Feasibility of Single-Beam Interference Alignment in Multi-Carrier Interference Channels

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***Abstract*— Sun and Luo recently showed that if the vector-space single-beam interference alignment problem for a K-user, L-carrier interference channel is feasible, then $K \leq 2L - 2$. We prove the converse, that if $K \leq 2L - 2$, then the problem is feasible, i.e., that the requisite beamformers do exist.**

***Index Terms*— Interference alignment, interference network, multi-carrier modulation.**

I. Introduction

CONSIDER a K-user, L-carrier interference channel, where for $1 \leq i, j \leq K$, transmitter j has M_j antennas, receiver i has N_i antennas, and the p^{th} transmitter-receiver pair from transmitter p to receiver p requires $d_p L$ degrees of freedom per channel use (across the L carriers). Let $H_{ij}(\ell) \in \mathbb{C}^{N_i \times M_j}$ be the channel from transmitter j to receiver i on the ℓ^{th} sub-channel, with each element of $H_{ij}(\ell)$ being drawn independently from a continuous distribution, so that with probability 1, we can consider the channel realization to be "generic"; that is, all the entries of all the $H_{ij}(\ell)$ are algebraically independent complex numbers over the field of rational numbers $\mathbb{Q}$ (i.e., there is no non-trivial polynomial equation with rational coefficients that the channel coefficients satisfy). The *interference alignment problem* is to find for $1 \leq i, j \leq K$, $1 \leq \ell \leq L$, beam-forming matrices $U_i(\ell)$ and $V_j(\ell)$, of size $d_i L \times N_i$ and $M_j \times d_j L$, respectively, $d_i \leq N_i$, $d_j \leq M_j$, so that

$$\sum_{\ell=1}^{L} U_i(\ell) H_{ij}(\ell) V_j(\ell) = 0, \text{ when } i \neq j, \quad \text{and}$$
$$\sum_{\ell=1}^{L} U_i(\ell) H_{ii}(\ell) V_i(\ell) = W_{d_i L} \tag{1}$$

where $W_{d_i L}$ denotes a $d_i L \times d_i L$ matrix of full rank. If such $U_i(\ell)$ and $V_j(\ell)$ exist, we say the system (1) is *feasible*. Here $d_i L$ and d_i are respectively the number of achievable degrees of freedom of the i^{th} transmitter-receiver pair per channel use and per sub-channel use, using linear transmit and receive functions. In the single-beam case, i.e., with $d_i L = 1$, a more general framework that incorporates the notions of finite diversity and time extensions is provided in [14]. The reader is referred to [14] and its references for a detailed discussion of non-asymptotic vector space interference alignment in interference channels.

The special case of the single antenna (or single-input, single-output (SISO)) interference channel and single-beam transmisson per channel use, i.e., with $M_i = N_i = 1$ and $d_i L = 1$ for all $1 \leq i \leq K$, has been well-studied. In this case (1) reduces to: given for each $1 \leq \ell \leq L$, a $K \times K$ generic channel realization matrix $[H_{ij}(\ell)]_{1 \leq i,j \leq K}$, find for each $1 \leq i, j \leq K$, beam-forming vectors U_i and V_j, of size $1 \times L$ and $L \times 1$ respectively, so that

$$\sum_{\ell=1}^{L} U_i(\ell) H_{ij}(\ell) V_j(\ell) = 0, \text{ when } i \neq j, \quad \text{and}$$
$$\sum_{\ell=1}^{L} U_i(\ell) H_{ii}(\ell) V_i(\ell) \neq 0. \tag{2}$$

Sun and Luo recently have shown (see the remark after [14, Th. 3.1]), that if (2) is feasible, then

$$K \leq 2L - 2.$$

Earlier, Shi *et al.* [11] showed the same result under the additional assumption that U_i, V_i, $1 \leq i \leq K$, are drawn uniformly and independently from the unit sphere in $\mathbb{C}^L$. The purpose of this note is to show that this bound is sharp. Namely, we prove the converse:

Theorem 1: The interference alignment problem in (2) *is feasible whenever* $K \leq 2L - 2$.

