Solutions to HW3
Introduction to Real Analysis 1. (MATH 6310)

p. 34 Problem 6. A single point has obviously zero outer measure and so this is also true for $B = \mathbb{Q} \cap [0, 1]$, by countable subadditivity. Again, by subadditivity, $m^*(A) = m^*(A) + m^*(B) \geq m^*([0, 1]) = 1$. But, by monotonicity, $m^*(A) \leq 1$. □

p. 34 Problem 7. Clearly, $\gamma := m^*(E) < \infty$, and thus, for each $n \geq 1$ there exists an open cover $G_n = \bigcup_{k=1}^{\infty} I_{k}^{(n)}$ (a union of countably many open intervals) such that $E \subset G_n$ and $\sum_{k=1}^{\infty} m(I_{k}^{(n)}) \leq \gamma + 1/n$. By definition, $G := \cap_n G_n$ is a $G_{\delta}$-set, and clearly, $E \subset G$. Now,

$$\gamma \leq m^*(G) \leq m^*(G_n) \leq \sum_{k=1}^{\infty} m(I_{k}^{(n)}) \leq \gamma + 1/n.$$  

Since $n \geq 1$ is arbitrary, we are done. □

p. 34 Problem 8. Prove by induction on $n$ (number of intervals). For $n = 1$ it is trivial. Let us assume the statement is true for $n-1$ and consider $I_n$. (Here we note that we may apply the induction hypothesis for intervals other than $[0, 1]$; it is obvious that if the statement is true for $[0, 1]$, then it is also true for any other closed bounded interval.) Then $I_n$ either covers $[0, 1]$ itself, or $[0, 1] \setminus I$ has two parts (each intervals), or it has three parts (each intervals). Let’s consider the third case (the other two cases are even simpler). That is, let $[0, 1] = [0, a] \cup I_n \cup [b, 1]$, where this is a disjoint union.

We know that both $[0, a]$ and $[b, 1]$ are covered by at most $n-1$ intervals each. If these two collections of intervals have a common element, then either it is not needed, or the interval $I_n$ is not needed, and we are done, by the induction hypothesis.

If these two collections have no common element, then by the induction hypothesis, the total sum of the outer measures of all of these intervals is at least $a + (1 - b)$. But the length of $I_n$ is $b-a$, so the gross total for the $n$ intervals is no less than $a + (1 - b) + (b - a) = 1$. □
p. 79 Problem 9.

\[ m^*(B) \leq m^*(A \cup B) \leq m^*(A) + m^*(B) = m^*(B). \]

\[ \square \]

p. 85 Problem 10. It is enough to prove that

\[ m^*(A \cup B) \geq m^*(A) + m^*(B). \]

Let us call a cover \( G \) of \( A \) ‘trimmed’ if it is a union of countably many open bounded nonempty intervals and satisfies that for all \( x \in G \) there exists a \( y \in A \) such that \( |x - y| < \alpha/2 \). Trimmed covers of \( B \) and trimmed covers of \( A \cup B \) are defined similarly.

Because of the condition and the triangle inequality, a trimmed cover of \( A \) and a trimmed cover of \( B \) has no common element. It is also clear that if \( G \) is a non-trimmed cover of \( A \), then \( G^* := G \cap (\bigcup_{x \in A} B_{\alpha/2}(x)) \) is a trimmed cover of \( A \) and \( G^* \subset G \). (Same is true for \( B \) and for \( A \cup B \).) Therefore, in the definition of the outer measure one may replace ‘infimum over all covers’ by ‘infimum over all trimmed covers.’

Next, the disjoint union of a trimmed cover of \( A \) and a trimmed cover of \( B \) is a trimmed cover of \( A \cup B \); conversely, a trimmed cover of \( A \cup B \) is the disjoint union of a trimmed cover of \( A \) and a trimmed cover of \( B \).

Because of the last paragraph, and since \( m^*(A \cup B) \) can be calculated using the infimum over all trimmed covers of \( A \cup B \), it is larger or equal than the sum of \( m^*(A) \) (calculated by using trimmed covers) and \( m^*(B) \) (calculated by using trimmed covers), which is the same as simply \( m^*(A) + m^*(B) \). \[ \square \]