CALCULUS 3

September 15, 2010

1st TEST

YOUR NAME: **T H E S O L U T I O N S**

- ◯ **002** J. HARPER .... (9AM)
- ◯ **003** C. MESA .... (10AM)
- ◯ **004** A. SPINA .... (11AM)
- ◯ **005** A. SPINA .... (12PM)
- ◯ **006** M. STACKPOLE .... (1PM)
- ◯ **007** P. NEWBERRY .... (3PM)

SHOW ALL YOUR WORK

final answers without any supporting work
will receive no credit *even if they are right!*

No cheat-sheets allowed.

Partial credit will be given for any **reasonable amount of work pointing in the right direction** towards the solution of your problem. You will not get any partial credit for memorizing formulas and not knowing how to use them, or for anything you write that is not directly related to the solution of your problem.

If your tests contains **more than one solution or answer** to a problem or part of a problem, and one of them is wrong, then it will be **the wrong one** the one that **counts** for your grading!

Make sure you write an arrow on top of vector quantities to differentiate them from scalar quantities (numbers). A word-processor and boldface fonts were used in writing test, but you are writing by hand! Remember that, within the same context, $\vec{v}$ (with the arrow) is a *vector* ($\vec{v} \equiv \mathbf{v}$) and v (without the arrow) is the *norm* of the previous vector ($v \equiv ||\vec{v}|| \equiv ||\mathbf{v}||$). If a vector is the null vector, write an arrow on top of the zero!

DO NOT WRITE INSIDE THIS BOX!

problem	points	score
1	15 pts	
2	15 pts	
3	15 pts	
4	15 pts	
5	15 pts	
6	10 pts	
7	15 pts	
8	15 pts	
9	15 pts	
TOTAL	100 pts	

1. **[15 pts]** Find the equation of the largest sphere with center $(5,4,9)$ that is contained in the first octant.

SOLUTION

The largest sphere contained in the first octant has a radius equal to the smallest distance from the center to the coordinates planes, Π_{xy}, Π_{yz}, and Π_{zx}.

The distances of the center to the coordinates planes are,

$$d(C,\Pi_{xy}) = 9\,, \qquad d(C,\Pi_{yz}) = 5\,, \qquad d(C,\Pi_{zx}) = 4\,.$$

The largest sphere is then

$$(x-5)^2 + (y-4)^2 + (z-9)^2 = 16\,.$$

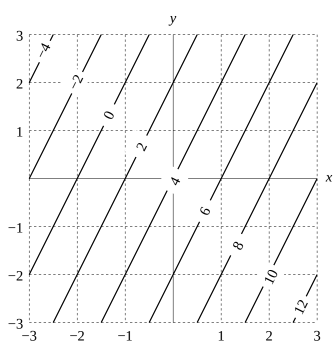
2. **[15 pts]** Consider the points $P(x,y,z)$ such that the distance from $P(x,y,z)$ to $A(4,2,6)$ is twice the distance from $P(x,y,z)$ to $B(4,5,-3)$. Show that the set of all such points is a sphere, and find its center and its radius.

SOLUTION

$$\begin{aligned}
d(P,A) = 2d(P,B) \quad &\Rightarrow \quad d^2(P,A) = 4d^2(P,B)\\
&\Rightarrow \quad (x-4)^2 + (y-2)^2 + (z-6)^2 = 4\left[(x-4)^2 + (y-5)^2 + (z+3)^2\right]\\
&\Rightarrow \quad 3x^2 + 3y^2 + 3z^2 - 24x - 36y + 36z = -144\\
&\Rightarrow \quad x^2 + y^2 + z^2 - 8x - 12y + 12z = -48\\
&\Rightarrow \quad (x-4)^2 + (y-6)^2 + (z+6)^2 = 40\cdot
\end{aligned}$$

Therefore, the set of points such that $d(P,A) = 2d(P,B)$ is a sphere of radius $R = \sqrt{40}$ and center $C(4,6,-6)$.