We note that the interference alignment problem is trivial when $K = 1$ (one can take $L = 1$) or $K = 2$ (one can take $L = 2$), since there is no opportunity for alignment. Hence, we will assume from now on that $K \geq 3$ and develop what we need to prove Theorem 1.

Like the groundbreaking papers of [1], [9], and [15], which study the feasibility problem for single-carrier ($L = 1$) interference channel with multiple antenna terminals, we will also employ techniques from algebraic geometry to tackle that question for the multi-carrier SISO interference channel, since the equations describing the interference alignment problem herein too are polynomial equations, and algebraic geometry is the tool that mathematicians have developed to understand the simultaneous solutions to sets of polynomial equations. We will make use of a classic theorem of Jacobi from 1841 [8] that states that the transcendence degree of a set

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of polynomials is given by the rank of its Jacobian matrix. In essence, this result was used in [9] in the special case that the Jacobian matrix has full rank, under the guise of the implicit function theorem. This approach was revisited in the setting of differential topology in [6] to study (1) in the single-carrier case, but we see no reason to resort to that framework to solve this algebraic geometric problem. Jacobi's Theorem is much stronger than the transcendence criterion invented in [10].

This entire note is dedicated to the proof of Theorem 1. We make some preliminary considerations in Section II. In Section III, for the readers' convenience, we first describe all the algebraic geometry we will need, and then apply the theory to our interference alignment problem. This lays the ground work for the final argument of the proof of Theorem 1, which is given in Section IV.

II. Preliminary Considerations

Our goal is to prove the feasibility of (2) under the hypothesis of Theorem 1, so there is no harm in adding an additional constraint which only makes the problem harder. We will now assume that there exists an ℓ', $1 \le \ell' \le L$, such that

$$U_i(\ell')V_i(\ell') \neq 0, \quad \text{for all } 1 \le i \le K. \tag{3}$$

We call (2) along with this constraint (3) the *nonsparse* interference alignment problem. What we will actually show in Section 3 is that the nonsparse interference alignment problem is feasible if $K \ge 3$ and $K \le 2L-2$, which will give us Theorem 1.

Given (3), exchanging $H_{ij}(\ell')$ and $H_{ij}(1)$ and $U_i(\ell')$ and $U_i(1)$ and $V_j(\ell')$ and $V_j(1)$ for all $1 \le i, j \le K$, we can assume without loss of generality that for each $1 \le i \le K$, we have $U_i(1)V_i(1) \neq 0$. Then rescaling the equations in (2), we can assume without loss of generality for all $1 \le i \le K$, that $U_i(1) = V_i(1) = 1$.

The equations in (2) can be stated more succinctly. Let U and V be respectively the block diagonal matrices of sizes $K \times KL$ and $KL \times K$ with U_i and V_i forming the blocks for $i = 1, \dots, K$. For every $1 \le i, j \le K$, let H_{ij} be the diagonal $L \times L$ matrix whose diagonal entries are sequentially $H_{ij}(1), \dots, H_{ij}(L)$. Then let H be the $KL \times KL$ matrix whose ij^{th} block is H_{ij} for $1 \le i, j \le K$. Then the system of equations (2) can be rewritten as

$$UHV = D, \tag{4}$$

where D is a $K \times K$ diagonal matrix with nonzero diagonal entries we denote by $D_1, \dots, D_K$.

Now for each $1 \le i \le K$, write $U_i = (1\ \alpha_i)$ and $V_i = (1\ \beta_i)^t$, where α_i and β_i are row vectors of length $L-1$, and t denotes taking the transpose.

Now with this holding, let us investigate how to solve (4) if a solution exists. Let R be the $KL \times KL$ "row reduction" matrix, which is block diagonal with blocks $R_1, \dots, R_K$, where for $1 \le i \le K$,

$$R_i = \begin{pmatrix} 1 & 0_{1,L-1} \\ -\beta_i^t & I_{L-1} \end{pmatrix},$$

where for natural numbers m and n, I_n denotes the $n \times n$ identity matrix, and $0_{m,n}$ denotes the $m \times n$ matrix of all zero entries. Then by design, RV is a $KL \times K$ block diagonal matrix whose i^{th} block for $i = 1, \dots, K$, is the standard basis (column) vector e_1. Likewise, let C be the $KL \times KL$ "column reduction" matrix, which is block diagonal with blocks $C_1, \dots, C_K$, where for $1 \le i \le K$,

