$\boxed{9.30}$ If $|z-z_0| < \liminf\limits_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$, then $\exists\ \gamma \in (0,1)$ s.t.

$|z-z_0| < \gamma \left|\frac{a_n}{a_{n+1}}\right|$ for all large n (say $n \geq N$.)

So $\frac{|a_{n+1}||z-z_0|^{n+1}}{|a_n|\,|z-z_0|^n} = \frac{|a_{n+1}|}{|a_n|}|z-z_0| < \gamma$, $n \geq N$.

Comparing with geom. series (quot <1), $\sum a_n(z-z_0)^n$ is absolutely convergent.

If $|z-z_0| > \limsup \left|\frac{a_n}{a_{n+1}}\right|$, then $\exists\ \hat{\gamma} > 1$ s.t.

$|z-z_0| > \hat{\gamma} \left|\frac{a_n}{a_{n+1}}\right|$ for all large n (say $n \geq N$).

So $\frac{|a_{n+1}|\,|z-z_0|^{n+1}}{|a_n|\,|z-z_0|^n} = \frac{|a_{n+1}|}{|a_n|}|z-z_0| > \hat{\gamma}$, $n \geq N$.

Comparing with geom. series (quot >1), $\sum a_n(z-z_0)^n$ diverges

$\boxed{9.31}$

Use the formula $R = \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}}$.

(a) $\frac{1}{\overline{\lim} \sqrt[n]{|a_n^k|}} = R^k = 2^k$

(b) $\sum\limits_0^\infty a_n z^{kn} = \sum\limits_0^\infty b_m z^m$, where $b_m = \begin{cases} a_n & \text{if } m=kn \\ 0 & \text{o.w.} \end{cases}$

$\frac{1}{\overline{\lim} \sqrt[m]{|b_m|}} = \frac{1}{\overline{\lim} \sqrt[kn]{|a_n|}} = R^{1/k} = 2^{1/k}$

(c) $\sum\limits_0^\infty a_n z^{n^2} = \sum\limits_0^\infty b_m z^m$, where $b_m = \begin{cases} a_n & \text{if } m=n^2 \\ 0 & \text{o.w.} \end{cases}$

$\frac{1}{\overline{\lim} \sqrt[m]{|b_m|}} = \frac{1}{\overline{\lim}\, |a_n|^{1/n^2}} = 1$ b/c $\overline{\lim}\, |a_n|^{1/n^2} = e^{\frac{1}{n}\ln\left(\overline{\lim}\, |a_n|^{1/n}\right)} = 1$

$\boxed{9.33}$ For both statements, it is enough to show that $f^{(n)}(0) = 0$, $n = 1, 2, \ldots$

(Taylor series in fact represents the fct $g \equiv 0$.)

Let's see it for $n = 1, 2$:

$$f'(0) = \lim_{x \to 0} \frac{1}{x} e^{-1/x^2} = \lim_{t \to \infty} t e^{-t^2} = 0$$

$$f''(0) = \lim_{x \to 0} \frac{1}{x}\left(-2x^{-3} e^{-1/x^2}\right) = \lim_{t \to \infty} -2t^{-4} e^{-t^2} = 0.$$

In general, for $x \neq 0$, $f^{(n)}(x)$ is of the form $p(x^{-1}) e^{-1/x^2}$, where p is a polynomial. This is true for $n = 1$, and if true for $f^{(n)}$, then for $x \neq 0$,

$$f^{(n+1)}(x) = \left(f^{(n)}(x)\right)' = \left(-p'(x^{-1}) \cdot x^{-2} - 2p(x^{-1}) \cdot x^{-3}\right) e^{-\frac{1}{x^2}} =: q(x^{-1}) e^{-\frac{1}{x^2}}$$

Hence, if we know that $f^{(n)}(0) = 0$, then

$$f^{(n+1)}(0) = \lim_{x \to 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x} = \lim_{x \to 0} \frac{f^{(n)}(x)}{x} = \lim_{x \to 0} x^{-1} q(x^{-1}) e^{-1/x^2}$$

$$= \lim_{t \to \infty} t q(t) e^{-t^2} = 0.$$ Since $f'(0) = 0$, we are done by induction.

9.34

<u>Divergence</u>: If $x = -1$, $\alpha < 0$, then $\left|\binom{\alpha}{n}(-1)^n\right| = \frac{1}{n!}\left|(-\alpha)(-\alpha+1)\ldots(-\alpha+n-1)\right| \overset{|\alpha|}{\geq} \frac{(n-1)!}{n!} = \frac{|\alpha|}{n}$, and so diverges by comparison with harmonic series.

If $x = 1$, $\alpha \leq -1$, then $\left|\binom{\alpha}{n} x^n\right| = \left|\frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}\right| \geq 1 \not\to 0$, so diverges.

<u>$x = -1$, $\alpha \geq 0$</u>: $\binom{\alpha}{n}(-1)^n = \frac{-\alpha}{n}(1-\alpha)\left(1-\frac{\alpha}{2}\right)\ldots\left(1-\frac{\alpha}{n+1}\right) = \frac{-\alpha}{n} \text{const}\left(1-\frac{\alpha}{n_0}\right)\ldots\left(1-\frac{\alpha}{n-1}\right)$, where $n_0 = \lceil \alpha \rceil$.

(WLOG, $\alpha \notin \mathbb{N}$, o/w trivial). The product $\leq \exp\Big(-\alpha\underbrace{\left(\frac{1}{n_0} + \ldots + \frac{1}{n-1}\right)}_{\text{const} + \log n + o(1)}\Big) = \left(n \,\text{const}\, e^{o(1)}\right)^{-\alpha}$.

So $\sum \left|\binom{n}{\alpha}(-1)^n\right| \leq \text{const} \cdot \sum \frac{1}{n^{1+\alpha}}$ ✓

<u>$x = 1$, $\alpha \geq 0$</u>: Same pf as previous shows $\sum \left|\binom{n}{\alpha} \cdot 1^n\right| \leq \text{const} \sum \frac{1}{n^{1+\alpha}}$. ✓

<u>$x = 1$, $-1 < \alpha < 0$</u>: Not AC, b/c $\left|\frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}\right| \geq |\alpha| \frac{1}{n}$, and by comparison with harmonic series. But convergent, b/c $\binom{\alpha}{n} = (-1)^n \frac{\beta}{n}(\beta+1)\left(\frac{\beta}{2}+1\right)\ldots\left(\frac{\beta}{n-1}+1\right)$, where $\beta := |\alpha|$. It's a Leibniz series, as the ratio between conseq. terms $= \left(\frac{\beta}{n}+1\right)\frac{n}{n+1} = \frac{\beta+n}{n+1} < 1$, and also the n^{th} term $\to 0$, as it can be written as $\frac{1}{n!}(1-\gamma)(2-\gamma)\ldots(n-\gamma)$; $\gamma := 1-\beta$. Use again $\left(1-\frac{\gamma}{k}\right) \leq e^{-\gamma/k}$ and that $1 + \ldots + \frac{1}{n} = \log n + C + o(1/n)$, together with $\gamma > 0$.