4. (18 points) Let T be the triangle with vertices $A = (1,2,3)$, $B = (3,3,6)$, and $C = (2,2,5)$.

(a) Calculate $\overrightarrow{AB} \times \overrightarrow{AC}$.

$\overrightarrow{AB} = 2i + j + 3k \quad \overrightarrow{AC} = i + 0j + 2k$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 2 & 1 & 3 \\ 1 & 0 & 2 \end{vmatrix} = 2i - j - k$$

(b) Find an equation of the form $ax + by + cz = d$ for the plane containing the triangle T.

$2(x-1) - (y-2) - (z-3) = 0$

$2x - 2 - y + 2 - z + 3 = 0$

$2x - y - z = -3$

(c) Find the area of T.

$$\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{4+1+1} = \frac{1}{2}\sqrt{6}$$

1. (25 points) (a) Find the area of the parallelogram determined by the following vectors: $\vec{v}_1 = \langle 2, 3, -3 \rangle$, $\vec{v}_2 = \langle 1, 5, 0 \rangle$

Solution: The area of the parallelogram is given by

$$\begin{aligned} |\vec{v}_1 \times \vec{v}_2| &= \left| \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -3 \\ 1 & 5 & 0 \end{pmatrix} \right| \\ &= \left| 15\vec{i} - 3\vec{j} + 7\vec{k} \right| \\ &= \sqrt{15^2 + 3^2 + 7^2} \\ &= \sqrt{283} \end{aligned}$$

(b) Using vectors, find the angle between the two planes below. Leave your answer in terms of inverse trigonometric functions.

$$3x + 6y + z = 5 \quad \text{and} \quad 2x - y + \frac{1}{2}z = -7$$

Solution: These planes have normal vectors $\vec{n}_1 = \langle 3, 6, 1 \rangle$ and $\vec{n}_2 = \langle 2, -1, \frac{1}{2} \rangle$. The angle θ between them can be found from the equation $\vec{n}_1 \cdot \vec{n}_2 = |\vec{n}_1||\vec{n}_2| \cos\theta$.

$$\begin{aligned} |\vec{n}_1| &= \sqrt{46} \\ |\vec{n}_2| &= \sqrt{\frac{11}{2}} \\ \vec{n}_1 \cdot \vec{n}_2 &= 3 \cdot 2 + 6 \cdot -1 + 1 \cdot \frac{1}{2} \\ &= \frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} \right) \\ &= \cos^{-1}\left(\frac{1}{2\sqrt{253}} \right) \end{aligned}$$

(c) Using vector products, find the volume of the parallelepiped determined by the following vectors: $\vec{v}_1 = \langle 2, 2, -3 \rangle$, $\vec{v}_2 = \langle 0, 2, -1 \rangle$, $\vec{v}_3 = \langle -3, 2, -1 \rangle$.

Solution: The volume of the parallelepiped is given by

$$\begin{aligned} |\vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3)| &= |\vec{v}_1 \cdot \langle 0, 3, 6 \rangle| \\ &= |-12| \\ &= 12 \end{aligned}$$

2. (25 points) (a) Find an equation of the line passing through the point $P_0(1, 2, -3)$ and parallel to the vector $\langle 1, -1, 2\rangle$.

Solution: The line can be expressed parametrically by

$$\vec{r}(t) = \vec{P}_0 + t\vec{v} = \langle 1+t, 2-t, -3+2t\rangle$$

or equivalently,

$$\begin{aligned} x &= 1+t \\ y &= 2-t \\ z &= -3+2t \end{aligned}$$

By solving for t we can also express this line symmetrically as

$$x - 1 = -y + 2 = \frac{z+3}{2}$$

(b) Find an equation of the plane passing through $P_0(1, 2, -3)$, $P_1(0, 2, 1)$, and $P_2(-1, 5, 2)$.

Solution: Consider for example the vectors $\overrightarrow{P_0P_1} = \langle -1, 0, 4\rangle$ and $\overrightarrow{P_0P_2} = \langle -2, 3, 5\rangle$. Then

$$\begin{aligned} \vec{n} &= \overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2} \\ &= \langle -12, -3, -3\rangle \end{aligned}$$

is a normal vector to the plane. Choosing P_0 for example, the plane is given by the equation

$$\vec{n} \cdot (\langle x, y, z\rangle - P_0) = 0$$

Hence the equation of the plane is

$$4x + y + z = 3$$

(c) What is the distance of the point $P_3(-1,1,1)$ from the plane passing through $P_0(1,2,-3)$ with normal vector $\langle 1,0,-4\rangle$?

