Math 4001 Sol's to HW3

8.30 $\sum_{m,n} a(n) x^{mn} = \sum_{n=1}^{\infty} a(n) \sum_{m=1}^{\infty} x^{mn} = \sum_{n=1}^{\infty} a(n) \frac{x^n}{1-x^n}$

for $|x|<1$, where we used Thm 8.42.
So $\sum^{\infty} a(n) \frac{x^n}{1-x^n}$ converges abs-ly to $S(x)$.
Since every term in $\sum a(n) x^{mn}$ appears exactly once in $\sum_1^{\infty} A(n) x^n$, by Thms 8.42 and 8.13,

$$\sum_{n=1}^{\infty} A(n) x^n = \sum a(n) x^{mn} = S(x).$$

8.35 $\left(\sum_1^{\infty} \frac{1}{n^s}\right)\left(\sum_1^{\infty} \frac{1}{n^s}\right) = \sum_1^{\infty} C_n$, where

$$C_n = \sum_{d|n} d^{-s} \cdot \left(\frac{n}{d}\right)^{-s} = n^{-s} \sum_{d|s} 1 = n^{-s} d(n).$$

8.37 (a) Let $S_0 := 0$.
Then $t_n = \sum_1^n k(s_k - s_{k-1})$

$$= \sum_1^n k s_k - \sum_{k=1}^{n-1} (k+1) s_k$$

$$= \sum_{k=1}^{n} k s_k - \sum_{k=1}^{n} (k+1) s_k + (n+1) s_n$$

$$= (n+1) s_n - \sum_1^n s_k = (n+1) s_n - n\sigma_n.$$

(b)

$\sum a_n$ conv $\Rightarrow \exists \lim s_n =: a$ and $\lim \sigma_n = a$

$\Rightarrow \lim \frac{t_n}{n} = \lim \left(\frac{n+1}{n} s_n - \sigma_n\right) = 0.$

Conversely, if $\frac{t_n}{n} \to 0$ then $\frac{n+1}{n} s_n - \sigma_n \to 0$, and so $s_n - \sigma_n \to 0$.