3. **[15 pts]** Find the equation for the linear function with the contour diagram



SOLUTION

$$\begin{aligned}
m &= \left(\frac{\Delta f}{\Delta x}\right)_y = \frac{+2}{1} = +2\,,\\
n &= \left(\frac{\Delta f}{\Delta y}\right)_x = \frac{-2}{2} = -1\,,
\end{aligned}$$

hence,

$$f(x,y) = c + 2x - y\,.$$

But

$$f(0,0) = c = 4\,,$$

therefore

$$f(x,y) = 4 + 2x - y\,.$$

4. **[15 pts]** Describe the level curves of the function $f(x,y) = |x| + |y|$,

SOLUTION

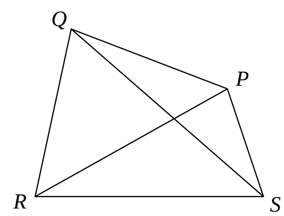
$$f(x,y)=k \quad\Rightarrow\quad |x|+|y|=k \quad\Rightarrow\quad \begin{cases} x+y=k & \text{1st quadrant}\\ -x+y=k & \text{2nd quadrant}\\ -x-y=k & \text{3rd quadrant}\\ x-y-k & \text{4th quadrant}\end{cases}$$

The level curves are squares whose vertexes are at $(\pm k, 0)$ and $(0, \pm k)$ for $k \geq 0$.

5. **[15 pts]** Write each combination of vectors as a single vector, e.g. $\vec{PQ}$.

(a) $\vec{PQ}+\vec{QR}$, (b) $\vec{RP}+\vec{PS}$, (c) $\vec{QS}-\vec{PS}$, (d) $\vec{RS}+\vec{SP}+\vec{PQ}$.

You can use the figure below as a guide, if you need.



SOLUTION

$$\begin{aligned}
&\text{(a)}\quad \vec{PQ}+\vec{QR} &&= \vec{PR}\\
&\text{(b)}\quad \vec{RP}+\vec{PS} &&= \vec{RS}\\
&\text{(c)}\quad \vec{QS}-\vec{PS} &&= \vec{QS}+\vec{SP} = \vec{QP}\\
&\text{(d)}\quad \vec{RS}+\vec{SP}+\vec{PQ} &&= \vec{RQ}
\end{aligned}$$

6. **[10 pts]** If $\mathbf{v}$ lies in the first quadrant and makes an angle $\pi/3$ with the positive y-axis and $|\mathbf{v}| = 4$, find $\mathbf{v}$ in component form.

SOLUTION

$$\begin{aligned}
\mathbf{v} &= |\mathbf{v}|\cos\theta\mathbf{i} + |\mathbf{v}|\sin\theta\mathbf{j}\\
&= 4\cos(\pi/2-\pi/3)\mathbf{i} + 4\sin(\pi/2-\pi/3)\mathbf{j}\\
&= 4\cos(\pi/6)\mathbf{i} + 4\sin(\pi/6)\mathbf{j}\\
&= 4\frac{\sqrt{3}}{2}\mathbf{i} + 4\frac{1}{2}\mathbf{j}\\
&= 2\sqrt{3}\mathbf{i} + 2\mathbf{j}
\end{aligned}$$

7. **[15 pts]** Show that

$$\frac{\mathbf{u}}{\|\mathbf{u}\|^2}-\frac{\mathbf{v}}{\|\mathbf{v}\|^2} \quad \text{and} \quad \frac{\mathbf{u}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}-\frac{\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$$

have the same magnitude, where $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$ are nonzero vectors.

