CALCULUS 3

February 2, 2011

1st TEST

YOUR NAME: **T H E S O L U T I O N S**

- ◯ **001** J. Keller .... (9am)
- ◯ **002** B. Purkis .... (10am)
- ◯ **003** A. Spina .... (11am)
- ◯ **004** A. Spina .... (12pm)
- ◯ **005** M. Noyes .... (1pm)

SHOW ALL YOUR WORK

final answers without any supporting work will receive no credit *even if they are right!*

No cheat-sheets allowed.

Partial credit will be given for any **reasonable amount of work pointing in the right direction** towards the solution of your problem. You will not get any partial credit for memorizing formulas and not knowing how to use them, or for anything you write that is not directly related to the solution of your problem.

If your tests contains **more than one solution or answer** to a problem or part of a problem, and one of them is wrong, then **the wrong one** will be **counted** for your grading!

Make sure you write an arrow on top of vector quantities to differentiate them from scalar quantities (numbers). Remember that, within the same context, $\vec{v}$ (with the arrow) is a *vector* and v (without the arrow) is the *norm* of the previous vector If a vector is the null vector, write an arrow on top of the zero!

DO NOT WRITE INSIDE THIS BOX!

problem	points	score
1	15 pts	
2	10 pts	
3	15 pts	
4	10 pts	
5	15 pts	
6	10 pts	
7	10 pts	
8	15 pts	
TOTAL	100 pts	

1. **[15 pts]** Given the following functions

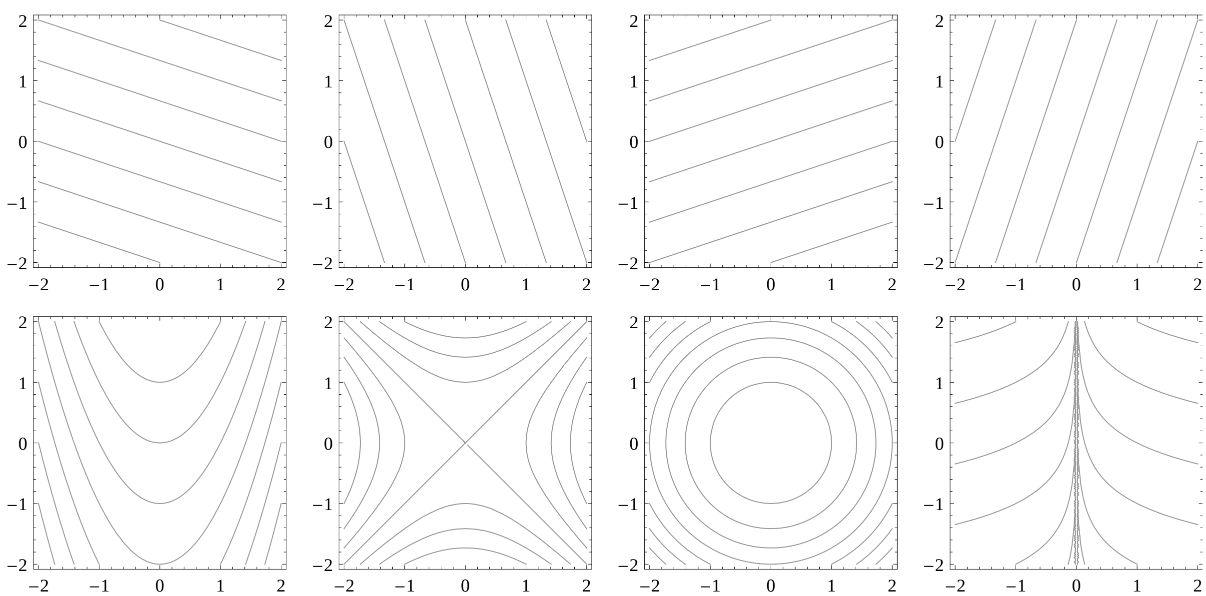
(**a**) $f(x,y) = 3x - y$	(**b**) $f(x,y) = x + 3y$
(**c**) $f(x,y) = x^2 - y$	(**d**) $f(x,y) = 2y + \ln|x|$
(**e**) $f(x,y) = x^2 - y^2$	(**f**) $f(x,y) = 3x + y$

write down the equations for their level-curvers.

