Research Statement: Florian Sobieczky

My main research interests lie in probability theory and the spectral and structural theory of graphs. Discrete probabilistic models such as percolation, branching processes, and random graphs are structures with properties that may undergo phase-transitions. Examples of such properties include the rate of volume-growth of subgraphs (such as balls with linearly growing radius), heat-conductivity (in the sense of the mobility of random walks), connectedness (such as the existence of an infinite cluster), and amenability. In particular, the question of how **random perturbations** affect these properties is the objective of my work. The goal is to understand the role of the randomness of these perturbations, in particular when the specific property reveals itself to be an **invariant**.

A now classical result by Barlow ([1], 2004) in this direction is the invariance of the asymptotic type of a simple random walk's return probability on a percolation cluster. This problem is archetpyical in that it is a random process defined on a *randomly* perturbed structure. Similar questions concerning my interest are the following:

i. Which change occurs in the asymptotics of the Integrated Density of States (IDS) of a graph Laplacian when passing from an infinite graph to a random subgraph?

ii. When is a percolation subgraph of an amenable graph still amenable?

iii. Under what circumstances does the rate of escape of a simple random walk on a random subgraph of an infinite graph vanish, if it is positive before the 'random perturbation'?

iv. For which of the questions above does the answer become different if either a random walk other than the simple random walk is considered, or if the measure generating the random subgraphs is alternated, possibly from Bernoulli to some invariant measure with strong correlations?

None of these questions is tangible with the aid of classical (Fourier-type) methods from harmonic analysis, due to the (almost sure) lack of transitivity of randomly generated subgraphs, such as a percolation cluster. Therefore, usually new types of approaches are asked for, in order to answer them. So far, I have obtained results in the following specific problems of this type:

- **Random Walks on Percolation Cluster**

  [FS1] 'An interlacing technique for spectra of random walks and its application to finite percolation clusters', Jour. Theor. Probab., 23, No. 3, (2010), 639-670

  [FS2] 'Bounds for the annealed return probability on large finite random percolation clusters', (2010), arXiv:0812.0117

  For illustration of the importance of the specific type of randomness of the perturbations (such as those defined by percolation), it may be asked of how relevant is the assumption of *invariance* of the perturbing measure under automorphisms of the underlying graph (a property which in the particular case of percolation on euclidean lattices is often discribed as 'stationarity under translations'). It may be suspected, that this quality of the perturbing measure has no further effect on properties which concern heat conduction (such as the return probability of a random walk). It turns out, however, that an additional downscaling of the upper bound of the *expected* return probability can be observed:

  Theorem (Theorem 2.2 in [FS1]) *Given subcritical invariant percolation on an infinite connected unimodular transitive graph, the expected return probability* $p_t(o,o)$

of a continuous time nearest neighbour delayed random walk (with equal transition probabilities to neighbours) back to its origin "o" on the almost surely finite cluster of random size N and random spectral gap λ is bounded from above as

$$p_t(o,o) \;\leq\; \mathbb{E}[1/N] \;+\; \frac{c}{t^{1/6}}\mathbb{E}\left[e^{-t\lambda}\right].$$

The prefactor $c/t^{1/6}$ is not present if the assumption of invariance of the graph 'hosting' the percolation is dropped. Thereby, this theorem supports a general folklore-result, that any additional symmetry in a random structure (in this case it is the invariance under automorphisms) *increases mobility*, which here is seen as the randomly migrating particle more quickly diffusing away from its origin (reflected by a stronger decrease of the return-probability). Similar observations are made in the theory of random Schroedinger operators ('Localization'). A result showing that randomness doesn't always decrease mobility has been given by [12]. To understand the nature of random perturbations that lead to such 'slowing down of random walk' is one of the central themes of my work.

