Math 4001 Sol. to HW4

9.1 Pick N s.t. for $n \geq N$, $\sup_{x\in S} |f(x) - f_n(x)| < 1$.
Since f_N is bdd, thus so is f. If $|f| < M$, then for $n \geq N$

$$|f_n(x)| \leq |f_n(x) - f(x)| + |f(x)| < M+1, \ \forall x.$$

And for $\{f_1, f_2, \dots, f_{N-1}\}$ we have a uniform bound, of course. ▨

9.2 $f_n(x) \rightrightarrows f(x) := x$, b/c $|f_n(x) - f(x)| = |x|/n$, and $x \in I$ ← bdd

$g_n(x) \rightrightarrows g(x) := \begin{cases} 0, & 0 \text{ or irrat. } x \\ b, & x = \frac{a}{b}, b>0 \end{cases}$, b/c

$$|g_n(x) - g(x)| = \begin{cases} \frac{1}{n}, & 0 \text{ or irrat. } x \\ \frac{1}{n}, & x = \frac{a}{b}, b>0 \end{cases} = \frac{1}{n}.$$

But $h_n(x) \not\rightrightarrows h(x) := \begin{cases} 0, & \text{or irrat } x \\ b\cdot\frac{a}{b} = a, & x = \frac{a}{b}, b>0 \end{cases}$

b/c $\frac{a}{b}(1+\frac{1}{n})(b+\frac{1}{n}) = a + \frac{a}{bn} + \frac{a}{n} + \frac{a}{bn^2} = a\left(1 + \frac{1}{bn} + \frac{1}{n} + \frac{1}{bn^2}\right)$,
and $a\left(\frac{1}{bn} + \frac{1}{n} + \frac{1}{bn^2}\right)$ is not going to zero uniformly, b/c $\exists$ arbitrarily large "a" in any int. and so $\frac{a}{n} \not\rightrightarrows 0$.

9.3 (a) $|f_n + g_n - (f+g)| \leq |f_n - f| + |g_n - g| \rightrightarrows 0$.

(b) $|fg - f_n g_n| = |(f-f_n)g_n + f(g-g_n)| \leq |(f-f_n)g_n| + |f(g-g_n)|$
First term $\rightrightarrows 0$ b/c $\{g_n\}$ is uniformly bdd by 9.1.
Second term $\rightrightarrows 0$ b/c f is bounded (see pf of 9.1).

9.4 g is infact uniformly cont. on closed disk (Heine's Thm).
So, $\forall \delta > 0 \ \exists \epsilon$ s.t. $|g(f_n(x)) - g(f(x))| < \delta$ if $|f_n(x) - f(x)| < \epsilon$.
But, since $f_n \rightrightarrows f$, for this $\epsilon > 0$ $\exists N$ s.t. for $n > N$, $|f_n(x) - f(x)| < \epsilon \ \forall x \in S$.
Hence, for $n > N$, $|h_n(x) - h(x)| < \delta \ \forall x \in S$. ▨