$$C_i = \begin{pmatrix} 1 & -\alpha_i \\ 0_{L-1,1} & I_{L-1} \end{pmatrix}.$$

Then by design, UC is an $K \times KL$ block diagonal matrix whose i^{th} block for $i = 1, \dots, K$ is e_1^t. Let $J = C^{-1}HR^{-1}$. Then (4) is equivalent to $(UC)J(RV) = D$, i.e., for all $1 \le i, j \le K$,

$$e_1^t J_{ij} e_1 = D_i \delta_{ij}, \tag{5}$$

where $J_{ij} = C_i^{-1} H_{ij} R_j^{-1}$, and δ_{ij} is the Kronecker δ. For each $1 \le i, j \le K$, writing each $L \times L$ matrix J_{ij} in terms of blocks:

$$J_{ij} = \begin{pmatrix} E_{ij} & F_{ij} \\ G_{ij} & P_{ij} \end{pmatrix},$$

where E_{ij} is 1×1, F_{ij} is $1 \times (L-1)$, G_{ij} is $(L-1) \times 1$, and P_{ij} is $(L-1) \times (L-1)$, equation (5) is just

$$E_{ij} = 0, \text{ for } i \neq j; \text{and } E_{ii} = D_i, \ 1 \le i, j \le K.$$

Since each H_{ij} is a diagonal matrix, for each $1 \le i, j \le K$, the equation $H_{ij} = C_i J_{ij} R_j$ implies that $F_{ij} = \alpha_i P_{ij}$, $G_{ij} = P_{ij}\beta_j^t$, and that P_{ij} is a diagonal matrix, and if we denote its diagonal entries as $P_{ij}(\ell)$, $2 \le \ell \le L$, then $P_{ij}(\ell) = H_{ij}(\ell)$ for $2 \le \ell \le L$. Finally we get that

$$H_{ij}(1) = D_i \delta_{ij} - \alpha_i P_{ij} \beta_j^t.$$

Note that the condition $D_i \neq 0$ can be described by picking a new variable ϵ_i and adding the constraint equation $D_i \epsilon_i = 1$. In what follows it will be convenient for all $1 \le i \le K$ to index the entries of α_i and β_i as $\alpha_i = (\alpha_i(2), \dots, \alpha_i(L))$ and $\beta_i = (\beta_i(2), \dots, \beta_i(L))$.

We have proved part (a) of the following (and since R and C are invertible, part (b) follows from retracing our steps):

Proposition 1: a) If the system of equations (2) *in the nonsparse interference alignment problem is feasible for a particular channel realization, then the following system of equations, in variables consisting of:* $(L-1)$*-tuples* α_i, β_i *for each* $1 \le i \le K$, $P_{ij}(2), \dots, P_{ij}(L)$ *for each* $1 \le i, j, \le K$, *and* D_i, ϵ_i *for* $1 \le i \le K$, *is feasible:*

$$H_{ij}(1) = D_i \delta_{ij} - \alpha_i P_{ij} \beta_j^t, \ 1 \le i, j \le K, \tag{6}$$

$$H_{ij}(\ell) = P_{ij}(\ell), \ 1 \le i, j \le K, \ 2 \le \ell \le L, \tag{7}$$

$$D_i \epsilon_i = 1, \ 1 \le i \le K. \tag{8}$$

b) Conversely, if (6)-(8) are feasible for a particular channel realization, then so is the system of equations (2) *when* $U_i(1) = V_i(1) = 1$ *for all* $1 \le i \le K$.

As we will see in the next section, the power of rewriting (2) in this way is that in (6) and (7) we have isolated the generic channel realization variables on one side of the equations.

We would like to note that if a system of polynomial equations like (6)-(8) is feasible (i.e., has a solution), then a solution can be effectively computed. This can be seen naively by using resultants (elimination theory) to reduce the problem to solving a single polynomial in a single variable [5, Ch. 2]. For state-of-the art efficient algorithms using Gröbner bases and univariate representations, see Chapter 12 on "Polynomial System Solving" in [2].

III. Algebraic Geometric Formulation and Groundwork for the Proof of Theorem 1

A good general reference for the background material on algebraic geometry in this section is [13].