SOLUTION

$$\begin{aligned}
\left\|\frac{\mathbf{u}}{\|\mathbf{u}\|^2}-\frac{\mathbf{v}}{\|\mathbf{v}\|^2}\right\|^2 &= \left(\frac{\mathbf{u}}{\|\mathbf{u}\|^2}-\frac{\mathbf{v}}{\|\mathbf{v}\|^2}\right)\cdot\left(\frac{\mathbf{u}}{\|\mathbf{u}\|^2}-\frac{\mathbf{v}}{\|\mathbf{v}\|^2}\right)\\
&= \frac{\|\mathbf{u}\|^2}{\|\mathbf{u}\|^4}+\frac{\|\mathbf{v}\|^2}{\|\mathbf{v}\|^4}-2\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2\|\mathbf{v}\|^2}\\
&= \frac{1}{\|\mathbf{u}\|^2}+\frac{1}{\|\mathbf{v}\|^2}-2\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2\|\mathbf{v}\|^2}
\end{aligned}$$

$$\left\|\frac{\mathbf{u}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}-\frac{\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right\|^2 = \left(\frac{\mathbf{u}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}-\frac{\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right)\cdot\left(\frac{\mathbf{u}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}-\frac{\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right)$$

$$\begin{aligned} &= \frac{\|\mathbf{u}\|^2}{\|\mathbf{u}\|^2\|\mathbf{v}\|^2} + \frac{\|\mathbf{v}\|^2}{\|\mathbf{u}\|^2\|\mathbf{v}\|^2} - 2\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \\ &= \frac{1}{\|\mathbf{u}\|^2} + \frac{1}{\|\mathbf{v}\|^2} - 2\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \end{aligned}$$

Therefore, the magnitudes are equal.

8. **[15 pts]** Use the vector product identity

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

to evaluate

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + \mathbf{b}\times(\mathbf{c}\times\mathbf{a}) + \mathbf{c}\times(\mathbf{a}\times\mathbf{b})\,.$$

SOLUTION

$$\begin{aligned} \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + \mathbf{b}\times(\mathbf{c}\times\mathbf{a}) + \mathbf{c}\times(\mathbf{a}\times\mathbf{b}) &= (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \\ &\quad +(\mathbf{b}\cdot\mathbf{a})\mathbf{c} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a} \\ &\quad +(\mathbf{c}\cdot\mathbf{b})\mathbf{a} - (\mathbf{c}\cdot\mathbf{a})\mathbf{b} \\ &= \mathbf{0}\,. \end{aligned}$$

9. **[15 pts]** Is the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+xy+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

continuous at $(0,0)$?

SOLUTION

To prove continuity we have to show $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

1st way.

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} f(x,y) = \lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{xy}{x^2+xy+y^2} = \lim_{x\to 0}\frac{0}{x^2} = 0\,,$$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}} f(x,y) = \lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{xy}{x^2+xy+y^2} = \lim_{x\to 0}\frac{x^2}{3x^2} = \frac{1}{3}\,.$$

Therefore,

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} f(x,y) \neq \lim_{\substack{(x,y)\to(0,0)\\ y=x}} f(x,y)$$

there is no limit and the function is not continuous at $(x,y) = (0,0)$.

2nd way.

$$\begin{aligned} \lim_{\substack{(x,y)\to(0,0)\\ y=mx}} f(x,y) &= \lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{xy}{x^2+xy+y^2} \\ &= \lim_{x\to 0}\frac{mx^2}{x^2+mx^2+m^2x^2} \\ &= \lim_{x\to 0}\frac{m}{1+m+m^2} \\ &= \lim_{x\to 0} g(m)\,. \end{aligned}$$

Therefore, there is no limit and the function is not continuous at $(x,y) = (0,0)$.

3rd way.

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+xy+y^2} \\
&= \lim_{r\to 0} \frac{r^2\cos\theta\,\sin\theta}{r^2+r^2\cos\theta\,\sin\theta} \\
&= \lim_{r\to 0} \frac{\cos\theta\,\sin\theta}{1+\cos\theta\,\sin\theta} \\
&= \lim_{r\to 0} g(\theta)\,.
\end{aligned}$$

Therefore, there is no limit and the function is not continuous at $(x,y)=(0,0)$.