Solution: We project the displacement vector $\overrightarrow{P_0P_3} = \langle -2,-1,4\rangle$ onto the normal vector $\vec{n} = \langle 1,0,-4\rangle$:

$$\begin{aligned}\frac{|\overrightarrow{P_0P_3}\cdot\vec{n}|}{|\vec{n}|} &= \frac{|-2+0-16|}{\sqrt{1^2+0^2+4^2}} \\ &= \frac{18}{\sqrt{17}}\end{aligned}$$

(d) Find an equation (or equations) representing all the lines passing through the point $P_0(1,2,-3)$ and perpendicular to the vector $\langle 1,-1,1\rangle$.

Solution: All such lines constitute the plane

$$\langle 1,-1,1\rangle\cdot(\langle x,y,z\rangle - \vec{P}_0) = 0$$

or equivalently

$$z = -4 - x + y$$

Hence, all such lines have parametric equations

$$\begin{aligned}x &= 1 + at \\ y &= 2 + bt \\ z &= -3 - at + bt\end{aligned}$$

for any real numbers a and b.

4. (25 points) Consider the equations

   (a) $x^2 + y^2 - z = 0$ in Cartesian (= rectangular) coordinates.
   (b) $r^2 - z^2 = 0$ in cylindrical coordinates.
   (c) $\rho^2(1 + \cos\phi) = 1$ in spherical coordinates.

   (i) Which quadric surface, if any, does equation (a) represent?

   Solution: The graph of $z = x^2 + y^2$ is an elliptic (in fact, circular) paraboloid.

   (ii) Give an equation in cylindrical coordinates representing the surface given by (a).

   Solution: $r^2 - z = 0$.

   (iii) Give an equation in spherical coordinates representing the surface given by (a).

   Solution:

$$
\begin{aligned}
x^2 + y^2 - z &= 0 \\
x^2 + y^2 + z^2 - z^2 - z &= 0 \\
\rho^2 - \rho^2 \cos^2\phi - \rho\cos\phi &= 0
\end{aligned}
$$

(iv) Give an equation in Cartesian (= rectangular) coordinates representing the surface given by (b).

Solution: $x^2 + y^2 - z^2 = 0$.

(v) Give an equation in Cartesian (= rectangular) coordinates representing the surface of equation (c).

Solution:

$$\begin{aligned} \rho^2(1 + \cos\phi) &= 1 \\ \rho^2 + \rho\cos\phi \cdot \rho &= 1 \\ x^2 + y^2 + z^2 + z\sqrt{x^2 + y^2 + z^2} &= 1 \end{aligned}$$

1. **(17 points)** Let P_0 be the point $(1, 1, 2)$ and let $\wp$ be the plane given by the equation $2x - y + 2z = 2$

(a) **(9 points)** Find parametric equations of the line L passing through the point P_0 and perpendicular to the plane $\wp$.

Solution: Since $\wp \perp \langle 2, -1, 2\rangle$, it follows that $L // \langle 2, -1, 2\rangle$. Thus L has parametric equation

$$x = 1 + 2t, y = 1 - t; z = 2 + 2t.$$

(b) **(8 points)** Find the intersection point of the line L in (a) above and the plane $\wp$.

Solution: We need to solve the equation

$$2(1 + 2t) - (1 - t) + 2(2 + 2t) = 2.$$

It gives $9t = -3$ or

$$t = -1/3.$$

Hence the desired intersection point is $(1/3, 4/3, 4/3)$.

3. **(16 points)** Which of the following is the angle between the (big) diagonal of a unit cube and one of its edges, where the diagonal and the edge start at the same point ? (Circle one of them and justify your answer. Show all work for full credit.)