It is standard to call the set which contain solutions to polynomial equations in n-variables *affine n-space*, and denote it as $\mathbb{A}^n$. If S is a set of polynomials in the polynomial ring $R = \mathbb{C}[x_1, \ldots, x_n]$ in the n indeterminates $x_1, \ldots, x_n$, we let

$$Z(S) = \{p \in \mathbb{A}^n | s(p) = 0, \forall s \in S\},$$

be the set of common zeros of all $s \in S$. This is called an *algebraic set* in $\mathbb{A}^n$: one can show that algebraic sets satisfy the necessary axioms to be the closed sets in a topology on $\mathbb{A}^n$, called the *Zariski topology* on $\mathbb{A}^n$. This is why we use the different symbol $\mathbb{A}^n$ rather that $\mathbb{C}^n$: they have the same elements as sets, but the former has the Zariski topology while the latter has its usual complex topology. A set that is closed under the Zariski topology is closed under the usual complex topology, but not vice versa. In particular, a proper closed subset of $\mathbb{A}^n$ considered as a subset of $\mathbb{C}^n$ has measure 0. Each subset U of $\mathbb{A}^n$ is given the subspace topology, called the Zariski topology on U.

A set in a topological space is called *irreducible* if it cannot be written as the union of two proper closed subsets. One can show that the closure of an irreducible set is irreducible, and that the image under a continuous map of an irreducible set is irreducible. An irreducible closed subset of $\mathbb{A}^n$ is called an *affine variety*. One can show that the affine varieties are precisely the algebraic sets of the form $V = Z(\mathfrak{p})$, where $\mathfrak{p} \subset R$ is a prime ideal. We call $\mathbb{C}[V] = R/\mathfrak{p}$ the *coordinate ring* of the affine variety V. The transcendence degree over $\mathbb{C}$ of the fraction field of $\mathbb{C}[V]$ is called the *dimension* of V.

Example 1: Let $G = Z(x_1x_2 - 1) \subset \mathbb{A}^2$. Since $x_1x_2 - 1$ is irreducible in $\mathbb{C}[x_1, x_2]$ it generates a prime ideal $(x_1x_2 - 1)$. Solving for $x_2 = 1/x_1$ shows that the fraction field of $\mathbb{C}[x_1x_2]/(x_1x_2 - 1)$ is isomorphic to $\mathbb{C}(x_1)$. Hence G is an affine variety of dimension 1, consisting of the points (a, a^{-1}), $a \in \mathbb{C}$, where $a \neq 0$. If we just looked at the first coordinates of the points in G, it is the set $G_1 \subset \mathbb{A}^1$, where $G_1 = \{a \in \mathbb{C} | a \neq 0\}$. We note that G_1 is not closed in $\mathbb{A}^1$, rather that its closure in the Zariski topology is all of $\mathbb{A}^1$. To see this, by the definition of the Zariski topology, it suffices to show that any polynomial $f(x_1)$ that vanishes on G_1 vanishes on all of $\mathbb{A}^1$, which is clear since $\mathbb{C}$ is infinite.

Let $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ be two affine varieties with coordinate rings $\mathbb{C}[V] = k[x_1, \ldots, x_n]/\mathfrak{p}$ and $\mathbb{C}[W] = \mathbb{C}[y_1, \ldots, y_m]/\mathfrak{q}$. If A is an open subset of V, then a function ϕ on V is called a regular function if for any $x \in A$, there is an open set U of A containing x such that ϕ restricted to U is given by by a ratio of polynomials f/g where g does not vanish on U. If A and B are open subsets of V and W, then a morphism from A to B is an m-tuple of regular functions $F = (F_1, \ldots, F_m)$ such that $F(A)$ is contained in B. If $A = V$ and $B = W$, then one can show that a morphism $\phi : V \to W$ is necessarily an m-tuple of polynomials $(f_1, \ldots, f_m) \in \mathbb{C}[x_1, \ldots, x_n]$, such that $g(f_1, \ldots, f_n) \in \mathfrak{p}$ for all $g \in \mathfrak{q}$ (so the morphism actually takes points in V to points in W). In particular, a morphism from V to $\mathbb{A}^m$ is just an m-tuple of polynomials. One can show that morphisms are continuous maps in the Zariski topology. The composition of morphisms is a morphism, and if $\phi : V \to W$ has a 2-sided inverse morphism $\phi^{-1} : W \to V$, we say that ϕ is an *isomorphism* and that V and W are *isomorphic*.

