mathscr{B}_1, \mathscr{B}_2\}$ is any other symplectic basis. This is equivalent to

$$\begin{pmatrix}\mathscr{B}_1\\ \mathscr{B}_2\end{pmatrix} = C\begin{pmatrix}B_1\\ B_2\end{pmatrix} + D\begin{pmatrix}A_1\\ A_2\end{pmatrix}, \quad\text{and}\quad \begin{pmatrix}\mathscr{A}_1\\ \mathscr{A}_2\end{pmatrix} = E\begin{pmatrix}B_1\\ B_2\end{pmatrix} + F\begin{pmatrix}A_1\\ A_2\end{pmatrix},$$

with $\begin{pmatrix}C & D\\ E & F\end{pmatrix} \in \mathrm{Sp}(2, \mathbb{Z})$. Then if we set

$$\Omega_{ij} = \int_{\mathscr{A}_j}\mu_i, \quad \Omega'_{ij} = \int_{\mathscr{B}_j}\mu_i, \quad (i, j = 1, 2), \quad\text{and}\quad T = \Omega^{-1}\Omega',$$

we get $\Omega = \omega' {}^tC + \omega {}^tD$, $\Omega' = \omega' {}^tE + \omega {}^tF$, and $T = (C\tau + D)(E\tau + F)^{-1}$.

Now by the functional equation of the theta function [7], pp. 85, 176, and 182,

$$\begin{aligned}\theta[\delta^*](\Omega^{-1}z, T) &= \theta[\delta^*]({}^t(E\tau+F)^{-1}\omega^{-1}z, (C\tau+D)(E\tau+F)^{-1})\\ &= \varrho\det(E\tau+F)^{1/2} e^{\pi i {}^t(\omega^{-1}z)(E\tau+F)^{-1}E(\omega^{-1}z)}\theta[\delta](\omega^{-1}z, \tau),\end{aligned}$$

where:

$$\varrho^8 = 1, \; \delta^* = \begin{pmatrix}F & -D\\ -E & C\end{pmatrix}\begin{pmatrix}\delta'\\ \delta''\end{pmatrix} + \frac{1}{2}\begin{pmatrix}(E\,{}^tF)_0\\ (C\,{}^tD)_0\end{pmatrix},$$

and M_0 denotes the column vector consisting of the diagonal entries of a matrix M.

So we have,

$$\left(\frac{\partial}{\partial z_1}\,\theta[\delta^*]\,(\Omega^{-1}z,\,T)|_{z=0}\right)^8 = \det(E\tau+F)^4\left(\frac{\partial}{\partial z_1}\,\theta[\delta]\,(\omega^{-1}z,\,\tau)|_{z=0}\right)^8$$
$$= \pm 16\pi^{12}\,(\det\Omega)^{-6}\,\det(E\tau+F)^{10}\prod_{\eta\,\text{even}}\theta[\eta]^2\,(0,\tau)$$
$$= \pm 16\pi^{12}\,(\det\Omega)^{-6}\prod_{\eta\,\text{even}}\theta[\eta]^2\,(0,T).$$

Since the map $\delta \longrightarrow \delta^*$ gives a transitive action of $\mathrm{Sp}(2,\mathbb{Z})$ on the odd theta characteristics mod 1, we see that 2. 11 holds for any odd theta characteristic δ with appropriate choice of symplectic basis for $H_1(\mathscr{C},\mathbb{Z})$.

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Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA

Eingegangen 20. August 1987, in revidierter Fassung 22. Februar 1988