(a) $\arcsin \frac{1}{\sqrt{3}}$ (b) $\arccos \frac{1}{\sqrt{3}}$ (c) $\arcsin \frac{2}{\sqrt{6}}$ (d) $\arccos \frac{2}{\sqrt{6}}$

Solution: We may assume that the cube has base with vertices $(0,0,0)$, $(1,0,0)$, $(1,1,0)$, $(0,1,0)$ on the xy-plane and the other four vertices $(0,0,1)$, $(1,0,1)$, $(1,1,1)$, $(0,1,1)$. We take the big diagonal to be

$$\vec{a} = \overrightarrow{(0,0,0),(1,1,1)} = \langle 1,1,1 \rangle$$

and the edge to be

$$\vec{b} = \overrightarrow{(0,0,0),(1,0,0)} = \langle 1,0,0 \rangle.$$

Suppose that θ is the angle between $\vec{a}$ and $\vec{b}$. Then we have

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} = \frac{1}{\sqrt{3}\cdot 1} = \frac{1}{\sqrt{3}}.$$

Thus we have $\theta = \arccos \frac{1}{\sqrt{3}}$, namely (b).

4. **(17 points)** Let C be the helix $\boldsymbol{r}(t) = \langle \sin(\pi t), \cos(\pi t), t \rangle$ and let S be the sphere $x^2+y^2+z^2 = 5$.

(a) **(8 points)** At what points do the helix C intersect the sphere S ?

Solution: We need to solve the equation

$$\sin^2(\pi t) + \cos^2(\pi t) + t^2 = 5.$$

This equation implies

$$t^2 = 4$$

since $\sin^2(\pi t) + \cos^2(\pi t) = 1$. So we have $t = \pm 2$ and it follows that the desired intersection points are $(\sin(\pm 2\pi), \cos(\pm 2\pi), \pm 2)$, namely

$$(0, 1, \pm 2).$$

(b) **(9 points)** Find the tangent line to the helix C at the intersection point having positive z-coordinate.

Solution: The intersection point with positive z-coordinate is $(0, 1, 2)$, namely the point coming from $t = 2$. The desired tangent line passes through $(0, 1, 2)$ and has directional vector $\boldsymbol{r}'(2)$. Look at

$$\boldsymbol{r}'(t) = \langle \pi \cos(\pi t), -\pi \sin(\pi t), 1 \rangle$$

and get

$$\boldsymbol{r}'(2) = \langle \pi \cos(2\pi), -\pi \sin(2\pi), 1 \rangle = \langle \pi, 0, 1 \rangle.$$

Therefore the tangent line is given by

$$x = 0 + \pi t = \pi t; y = 1 + 0t = 1; z = 2 + 1 \cdot t = 2 + t.$$

5. (16 points)

(a) **(8 points)** Find the spherical coordinates of the point given by $(1, 1, -\sqrt{2})$ in rectangular coordinates.

Solution: Recall the general formulae for $(x, y, z) \leftrightarrow (\rho, \theta, \phi)$

$$x = \rho \sin \phi \cos \theta; y = \rho \sin \phi \sin \theta; z = \rho \cos \phi.$$

Now look at

$$\rho = \sqrt{1^1 + 1^2 + (-\sqrt{2})^2} = \sqrt{4} = 2;$$

$$\cos \phi = \frac{-\sqrt{2}}{2}, \quad \text{hence} \quad \phi = \frac{3\pi}{4} \quad \text{and;}$$

$$\sin \theta = \frac{1}{2 \cdot \sin \frac{3\pi}{4}} = \frac{1}{2 \cdot \frac{\sqrt{2\pi}}{2}} = \frac{1}{\sqrt{2}}, \quad \text{hence} \quad \theta = \frac{\pi}{4} \quad \text{or} \quad \frac{3\pi}{4}.$$

But the y-coordinate of $(1, 1, -\sqrt{2})$ has positive sign, so $\theta = \frac{\pi}{4}$. Therefore the desired spherical coordinate is $(2, \frac{\pi}{4}, \frac{3\pi}{4})$.

(b) **(8 points)** In Cartesian coordinates, write down the equation of the surface given by the equation $r = 2 \cos \theta$ in cylindrical coordinates and describe the surface in words or in a picture.