Again let $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ be two affine varieties with coordinate rings $\mathbb{C}[V] = \mathbb{C}[x_1, \ldots, x_n]/\mathfrak{p}$ and $\mathbb{C}[W] = \mathbb{C}[y_1, \ldots, y_m]/\mathfrak{q}$. For any $f \in k[x_1, \ldots, x_n]$ or $g \in \mathbb{C}[y_1, \ldots, y_m]$ let $\tilde{f}$ and $\tilde{g}$ denote f and g thought of as polynomials in $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_m]$. Then the *product variety* $V \times W \subseteq \mathbb{A}^{n+m}$ of V and W is the set of zeros of all $\{\tilde{f}, \tilde{g} | f \in \mathfrak{p}, g \in \mathfrak{q}\}$.

Now let us apply this to the situation described in Proposition 1. The equations $D_i\epsilon_i - 1$, $1 \le i \le K$, in

$$\mathbb{C}[\ldots, P_{ij}(\ell), \ldots, \alpha_i(\ell), \ldots, \beta_i(\ell), \ldots, D_i, \ldots, \epsilon_i, \ldots],$$

for $1 \le i, j \le K$, $2 \le \ell \le L$, define a variety X in $\mathbb{C}^{K^2(L-1)+2KL}$ isomorphic to $\mathbb{A}^{K^2(L-1)+2K(L-1)} \times G^K$, with G as in Example 1. The polynomials in the righthand side of the equations (6) and (7) define a morphism ϕ from X to $\mathbb{A}^{K^2L}$.

In our algebraic geometric language, the feasibility of the equations in Proposition 1 is precisely the following:

Question 1: Let $\eta = (\ldots, H_{ij}(\ell), \ldots)_{1 \le i,j \le K, 1 \le \ell \le L} \in \mathbb{A}^{K^2L}$ be a channel realization. With probability 1, is $\eta \in \phi(X)$?

A morphism f from an affine variety V to an affine variety W is said to be *dominant* if the closure of $f(V)$ is all of W. We will now show that the answer to Question 1 is "yes" precisely when ϕ is a dominant morphism. Note first that if ϕ were not dominant, then there would be a nonzero polynomial in K^2L variables that vanishes on $\phi(V)$, so $\phi(V)$ would have measure 0, and $\eta \in \phi(V)$ with probability 0.

For the converse, we need the following special case of a fundamental theorem of Chevalley that shows that the image $f(V)$ of an affine variety V under a morphism f contains a non-empty open subset of its closure. (See [7, Example 3.19])

Theorem 2: Let V be an affine variety, and f be a dominant morphism from V to $\mathbb{A}^m$. Then there is a nonzero polynomial g in m-variables such that g vanishes on the complement of $f(V)$ in $\mathbb{A}^m$.

Applying the theorem to X and ϕ, we see that if ϕ is dominant, then the complement of $\phi(V)$ has measure 0, so indeed $\eta \in \phi(V)$ with probability 1.

So the remainder of the paper is dedicated to determining precisely when ϕ is dominant, i.e, when the closure $\overline{\phi(X)}$ is all of $\mathbb{A}^{K^2L}$.

We can simplify this before we begin the task. We first note that if we let f be the morphism from $\mathbb{A}^{K^2(L-1)+(2L-1)K}$ to $\mathbb{A}^{K^2L}$ given by the K^2L polynomials from the right-hand side of (6) and (7):

$$D_i\delta_{ij} - \alpha_i P_{ij}\beta_j^t,\ 1 \le i, j \le K,$$
$$P_{ij}(\ell),\ 1 \le i, j \le K,\ 2 \le \ell \le L, \tag{9}$$

