4001 HW 1 (SOL.)

8.4 Enough liminf (limsup similar)

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{a_1}{1}\frac{a_2}{a_1}\frac{a_3}{a_2}\cdot\ldots\cdot\frac{a_n}{a_{n-1}}}$$

Enough when $\liminf \frac{a_{n+1}}{a_n} > -\infty$, o/w it is trivial.

Let $\liminf \frac{a_{n+1}}{a_n} = L \in (-\infty, \infty]$. (*)

We prove that $\liminf \sqrt[n]{a_n} > k$ for any $k < L$.

Indeed, $\sqrt[n]{a_n} = \sqrt[n]{\frac{a_1}{1}\frac{a_2}{a_1}\cdot\ldots\cdot\frac{a_N}{a_{N-1}}}\cdot\sqrt[n]{\frac{a_{N+1}}{a_N}\cdot\ldots\cdot\frac{a_n}{a_{n-1}}}$

where N is such that $\frac{a_{i+1}}{a_i} > \hat{k}$ for $i \geq N$, where $k < \hat{k} < L$.

($\exists N$ because of (*).)

If $n \to \infty$, the first $\sqrt[n]{\ }$ tends to 1, and the second is $> \hat{k}$, so if n is large, $\sqrt[n]{a_n} > k$. ▨

8.5 $\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!}\frac{n!}{n^n} = \left(\frac{n+1}{n}\right)^n\cdot\frac{\cancel{n+1}}{\cancel{n+1}} \to e$

By (8.4), $e = \liminf \frac{a_{n+1}}{a_n} \leq \liminf \frac{n}{\sqrt[n]{n!}} \leq \limsup \frac{n}{\sqrt[n]{n!}} \leq \limsup \frac{a_{n+1}}{a_n} = e.$

So, equality throughout. ▨

8.6 $b_n := \min\{a_1,\ldots,a_n\}$, $B_n := \max\{a_1,\ldots,a_n\}$

Then $b_n \searrow$, $B_n \nearrow$, and $b_n \leq \sigma_n \leq B_n$.

So $\liminf a_n \leq \lim b_n \leq \liminf \sigma_n \leq \limsup \sigma_n \leq \lim B_n \leq \limsup a_n$ ▨

8.8 Apply Thm 8.23 to $f(x) = \frac{1}{\sqrt{x}}$. Then $\int_1^n f(x)\,dx = 2(\sqrt{n}-1)$. By Thm 8.23, $\exists \lim d_n$. Also, $0 < \sum_1^n \frac{1}{\sqrt{i}} - 2(\sqrt{n}-1) \leq 1 \Rightarrow 1 \leq 2\sqrt{n} - \sum_1^n \frac{1}{\sqrt{i}} < 2.$ ▨