(Contradiction)

1. (a) If x and y are rational, then $x+y$ is rational.

proof. suppose ~~to~~o the contrary $x+y$ is irrational.

since x and y are ratinal, there are integers $a, b, c \neq 0, d \neq 0$ such that $x = \frac{a}{c}$ and $y = \frac{b}{d}$. Then

$x+y = \frac{a}{c} + \frac{b}{d} = \frac{ad+bc}{cd}$. Note that $cd \neq 0$ because $c \neq 0$ and $d \neq 0$, also $ad+bc \in \mathbb{Z}$.

Therefore $x+y$ is rational, a contradiction to $x+y$ is irrational.

1 (b). If x is rational and y is irrational, then $x+y$ is irrational.

proof. suppose to the contrary that $x+y$ is rational. Then there exists integers a and $b \neq 0$ ~~integers~~ such that $x+y = \frac{a}{b}$. Moreover, we have x is rational, so there exists integers $c, d \neq 0$ such that $x = \frac{c}{d}$. note that

$$\frac{c}{d} + y = \frac{a}{b} \Rightarrow y = \frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}.$$

Since $ad-bc$ and $bd \neq 0$ are integers, y is a rational number, a contradiction since y is rational.

2. prove that $\sqrt{2}$ is irrational.

suppose to contrary that $\sqrt{2}$ is rational. So there are integers $a, b \neq 0$ and $(a, b) = 1$ such that $\sqrt{2} = \frac{a}{b}$.

Thus, $2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2 \Rightarrow a^2$ is even $\Rightarrow$ a is even $\Rightarrow \exists\, k \in \mathbb{Z}$ such that $a = 2k$.

So $2b^2 = (2k)^2 = 4k^2 \Rightarrow b^2 = 2k^2 \Rightarrow$ b^2 is even $\Rightarrow$ b is even.

Therefore $\gcd(a,b) = 2$, yielding a contradiction since $\gcd(a,b) = 1$.

3. Direct proof.

Suppose that $5m+3n$ is even. Then $5m+3n=2k$ for some integer k. So we have

$$m = 2k - 4m - 3n.$$

It must be the case that n is either odd or even.

Case 1. n is even. Then $n=2l$ for some integer l.

Thus $m = 2k - 4m - 6l = 2(k-2m-3l)$.

Since $k-2m-3l$ is an integer, we can conclude that m is even.

Case 2. n is odd. so we have $n = 2t+1$ for some integer t. Therefore,

$$m = 2k - 4m - 3(2t+1) = 2k - 4m - 6t + 1 =$$

$$2(k-2m-3t)+1.$$

Since $k-2m-3t$ is an integer, we can conclude that m is odd.

3. Contrapositive

It is logically equivalent to prove that if m is even and n is odd or m is odd and n is even, then $5m+3n$ is odd.

Suppose m is even and n is odd. Then $m = 2k$ and $n = 2t+1$ for some $k, t \in \mathbb{Z}$.

Thus

$$5m+3n = 5(2k)+3(2t+1) =$$

$$10k + 6t + 3 =$$

$$2(5k+3t+1)+1$$

Therefore, by definition, $5m+3n$ is odd since $5k+3t+1$ is an integer.

An analogous argument shows that if m is odd and n is even, then $5m+3n$ is odd.

3. Contradiction

suppose to the contrary that $5m+3n$ is even when ① m is even and n is odd or ② m is odd and n is even.

① consider $5m+3n$ and let $m=2k$ and $n=2t+1$ for some $k,t \in \mathbb{Z}$. Thus

$$5m+3n = 5(2k)+3(2t+1) =$$
$$10k+6t+3 =$$
$$2(5k+3t+1)+1.$$

we can say $5m+3n$ is odd since $5k+3t+1$ is an integer. Yielding a contradiction since by assumption $5m+3n$ is even.

An analogous argument can be used for the case when m is odd and n is even.