in the variables $(\ldots, P_{ij}(\ell), \ldots, \alpha_i(\ell), \ldots, \beta_i(\ell), \ldots, D_i, \ldots,)$ (for $1 \le i, j \le K$, $2 \le \ell \le L$), then as in Example 1, if we let $W = \mathbb{A}^{K^2(L-1)+(2L-2)K} \times G_1^K$, the subset of $\mathbb{A}^{K^2(L-1)+(2L-1)K}$ where all the $D_i \neq 0$, then f restricts to a morphism on W. Furthermore, $f(W) = \phi(X)$, so ϕ is dense in $\mathbb{A}^{K^2 L}$ if and only if $f(W)$ is dense. But since f is continuous, it maps the closure $\overline{W}$ of W into the closure of $f(W)$, so $f(W)$ is dense when $f(\overline{W})$ is. But as in Example 1, $\overline{W}$ is all of $\mathbb{A}^{K^2(L-1)+(2L-1)K}$, so we have reduced our fundamental question to answering when

$$f:\ \mathbb{A}^{K^2(L-1)+(2L-1)K} \to \mathbb{A}^{K^2 L}$$

is a dominant morphism.

To further simplify we have:

Proposition 2: Since in (6) we can solve for D_i when $j = i$, f will be dominant if and only if the morphism g from $\mathbb{A}^{(K^2+2K)(L-1)}$ to $\mathbb{A}^{LK^2-K}$ defined by the polynomials

$$f_{ij1} := -\alpha_i P_{ij}\beta_j^t,\ 1 \le i \neq j \le K,$$
$$f_{ij\ell} := P_{ij}(\ell),\ 1 \le i, j \le K,\ 2 \le \ell \le L,$$

is dominant.

Furthermore, since in (6) and (7) we can solve for $P_{ii}(\ell)$, $1 \le i \le K, 2 \le \ell \le L$, g will be dominant precisely when the morphism from $\mathbb{A}^{(K^2+K)(L-1)}$ to $\mathbb{A}^{L(K^2-K)}$ defined by the polynomials

$$f_{ij\ell},\ 1 \le i \neq j \le K,\ 1 \le \ell \le L, \tag{10}$$

is dominant.

There is a classical theorem of Jacobi [8] that says a set of polynomials has transcendence degree equal to the rank of its Jacobian matrix. In our algebraic geometric language, this provides a test for whether a morphism from complex affine space to an affine variety is dominant. To wit:

Theorem 3: (Jacobi) Suppose $W \subseteq \mathbb{A}^m$ is a complex affine variety and n is at least the dimension of W. If $f : \mathbb{A}^n \to W$ is a morphism given by the polynomials $f_1(x_1, \ldots, x_n), \cdots, f_m(x_1, \ldots, x_n)$, then f is dominant precisely when the rank of the Jacobian matrix $\frac{\partial f}{\partial x} := [\frac{\partial f_i}{\partial x_j}]_{1\le i\le m, 1\le j\le n}$ is the dimension of W.

We will think of the Jacobian matrix $\frac{\partial f}{\partial x}$ as having its columns indexed by the polynomials f_i and the rows indexed by the variables x_j. Since we only care about the rank of the matrix, we will be free to specify an ordering for the rows or the columns. In conclusion:

Proposition 3: To prove Theorem 1, it suffices to compute the rank of the Jacobian matrix of the polynomials in (10) and show that for $K \ge 3$, it is of full rank when $K \le 2L - 2$.

To carry this out in the next section, we will need to compute the rank of a couple of matrices.

Example 2: For any $K \ge 2$, let $x_1, \ldots, x_{K-1}$ and $y_1, \ldots, y_K$ be independent indeterminates, whose indices are considered modulo K. Define the $2K - 2$ polynomials $f_i = x_i y_{i+1}$, $g_i = x_i y_{i+2}$, $1 \le i \le K - 1$. We will show that the rank of the Jacobian matrix J_K of the f_i and g_i (thought of as columns) with respect to the x_i and y_i (thought of as rows) has full rank $2K - 2$.