In [FS2], the results obtained for subcritical percolation are extended to critical percolations with almost surely finite clusters. The technical challenge manifests itself in the fact that here the cluster-size-distribution doesn't have integral moments. In this case, the prefactor increases in its importance, as the rate of decay of the term $\mathbb{E}[\exp(-t\lambda)]$ is itself polynomial instead of subexponential (as for subcritical percolations). There are no precise asymptotics of the expected return probability for critical percolation. The upper bound derived in [FS2], with a novel estimation technique, gives the following result for homogeneous trees:

Theorem: (in [FS2]) *The expected return probability $p_t(o,o)$ of continuous time nearest neighbour random walk with equal probability to jump to its neighbours on the almost surely finite cluster of random size N in critical invariant percolation on a homogeneous infinite tree is bounded above by*

$$p_t(o,o) \;\leq\; \mathbb{E}[\frac{1}{N}] \;+\; \frac{C(\epsilon)}{t^{3/4+\epsilon}},$$

where $C(\epsilon)$ exists for every given $\epsilon > 0$.

This result is strong enough to prove that $p_t(o,o)$ is different for the almost surely finite critical clusters, as opposed to the so called incipient infinite cluster. For the latter the Alexander-Orbach conjecture has been shown to hold, implying that $p_t(o,o) \sim t^{-2/3}$ [11]. A practical implication of this result is that is possible, given a simulation of random walks on percolation graphs in the critical case, to distinguish just by measurements of the asymptotics of $p_t(o,o)$ (or equivalently by its Laplace-transform, the Integrated Density of States - IDS), if the simulation was run on the finite clusters, or (as often intended [3]) on the incipient infinite cluster.

The methodology used here are mainly **comparison techniques**: with the help of eigenvalue bounds from finite graph spectral theory ('interlacing'), comparison between Dirichlet forms, and semi-groups is possible, as long as the random subgraphs are finite [FS1]. Their specific geometry determines the quenched mixing rate. Asking for the *expected* return probability of a simple random walk means summing up the contributions from the different realisations of the random state spaces. Thus, the two sources of probability compete. For critical percolation [FS2], it is an open problem if there is a single type of cluster-configuration that predominantly defines the asymptotics of the annealed return-probability. This it is true for the sub-critical case, where the "linear" path-like sub-graphs, with their small spectral gap characterise the low-lying spectrum of the graph Laplacian.

Further research in this direction is currently undertaken, see below projects A, B and E.

- **Amenability of random partial graphs**

  [VK+FS] with V. Kaimanovich: 'Stochastic homogenization of horospheric tree products' in: Probabilistic approach to geometry, 199–229, Adv. Stud. Pure Math., 57, Math. Soc. Japan, 2010

  [FS3] 'Amenability of horocyclic products of percolation trees', (2009), arXiv:0903.3140

  Amenability of a graph is the quality to be able to introduce a *mean* on its structure. As is well known, the volume growth-properties of finite balls is intimitely connected with amenability. We say that a graph has exponential/polynomial volume growth, if the rate at which the volume of the finite ball grows in the size of its radius (measured with the graph metric) grows is exponential/polynomial. While all graphs with polynomial volume growth are amenable, every non-amenable graph has exponential volume-growth. The interesting intermediate regime is determined by amenable graphs with exponential growth. As this is an untypical property, it is interesting to ask about its stability under random perturbations, such as percolation[7]. **Horocyclic products** of homogeneous trees are examples of such graphs, and it is natural to consider horocyclic products of random trees, such as Galton-Watson trees. In [FS3] it has been shown, that amenable horocyclic products of Galton Watson trees with finite-support offspring distribution exist, as long as both of the trees have almost surely the same expected number of offspring. [VK+FS] is a generalisation of these results to so called conformal trees, in which only the equality of the asymptotic exponential growth rate is required.

  The results are derived in the framework of measurable equivalence relations [4], a groupoid generalisation of random graphs [5]. In this abstract approach, it is of interest to find alternative examples to the Galton-Watson trees considered in [FS3]. This is an open question and the subject of a collaboration with Prof. Vadim Kaimanovich (see project D.). Under additional assumtions about the rate at which the asymptotic exponential grwoth is attained (so called 'uniformly growing trees'), it has been achieved to give sufficient and necessary criteria of amenability for the corresponding horocyclic products of these trees (project C).

