MATH 4001

SOL's to HW #5

9.5 (a) Clearly, $f_n(x) \to 0\ \forall x$, but $\sup_{x\in(0,1)} \frac{1}{nx+1} = 1$.
(b) $\frac{x}{n+1} = \frac{1}{n+1/x} < \frac{1}{n+1} \to 0$.

9.6 $h_n(x) = g(x)x^n$. Fix $\epsilon > 0$.
Let $\delta > 0$ be so small that $|g(x)| < \epsilon$ for $x \in [1-\delta, 1]$.
Then
$$|h_n(x)| \le \begin{cases} |g(x)|(1-\delta)^n < M(1-\delta)^n, & x\in[0,1-\delta) \\ \epsilon, \text{ if } x\in[1-\delta,1]. \end{cases}$$
(g is bdd b/c contin. on $[a,b]$)
Hence
$$\overline{\lim_{n\to\infty}} |h_n(x)| \le \epsilon;$$ since $\epsilon > 0$ was arbitrary, we are done. ▨

9.12 We know that $\sum^{\infty} (-1)^{n+1} g_n(x)$ exists for each x, by Leibniz. So, we need that
$$\lim_{n\to\infty} \sum_{n+1}^{\infty} (-1)^{k+1} g_k(x) \rightrightarrows 0. \quad (\#)$$
In the pf of the Leibniz crit (8.16), we saw that for a fix $x \in T$, we have
$$\left|\sum_{n+1}^{\infty} (-1)^{k+1} g_k(x)\right| < g_{n+1}(x).$$
So (#) follows from the condition $g_n \rightrightarrows 0$.

9.16 $$\left|\int_0^1 f(x)dx - \int_0^{1-1/n} f(x)dx\right| = \left|\int_{1-\frac{1}{n}}^1 f(x)dx + \int_0^{1-1/n}(f(x)-f_n(x))dx\right|$$
$$\le \int_{1-\frac{1}{n}}^1 |f(x)|dx + \int_0^{1-1/n} |f(x) - f_n(x)|dx =: \mathrm{I} + \mathrm{II}$$
$\mathrm{I} \le \frac{1}{n}\cdot M$, where $|f| < M$ (f is bdd b/c contin.)

$f_n \rightrightarrows f$

$\mathrm{II} \le \sup_{x\in[0,1]} |f(x) - f_n(x)| \to 0$.

Hence $\mathrm{I} + \mathrm{II} \to 0$. ▨