Use the equations you just wrote to match the functions with the contour plots below by filling the appropriate circle.

Show all your work, no work = no credit, . . . no excuses!

(**a**) $f(x,y) = 3x - y$	○ ○ ○ ♠ ○ ○ ○ ○	(**b**) $f(x,y) = x + 3y$	♠ ○ ○ ○ ○ ○ ○ ○
(**c**) $f(x,y) = x^2 - y$	○ ○ ○ ○ ♠ ○ ○ ○	(**d**) $f(x,y) = 2y + \ln|x|$	○ ○ ○ ○ ○ ○ ○ ♠
(**e**) $f(x,y) = x^2 - y^2$	○ ○ ○ ○ ○ ♠ ○ ○	(**f**) $f(x,y) = 3x + y$	○ ♠ ○ ○ ○ ○ ○ ○



(a) $f(x,y) = k \quad \Rightarrow \quad 3x - y = k \quad \Rightarrow \quad y = 3x - k$
straight lines of slope $m = +3$;

(b) $f(x,y) = k \quad \Rightarrow \quad x + 3y = k \quad \Rightarrow \quad y = -x/3 + k/3$
straight lines of slope $m = -1/3$;

(c) $f(x,y) = k \quad \Rightarrow \quad x^2 - y = k \quad \Rightarrow \quad y = x^2 - k$
upward opening parabolae;

(d) $f(x,y) = k \quad \Rightarrow \quad 2y + \ln|x| = k \quad \Rightarrow \quad y = k/2 - (\ln|x|)/2$
upside-down logs both sides of the y-axis;

(e) $f(x,y) = k \quad \Rightarrow \quad x^2 - y^2 = k$
equliateral hyperbolae ($k \neq 0$) and quadrant bisectors ($k = 0$);

(f) $f(x,y) = k \quad \Rightarrow \quad 3x + y = k \quad \Rightarrow \quad y = -3x + k$
straight lines of slope $m = -3$.

2. **[10 pts]** Find an equation for the surface whose points $P(x,y,z)$ are equidistant from $O(0,0,0)$ and $Q(1,1,1)$.

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SOLUTION

$$\begin{aligned} d(P,O) = d(P,Q) \ &\Rightarrow\ d^2(P,O) = d^2(P,Q) \\ &\Rightarrow\ (x-0)^2 + (y-0)^2 + (z-0)^2 = (x-1)^2 + (y-1)^2 + (z-1)^2 \\ &\Rightarrow\ x^2 + y^2 + z^2 = x^2 - 2x + 1 + y^2 - 2y + 1 + z^2 - 2z + 1 \\ &\Rightarrow\ 0 = -2x + 1 - 2y + 1 - 2z + 1 \\ &\Rightarrow\ \boxed{2x + 2y + 2z = 3} \end{aligned}$$

3. **[15 pts]** Find the limit

$$\lim_{(x,y)\to(0,0)} y \sin \frac{1}{x^2+y^2}\,.$$

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SOLUTION

$$\begin{aligned} &-1 \ \le\ \sin \frac{1}{x^2+y^2} \ \le\ 1\,, \quad \forall\,(x,y) \neq (0,0) \\ &\Rightarrow\ -y \ \le\ y \sin \frac{1}{x^2+y^2} \ \le\ y\,, \quad \forall\,(x,y) \neq (0,0) \\ &\Rightarrow\ \lim_{(x,y)\to(0,0)} (-y) \ \le\ \lim_{(x,y)\to(0,0)} y \sin \frac{1}{x^2+y^2} \ \le\ \lim_{(x,y)\to(0,0)} y \\ &\Rightarrow\ 0 \ \le\ \lim_{(x,y)\to(0,0)} y \sin \frac{1}{x^2+y^2} \ \le\ 0 \\ &\Rightarrow\ \boxed{\lim_{(x,y)\to(0,0)} y \sin \frac{1}{x^2+y^2} \ =\ 0} \end{aligned}$$

4. **[10 pts]** Let

$$f(x,y) = \begin{cases} \dfrac{x^2-4y^2}{x-2y}\,, & \text{if } x \neq 2y, \\ g(x)\,, & \text{if } x = 2y. \end{cases}$$

If f is continuous on the whole plane, find a formula for $g(x)$.