  The methods involved rest on the fact that for invariant percolation, the theory of measured equivalence relations by Feldman and Moore ([4], 1977) can be used to represent the randomly generated graphs as equivalence classes together with a 'graphing' [3]. This approach allows using the results from ergodic theory, and the theory of von Neumann algebras to obtain structural theorems about graphs for which there is a so called **invariant measure**. The invariance here is with respect to a set of partial isometries generating the equivalence relation. Amenability is one of the properties that can be proven by results about the amenability of equivalence realtions, which otherwise is possible only by using less direct combinatorial arguments [FS3].

- **Structural properties of randomly perturbed queueing systems**

  [FS+GR+ES] with G. Rappitsch, E. Stadlober: 'Tandem Queues for Inventory Management under Random Perturbations', Quality and Reliability Eng. Int. 26(8): 899-907 (2010)

  Queueing-networks have traditionally been used in applications such as for supply-chain modeling. As it is well know, there is an immediate connection of tandem-queues with random walks in the quarter plane ($\mathbb{Z}^2$ with edges given by the nearest neighbours), for which the question about recurrence/transience has been solved by Malyshef in a revolutionary method [14] in the homogeneous case, not in any randomly perturbed case. It was the likeness of these two problems and strong difference in methods which gave rise to a workshop at the University of Jena (`http://users.minet.uni-jena.de/~se64hod/riemann.html`), adressing possible common questions with Kilian Rachel, a leading researcher in this field. The approach used in [FS+GR+ES] to give recurrence criteria in the randomly perturbed case is by means of Burkes theorem, which assures asymptotic independence of the involved, serially aligned queues. (In the application of the supply chains considered in the paper, recurrence represents the stability of an order-demand-equilibrium.)

A further 'random perturbation' is also considered in [FS+GR+ES]: In mimmicking a time-inhomogeneous process a hidden parameter decides which set of transition rates are currently active. Such models were considered in a series of papers by H.D. Miller [15]. The idea that for a time-inhomogeneous random walk, in which concurrently to the movement of the random walk, a structurual change takes place is developed jointly in work with Prof. Steven Lalley in project F (see below).

Current projects:

The following currently pursued projects resulted from this earlier work:

A. **Extending the comparison techniques [FS1, FS2] to infinitely large graphs**: Supercritical percolation as well as the incipient infinite cluster deals with infinitely large random subgraphs. The conventional 'combinatorial approach' allows only using the methods from finite graphs spectral theory together with an appropriate limiting process. However, in the theory of measured equivalence relations, also infinite equivalence classes may carry a finite measure. This together with the theory of graph limits (involving the Szemerédi regularity lemma) has recently lead to important results about the integrated density of states (**IDS**) of random percolation graphs. The goal is to determine the expected return-probability on the incipient infinite cluster on Euclidean lattices in high dimensions. Kozma and Nachmias have recently proven the Alexander-Orbach conjecture in this case [11].

B. **Mixing time for finite percolation cluster**: Benjamini and Mossel (2003) have determined the mixing time of the finite graph obtained by restricting the supercritical percolation cluster to a finite box with given diameter. Similarily, it is of interest to study the finite percolation cluster under the condition that they 'touch the boundary of the box'.

C. with D. Lenz and I. Veselić, **Criteria for non-amenability**: The results obtained in [VK+FS, FS3] refer to randomly generated trees, and make no assumption on the variance of the n-th generation's population. By making assumptions on the uniformity of the involved trees' growth rate, sufficient and necessary criteria for amenability can be given. (This work is almost finished.)

D. with V. Kaimanovich: **Conformality of graphs** In [VK+FS], 'conformal trees' are defined to be the measure-space of random, compactified, rooted trees such that the corresponding probability measure ist absolutely continuous with respect to the map given by 'moving the root' to another vertex of the given tree. The only example so far is the so called Augmented Galton-Watson measure. Constructing other conformal tree-measure leads to answers of question ii. and a better understanding of 'dependent percolation'.

