Solutions to HW6 (Math 4001)

[9.23's solution [redacted] : **next page**]

$\boxed{9.18}$ $f_n \to 1$ ptwise b/c $nx^2 \to \infty$ $\forall x > 0$. But not unif-ly, b/c $1 - f_n\left(\frac{1}{\sqrt{n}}\right) = \frac{1}{2}$ and so $\sup_{[0,1]} |1 - f_n(x)| \geq \frac{1}{2} \not\to 0$.

Term-by-term int. is OK by Arzela's thm. ($f_n \leq 1$ $\forall n$)

$\boxed{9.22}$ Apply the M-test (Thm 9.6), using that

$|a_n \sin nx|, |a_n \cos nx| \leq a_n$.

$\boxed{9.27}$ It's enough to check it for a unit interval, b/c finite intervals are obtained by scaling and infinite intervals by countable union. So let $I = [(a,b)]$, $b - a = 1$.

Let us use the shorthand $\lambda(A) := \int_I 1_A$ for $\mathbf{1_A \in R(I)}$.

Define $A_n^\epsilon := \{x \in I : |f_n(x) - g(x)| > \epsilon\}$, $B_n^\epsilon := \{x \in I : |f_n(x) - f(x)| >$

Then, one can prove that $1_{A_n^\epsilon}, 1_{B_n^\epsilon} \in \mathbf{R}(I)$ and

$\lambda(A_n^\epsilon) \leq \frac{1}{\epsilon^2} \int_I (f_n(x) - g(x))^2$. Hence $\lambda(A_n^\epsilon) \to 0$ $\forall \epsilon > 0$.

By **BCT**, $\lambda(B_n^\epsilon) \longrightarrow \int_I 0 = 0$. By the triangle inequality,

This is Arzela for Riemann

$\lambda\left(\{x \in I \mid |f(x) - g(x)| > \epsilon\}\right) \leq \lambda(A_n^{\epsilon/2}) + \lambda(B_n^{\epsilon/2})$ and since this holds for $\forall n \geq 1$, therefore the LHS $= 0$.

Setting $\epsilon = \frac{1}{m}$, we have $\lambda\left(\{x \in I \mid |f(x) - g(x)| > \frac{1}{m}\}\right) = 0$ $\forall m = 1, 2, \ldots$

By **BCT**, $\lambda\left(\{x \in I : |f(x) - g(x)| > 0\}\right) = 0$. Now suppose that $\exists x_0$ s.t. $f(x_0) \neq g(x_0)$. Then, since f, g are continuous, $\exists I_0 \subset I$, s.t. $|f - g| > 0$ on I_0. But then $\lambda\left(\{x \in I : |f(x) - g(x)| > 0\}\right) > 0$; contradiction. $\boxtimes$

9.23

Let $F_n(x) := \sum_{k=1}^{n} \frac{\sin kx}{k}$. First we show that

Claim: $\{F_n(x)\}$ is unif-ly bdd in n and x.

pf: By periodicity and by being an odd fct, it is enough to consider the interval $(0,\pi)$. Let $\delta>0$.

(a) If $x \in [\delta, \pi]$, then we've seen (Thm 8.30) that $|F_n(x)| \le 1/\sin(\delta/2)$.

(b) If $x \in (0,\delta)$, then let $N := \lfloor \frac{1}{x} \rfloor$.

- $n<N \Rightarrow |F_n(x)| \le n|x| < N|x| \le 1$, using $\left|\frac{\sin\alpha}{\alpha}\right| \le 1$.
- $n \ge N \Rightarrow |F_n(x)| \le \underbrace{\left|\sum_{1}^{N-1} \frac{\sin kx}{k}\right|}_{\le 1} + \left|\sum_{k=N}^{n} \frac{\sin kx}{k}\right|$

Second term $\le$

$$\le \frac{1}{n+1}\left|\sum_{1}^{n} \frac{\sin kx}{k}\right| + \frac{1}{N}\left|\sum_{1}^{N-1}\frac{\sin kx}{k}\right| + \left|\sum_{k=N}^{n}\left(\sum_{j=1}^{k} \sin jx\right)\left(\frac{1}{k+1}-\frac{1}{k}\right)\right|$$

where we applied "summation by parts," which we used several times in the course.

$$\le \frac{1}{(n+1)\sin\frac{x}{2}} + \frac{1}{N\sin\frac{x}{2}} + \left(\frac{1}{N}-\frac{1}{n+1}\right)\frac{1}{\sin\frac{x}{2}}$$

$$= \frac{2}{\lfloor\frac{1}{x}\rfloor \sin\frac{x}{2}}$$, which tends to 4 at 0.

So, if δ is sufficiently small, $|F_n(x)| \le 5$ in $(0,\delta)$. ▨

Next, assume $na_n \to 0$. By the uniform version of Dirichlet's test, it follows that $\sum a_n \sin nx = \sum n a_n \frac{\sin nx}{n}$ conv. unif-ly. (Thm 9.15.)

Conversely, assume $\sum a_n \sin nx$ is uniformly conv, and so unif-ly Cauchy. Fix $\epsilon>0$. Then $\exists N$ s.t. for $n \ge N$, $\left|\sum_{n}^{2n-1} a_k \sin kx\right| < \epsilon$, for all $x \in \mathbb{R}$. Pick $x := \frac{1}{2n}$. Then $\sin(1/2) \le \sin(kx) \le \sin(1)$, and this is $\ge \sum_{n}^{2n-1} a_{2n} \sin(1/2)$ b/c $a_k \downarrow$, which is $n\, a_{2n} \sin(1/2) = 2n\, a_{2n} \cdot \text{const}$, and so $2n\, a_{2n} < \epsilon$ if $n \ge N$. For $(2n-1)\, a_{2n-1}$ it's similar. ▨

contradiction. ▨