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SOLUTION

$$\frac{x^2-4y^2}{x-2y} = \frac{(x-2y)(x+2y)}{x-2y} = x + 2y\,, \qquad \underline{\text{if } x \neq 2y}$$

$$\boxed{g(x) = \Big(x + 2y\Big)_{x=2y} = 2x}$$

CHECK (<u>NOT REQUIRED FOR CREDIT</u>): Each point on the line $x = 2y$ is of the form $(x_0, y_0) = (x_0, x_0/2)$ and the claim is that

$$\lim_{(x,y)->(x_0,y_0)} f(x,y) = g(x_0) = 2x_0\,.$$

This has to be veryfied:

$$\begin{aligned} \lim_{(x,y)->(x_0,y_0)} \Big| f(x,y) - 2x_0 \Big| \ &=\ \lim_{(x,y)->(x_0,y_0)} \left| \frac{x^2-4y^2}{x-2y} - 2x_0 \right| \\ &=\ \lim_{(x,y)->(x_0,y_0)} \left| \frac{(x-2y)(x+2y)}{x-2y} - 2x_0 \right| \\ &=\ \lim_{(x,y)->(x_0,y_0)} |(x+2y) - 2x_0| \\ &=\ |(x_0 + 2y_0) - 2x_0| \\ &=\ |2y_0 - x_0| \\ &=\ 0\,, \qquad \text{when } x_0 = 2y_0. \end{aligned}$$

5. **[15 pts]** Find the area of the triangle with vertices $P(-2,2,0)$, $Q(1,3,-1)$, $R(-4,2,1)$.
Show all your work, no work = no credit, . . . no excuses!
SOLUTION

$$\begin{aligned}\vec{PQ} &= (1+2)\vec{i} + (3-2)\vec{j} + (-1-0)\vec{k}\,,\\ \vec{PR} &= (-4+2)\vec{i} + (2-2)\vec{j} + (1-0)\vec{k}\,.\end{aligned}$$

$$\vec{PQ}\times\vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & -1 \\ -2 & 0 & 1 \end{vmatrix} = \vec{i} - \vec{j} + 2\vec{k}\,.$$

$$\text{Area}(OPQ) = \frac{1}{2}\|\vec{PQ}\times\vec{PR}\| = \frac{1}{2}\sqrt{(1)^2 + (-1)^2 + (2)^2} = \frac{\sqrt{6}}{2}\,.$$

6. **[10 pts]** Suppose that $\vec{u}\cdot\vec{w} = 8$ and $\vec{u}\times\vec{w} = 12\vec{i} - 3\vec{j} + 4\vec{k}$, and that the angle between $\vec{u}$ and $\vec{w}$ is θ, find $\tan\theta$.
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SOLUTION

$$\vec{u}\cdot\vec{w} = \|\vec{u}\|\,\|\vec{w}\|\,\cos\theta\,, \qquad \text{and} \qquad \|\vec{u}\times\vec{w}\| = \|\vec{u}\|\,\|\vec{w}\|\,\sin\theta\,.$$

$$\Rightarrow \qquad \tan\theta = \frac{\|\vec{u}\times\vec{w}\|}{\vec{u}\cdot\vec{w}}\,, \qquad \vec{u}\cdot\vec{w} \neq 0\,.$$

$$\|\vec{u}\times\vec{w}\| = \|12\vec{i} - 3\vec{j} + 4\vec{k}\| = \sqrt{(12)^2 + (-3)^2 + (4)^2} = \sqrt{169} = 13\,;$$

$$\boxed{\tan\theta = \frac{\|\vec{u}\times\vec{w}\|}{\vec{u}\cdot\vec{w}} = \frac{13}{8}}$$

7. **[10 pts]** Given the fixed points $A(-1,0)$, and $B(1,0)$, find the set of all points $P(x,y)$ such that $\vec{AP}\cdot\vec{BP} = 0$ and write it as an equation of the form $f(x,y) = k$. Describe the graph of this equation.
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SOLUTION

$$\begin{aligned}\vec{AP}\cdot\vec{BP} = 0 \quad &\Rightarrow \quad \Big((x+1)\vec{i} + y\vec{j}\Big)\cdot\Big((x-1)\vec{i} + y\vec{j}\Big) = 0\\ &\Rightarrow \quad (x+1)(x-1) + y^2 = 0\\ &\Rightarrow \quad \boxed{x^2 + y^2 = 1}\end{aligned}$$

$\boxed{\text{a circle of radius one centered at the origin}}$

8. **[15 pts]** The following statements are either **true** or **false**. If true, then say so and explain why. If false, then say so and give a simple counter-example to show why the statement is false.

