Proof of MCT

It is easy to see that it is enough for nondecreasing sequences, and when a.e. is replaced by everywhere convergence. Also, it is easy to see that \( f_n^+ \nearrow f^+ \) and \( f_n \searrow f^- \). Our goal is thus to prove these:

(i)
\[
\int_E f_n^- \, dx \nearrow \int_E f^- \, dx
\]

(ii)
\[
\int_E f_n^+ \, dx \searrow \int_E f^+ \, dx
\]

Proof of (i)

Clearly, it is enough to show that
\[
\liminf_{n \to \infty} \int_E f_n^+ \, dx \geq \int_E f^+ \, dx.
\]

Because of the definition of the Lebesgue integral for nonnegative functions, it is easy to see that it is enough to prove the following. If \( \phi \) is a simple function and \( \phi \leq f^+ \), then
\[
\liminf_{n \to \infty} \int_E f_n^+ \, dx \geq \int_E \phi \, dx.
\]

Let \( \phi = \sum_{i=1}^k a_i 1_{E_i} \), where \( E \) is the disjoint union of the \( E_i \) and \( f^+ \geq a_i \) on \( E_i \), \( 1 \leq i \leq k \). Fix \( \epsilon > 0 \). Then
\[
\sum_{i=1}^k \int_{E_i} f_n^+ \geq \sum_{i=1}^k \int_{E_i \cap \{f_n^+ \geq a_i - \epsilon\}} f_n^+ \geq \sum_{i=1}^k (a_i - \epsilon)m(E_i \cap \{f_n^+ \geq a_i - \epsilon\}).
\]

Taking \( \liminf \) on both sides, on the RHS we actually have a limit:
\[
\liminf_{n \to \infty} \int_E f_n^+ = \liminf_{n \to \infty} \sum_{i=1}^k \int_{E_i} f_n^+ \geq \sum_{i=1}^k (a_i - \epsilon)m(E_i).
\]

Letting \( \epsilon \downarrow 0 \), we are done.

Proof of (ii) We are now going to use the assumption in the statement of MCT. Because of that assumption, \( \int_E f_1 > -\infty \), that is, \( \int_E f_1^- < \infty \) and so \( \int_E f_n^- < \infty \), \( n \geq 1 \). We use these in the following calculations. First,
\[
\int_E f_n^- - \int_E f^- = \int_E (f_n^- - f^-).
\]

Hence, it is enough to check the following statement:

if \( f_n \searrow 0 \) and \( \int_E f_1 < \infty \), then \( \int_E f_n \searrow 0 \).

Let \( g_n := f_1 - f_n \). Then \( g_n \nearrow f_1 \), and just like in part (i), we can show that \( \int_E g_n \nearrow \int_E f_1 \), that is \( \int_E f_1 = \int_E f_n \nearrow \int_E f_1 \), and so \( \int_E f_n \searrow 0 \).