The proof is by induction, the base case of $K = 2$ being easy to verify. Now suppose $K \ge 3$ and assume the result for $K - 1$. Since the row R in J corresponding to the variable y_1 has a single nonzero entry x_{K-1} in the column C corresponding to g_{K-1}, the rank of J_K is one more than the rank of J'_K, the minor gotten by removing R and C from J_K. Then the row R' of J'_K corresponding to the variable x_{K-1} has a single nonzero entry y_K, in the column C' corresponding to the equation f_{K-1}, so the rank of J'_K is one more than the rank of the minor J''_K gotten by removing R' and C' from J'_K. But J''_K is identical to J_{K-1}, once the variable y_K is renamed y_1. Hence by induction, the rank of J_{K-1} is $2(K-1)-2$, so the rank of J_K is $2K - 2$ as desired.

Example 3: For any $K \ge 2$, let $x_1, \ldots, x_K$ and $y_1, \ldots, y_K$ be independent indeterminates, whose indices are considered modulo K. Define the $2K - 2$ polynomials $f_i = x_K y_i$, $g_i = x_i y_{i-1}$, $1 \le i \le K - 1$. We will show that the rank of the Jacobian $\mathcal{J}_K$ of the f_i and g_i (thought of as columns) with respect to the x_j and y_j (thought of as rows), $1 \le i, j \le K-1$, has full rank $2K - 2$.

Indeed, the Jacobian of the f_i with respect to the y_j has rank $K-1$, and the Jacobian of the f_i with respect to $x_1, \ldots, x_{K-1}$ vanishes. So the result follows by noticing that the Jacobian of the g_i with respect to $x_1, \ldots, x_{K-1}$ has rank $K - 1$.

IV. Completion of the Proof of Theorem 1

Let $K \ge 3$ and $K \le 2L - 2$, and $f_{ij\ell}$ as in (10). By Proposition 3, to finish the proof of Theorem 1, we need only show that the Jacobian matrix

$$\left[\frac{\partial f_{ijm}}{\partial p_{i'j'}(\ell), \partial \alpha_{i'}(\ell), \partial \beta_{i'}(\ell)}\right]_{1\le i\neq j, i'\neq j'\le K, 1\le m\le L, 2\le \ell\le L}$$

has full rank. Since with appropriate ordering of rows and columns,

$$\left[\frac{\partial f_{ijm}}{\partial p_{i'j'}(\ell)}\right]_{1\le i\neq j, i'\neq j'\le K, 2\le \ell, m, \le L}$$

is the identity matrix, this happens precisely when

$$J := \left[\frac{\partial f_{ij1}}{\partial \alpha_{i'}(\ell), \partial \beta_{i'}(\ell)}\right]_{1\le i\neq j, i'\le K, 2\le \ell\le L},$$

which is a $(2(L-1)K) \times (K^2 - K)$ matrix, has full rank. Since $K \le 2L - 2$, we have $2(L - 1)K \ge K^2 - K$, so we need only verify that J has rank $K^2 - K$. To simplify this calculation, let $z_{ij}(\ell) = \alpha_i(\ell)\beta_j(\ell)$, for $1 \le i \neq j \le K, 2 \le \ell \le L$, and set $g_{ij} = -\sum_{\ell=2}^{L} P_{ij}(\ell) z_{ij}(\ell)$. Then by the chain rule,

$$J = \left[\frac{\partial z_{ij}(\ell)}{\partial \alpha_{i'}(\ell'), \partial \beta_{i'}(\ell')}\right]_{1\le i\neq j, i'\le K, 2\le \ell, \ell'\le L} \times \left[\frac{\partial g_{ij}}{\partial z_{i'j'}(\ell)}\right]_{1\le i\neq j, i'\neq j'\le K, 2\le \ell\le L}.$$

We denote the first factor as J' and the second factor as J''.

If we order the rows and columns of J' and J'' so that the indices (ℓ, i, j) and (ℓ', i', j') are ordered lexicographically, then J' is a block diagonal matrix with $L-1$ blocks B_ℓ, $2 \le \ell \le L$, each of the form

$$\left[\frac{\partial(\alpha_i \beta_j)}{\partial \alpha_{i'}, \partial \beta_{i'}}\right]_{1 \le i \ne j, i' \le K},$$

a $2K \times (K^2 - K)$ matrix, and J'' is a partitioned matrix

$$-\left[[P_{ij}(2)]^t_{i \ne j} \mid \cdots \mid [P_{ij}(L)]^t_{i \ne j}\right]^t,$$

where for each $2 \le \ell \le L$, $[P_{ij}(\ell)]_{i \ne j}$ is a $(K^2 - K) \times (K^2 - K)$ matrix, all of whose entries are independent indeterminates.