(a) $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = ||\vec{a}||^2 - ||\vec{b}||^2$;
(b) $(\vec{a}+\vec{b})\times(\vec{a}-\vec{b}) = -2\vec{a}\cdot\vec{b}$;
(c) If $\vec{a} \neq \vec{0}$ and $\vec{b} \neq \vec{0}$, then $(\vec{a}\times\vec{b})\times\vec{a}$ is parallel to $\vec{b}$;
(d) If $\vec{a} \neq \vec{0}$ and $\vec{b} \neq \vec{0}$, then $(\|\vec{b}\|\vec{a} + \|\vec{a}\|\vec{b})$ and $(\|\vec{b}\|\vec{a} - \|\vec{a}\|\vec{b})$ are orthogonal.

NOTE: Just *one lucky* example doesn't prove a statement right, but it can prove it false!
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SOLUTION

(a) **True**. The expression on the left can be easily expanded, yielding

$$(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = \underbrace{\vec{a}\cdot\vec{a}}_{||\vec{a}||^2} - \vec{a}\cdot\vec{b} + \underbrace{\vec{b}\cdot\vec{a}}_{\vec{a}\cdot\vec{b}} - \underbrace{\vec{b}\cdot\vec{b}}_{||\vec{b}||^2}\,,$$

therefore

$$\boxed{(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = ||\vec{a}||^2 - ||\vec{b}||^2}$$

(b) **False**. The left-hand side is a vector, the right hand side is a scalar. The correct answer to the expansion of the expression on the left is

$$(\vec{a}+\vec{b})\times(\vec{a}-\vec{b}) = \underbrace{\vec{a}\times\vec{a}}_{\vec{0}} - \vec{a}\times\vec{b} + \underbrace{\vec{b}\times\vec{a}}_{-\vec{a}\times\vec{b}} - \underbrace{\vec{b}\times\vec{b}}_{\vec{0}}$$

therefore

$$\boxed{(\vec{a}+\vec{b})\times(\vec{a}-\vec{b}) = -2\,\vec{a}\times\vec{b}}$$

(c) **False**. As a counter-example take $\vec{a}=\vec{i}+\vec{j}$ and $\vec{b}=\vec{j}$, then

$$\begin{aligned}
\Big((\vec{i}+\vec{j})\times\vec{j}\Big)\times(\vec{i}+\vec{j}) &= (\vec{i}\times\vec{j}+\vec{j}\times\vec{j})\times(\vec{i}+\vec{j}) \\
&= (\vec{k}+\vec{0})\times(\vec{i}+\vec{j}) \\
&= \vec{k}\times(\vec{i}+\vec{j}) \\
&= \vec{k}\times\vec{i}+\vec{k}\times\vec{j} \\
&= \vec{j}-\vec{i}
\end{aligned}$$

which is not parallel to $\vec{j}$.

(d) **True**. If two vectors are orthogonal then their dot product is zero. We can check the orthogonality of the given vectors by direct computation,

$$\begin{aligned}
\Big(\|\vec{b}\|\vec{a}+\|\vec{a}\|\vec{b}\Big)\cdot\Big(\|\vec{b}\|\vec{a}-\|\vec{a}\|\vec{b}\Big) &= \|\vec{b}\|^2\vec{a}\cdot\vec{a} - \|\vec{a}\|\|\vec{b}\|\vec{a}\cdot\vec{b} + \|\vec{a}\|\|\vec{b}\|\vec{b}\cdot\vec{a} - \|\vec{a}\|^2\vec{b}\cdot\vec{b} \\
&= \|\vec{b}\|^2\vec{a}\cdot\vec{a} - \|\vec{a}\|^2\vec{b}\cdot\vec{b} \\
&= \|\vec{b}\|^2\|\vec{a}\|^2 - \|\vec{a}\|^2\|\vec{b}\|^2 \\
&= 0\,.
\end{aligned}$$