E. with T. Turova **IDS of the inhomogeneous random graph model**: The goal is to prove that the so called Lifshitz-tails of the IDS of Bernoulli percolation subgraphs on the Euclidean Lattice also exist almost surely for randomly generated graphs with 'randomly colored' vertices and a retention probability of the edges depending on the incident vertices' colors.

F. with S. Lalley **Reccurence of time-inhomogeneous random walks** : Is the time-inhomogeneous nearest-neighbour random walk on the subgraph of an infinite graph induced by a ball with a concurrently growing radius recurrent while it is transient on the infinite graph? Steve Lalley coined this problem the 'Bee in the Balloon problem' and solved it for Brownian Motion in $\mathbb{R}^d$. F.S has shown how to answer the question for the lazy delayed random walk on the euclidean lattice and on homogeneous trees.

G. with I. Krasovsky: **Stationary Determinental Percolation** A specific type of dependent percolation is obtained by chosing the conditional probabilities of single edges to be open, given that their neighbours are, by a determinant. These ensembles are well known in Random Matrix theory and intimetly related to the theory of Toeplitz operators [13]. Using results by Lyons and Steif ([6], 2003), we study various graph properties of the percolation clusters and compare them with Bernoulli percolation.

Bibliography:

1. M. T. Barlow: 'Random walks on supercritical percolation clusters'. Ann. Probab. 32 (2004), no. 4, 3024-3084
2. M. T. Barlow, T. Kumagai: 'Random walk on the incipient infinite cluster on trees', Illinois Journ. Math., 50, 33-65, (2006)
3. P. Argyrakis, R. Kopelman: 'Random walk on percolation clusters', Phys. Rev. B, 29, 1, (1984)
4. M. Feldman, C. Moore: 'Ergodic equivalence relations, cohomology, and von Neumann algebras'. I& II. Trans. Amer. Math. Soc. 234 (1977), no. 2, 289-359
5. Kaimanovich, Vadim A.: 'Equivalence relations with amenable leaves need not be amenable', Amer. Math. Soc. Transl. Ser. 2, 202, 151-166
6. R. Lyons, J. Steif: 'Stationary determinantal processes: phase multiplicity, Bernoullicity, entropy, and domination', Duke Math. J. 120, no. 3 (2003), 515–575
7. D. Chen, Y. Peres, G. Pete: 'Anchored expansion, percolation and speed', Ann. of Probab. 32, no. 4, 2978-2995, 2004
8. D. Aldous, R. Lyons: 'Processes on unimodular random networks', Electron. J. Probab. 12 (2007), no. 54, 1454-1508
9. I.Benjamini and E.Mossel: 'On the mixing time of a simple random walk on the super critical percolation cluster', Probab. Theory Related Fields 125 (2003), no. 3, 408-420
10. L.R.G. Fontès, P. Mathieu: 'On symmetric random walks with random conductances on Z^d' Probab. Theory Related Fields 134 (2006), no. 4, 565-602
11. G. Kozma, A. Nachmias: 'The Alexander-Orbach conjecture holds in high dimensions', Invent. Math. 178 (2009), no. 3, 635-654
12. C. Takacs: 'More randomness of environment does not always slow down a random walk', J. Theoret. Probab. 14 (2001), no. 3, 699-715
13. I. Krasovsky: 'Aspects of Töplitz determinants', in: Random Walks, Boundaries and Spectra, edt. by D.Lenz, F.Sobieczky, W.Woess, (2011), Progress in Probability, Vol. 64, 305-324
14. V. A. MALYSHEV, 'Analytic method in the theory of two-dimensional random walks', (1972), Sib. Math. J. 13 1314-1327
15. H.D. Miller: 'A convexivity property in the theory of random variables defined on a finite Markov chain. Ann. Math. Statist. 32 1961 1260-1270