Therefore, $J = -[[B_2 P_{ij}(2)]^t_{i \ne j} \mid \cdots \mid [B_L P_{ij}(L)]^t_{i \ne j}]^t$. Hence, if we partition the set $\{i \ne j | 1 \le i, j \le K\}$ into $L-1$ disjoint sets S_ℓ, $\ell \le 2 \le L$, and specialize $[P_{ij}(\ell)]_{i \ne j}$ to be a diagonal matrix, whose diagonal entries are -1 for columns in S_ℓ and 0 otherwise, then J specializes to a partitioned matrix J_0, with verticle blocks Q_ℓ, $2 \le \ell \le L$, consisting of the columns of B_ℓ for indices which lie in S_ℓ and has 0 columns for indices which do not. Let C_ℓ be the $(2K-2) \times |S_\ell|$ submatrix of Q_ℓ consisting of the columns of Q_ℓ whose indices lie in S_ℓ. To finish our proof, it suffices to show that J_0 is of full rank, which happens precisely when each Q_ℓ has column rank $|S_\ell|$, which in turn happens precisely when C_ℓ has full column rank.

Now we must specify S_ℓ and check that the resulting C_ℓ are of full column rank. We first consider the case when K is even. Allowing S_ℓ to be empty for $\ell > (K+2)/2$, it suffices to prove the result when $K = 2L - 2$, which we will now assume. Hence $L - 1 = K/2$, so we need to find $K/2$ sets S_ℓ, $2 \le \ell \le K/2 + 1 = L$.

Assume in what follows that the ordered pairs of indices are defined modulo K. We set

$$\begin{aligned} S_2 &= \{(i,j) | j = i+1, i+2, i \ne K\}, \\ S_3 &= \{(i,j) | j = i+3, i+4, i \ne K\}, \ldots, \\ S_{K/2} &= \{(i,j) | j = i+(K-3), i+(K-2), i \ne K\}, \\ S_{K/2+1} &= \{(i,j) | j = i+K-1 \text{ or } i = K\}, \end{aligned}$$

which are $(K/2)$ disjoint sets each of size $2K - 2$. Now we must show that the corresponding C_ℓ are of full column rank (which is the same as full rank, since they are of size $2K \times 2(K-1)$).

Note that if from C_2 we omit the row corresponding to partial differentiation with respect to x_K, we get the matrix J_K of Example 2, so C_2 has full rank $2K - 2$. In fact, each of $C_3, \ldots, C_{\frac{K}{2}}$ is the same as C_2 after a permutation of the indices of $y_1, \ldots, y_K$, so each is of full rank $2K - 2$. Finally, $C_{\frac{K}{2}} + 1$ is the matrix $\mathcal{J}_K$ of Example 3, so it has full rank $2K - 2$. This completes the proof of Theorem 1 in the case that K is even.

The case where K is odd is similar. As above, we might as well assume $K = 2L - 3$, so we need to find $(K+1)/2$ sets S_ℓ, $2 \le \ell \le (K+3)/2$.

We set

$$\begin{aligned} S_2 &= \{(i,j) | j = i+1, i+2, i \ne K\}, \\ S_3 &= \{(i,j) | j = i+3, i+4, i \ne K\}, \ldots, \\ S_{(K+1)/2} &= \{(i,j) | j = i+(K-2), i+(K-1), i \ne K\}, \\ S_{(K+3)/2} &= \{(i,j) | \ i = K\}, \end{aligned}$$

which are $(K+1)/2$ disjoint sets. As above, the corresponding C_ℓ, $2 \le \ell \le (K+1)/2$ are of full column rank. It is easy to check that $C_{(K+3)/2}$ is as well: indeed we already used this in Example 3. This completes the proof of Theorem 1.

V. A Concluding Remark

In principle, the method given in this work should be applicable to any problem (that may involve multiple antennas, time extensions, channel diversity, etc.) in which the channel matrices can be written in terms of the variables in the beamforming matrices. In practice, one needs to be somewhat canny in the calculation of the rank of Jacobian matrices. We would be curious to see how widely applicable the method turns out to be.

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