Solution. Recall the general formulae for $(x, y, z) \leftrightarrow (r, \theta, z)$

$$x = r \cos \theta; y = r \sin \theta; z = z.$$

Look at

$$r^2 = r \cdot r = r \cdot 2 \cos \theta = 2 \cdot r \cos \theta.$$

Note that $r^2 = x^2 + y^2$ and $x = r \cos \theta$. So we have

$$x^2 + y^2 = 2x \quad \text{or} \quad (x - 1)^2 + y^2 = 1.$$

This is the cylinder that is vertical along the z-axis and that has radius 1 centered at $(1, 0, z_0)$ on the plane defined by $z = z_0$, where z_0 is an arbitrary real number.

6. **(17 points)** Let C be the curve given by $\boldsymbol{r}(t) = \langle 2t, \ln t, t^2 \rangle$, where ln stands for the natural logarithm.

(a) **(9 points)** Find the arc length of the curve C for $1 \le t \le 4$.

Solution: Note that $\boldsymbol{r}(t)$ traces points only once when t runs over the interval $[1, 4]$ and that $\boldsymbol{r}$ has its derivative $\boldsymbol{r}'(t) = \langle 2, \frac{1}{t}, 2t \rangle$. Now we have the desired arc length

$$\begin{aligned}
\int_1^4 \sqrt{|\boldsymbol{r}'(t)|}\, dt &= \int_1^4 \sqrt{2^2 + \left(\frac{1}{t}\right)^2 + (2t)^2}\, dt \\
&= \int_1^4 \sqrt{4 + \frac{1}{t^2} + 4t^2}\, dt \\
&= \int_1^4 \sqrt{\left(\frac{1}{t} + 2t\right)^2}\, dt \\
&= \int_1^4 \left(\frac{1}{t} + 2t\right) dt \\
&= [\ln t + t^2]_1^4 \\
&= \{\ln 4 + 4^2\} - \{\ln(1) + 1^2\} \\
&= \ln 4 + 15.
\end{aligned}$$

(b) **(8 points)** Find the curvature of the curve C at $t = 1$.

Solution: The curvature is given by

$$\kappa = \frac{|\boldsymbol{r}'(1) \times \boldsymbol{r}''(1)|}{|\boldsymbol{r}'(1)|^3}.$$

Now recall the formula for $\boldsymbol{r}'(t)$ in (a) above and look at

$$\boldsymbol{r}'(1) = \langle 2, 1, 2 \rangle \text{ and } |\boldsymbol{r}'(1)| = \sqrt{2^2 + 1^2 + 2^2} = 3.$$

Since $\boldsymbol{r}''(t) = \langle 0, -\frac{1}{t^2}, 2 \rangle$, we have

$$\boldsymbol{r}''(1) = \langle 0, -1, 2 \rangle.$$

Observe

$$\boldsymbol{r}'(1) \times \boldsymbol{r}''(1) = \langle 2, 1, 2 \rangle \times \langle 0, -1, 2 \rangle = \langle 4, -4, -2 \rangle.$$

(You need to show detailed work for this in the exam.) Therefore we have

$$\kappa = \frac{\sqrt{4^2 + (-4)^2 + (-2)^2}}{3^3} = \frac{6}{27} = \frac{2}{9}.$$

1. **(17 points)**

(a) **(12 points)** Find an equation of the plane that passes through the three points $P = (1, 1, 0)$, $Q = (0, 2, 1)$, and $R = (3, 2, -1)$.

$\vec{PQ} = \langle -1, 1, 1 \rangle$, $\vec{PR} = \langle 2, 1, -1 \rangle$

$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = \langle -2, 1, -3 \rangle$$ is

normal to the plane.

Using $\vec{n} = \langle -2, 1, -3 \rangle$ and $P = (1,1,0)$, the equation of the plane is

$$-2(x-1) + 1(y-1) - 3(z-0) = 0$$
$$-2x + 2 + y - 1 - 3z = 0$$
$$2x - y + 3z - 1 = 0$$

(b) **(5 points)** Give the parametric equations of the line perpendicular to the plane from part (a) that passes through P.

The vector $\vec{n} = \langle -2, 1, -3 \rangle$ from part (a) is parallel to the line. Thus, parametric equations for the line are

$$x = 1 - 2t$$
$$y = 1 + t$$
$$z = -3t$$

2. **(17 points)**

(a) **(12 points)** Consider the surface S given by the equation

$$x^2 - 2y^2 + z^2 = 1.$$

Sktech the intersection of S with the planes

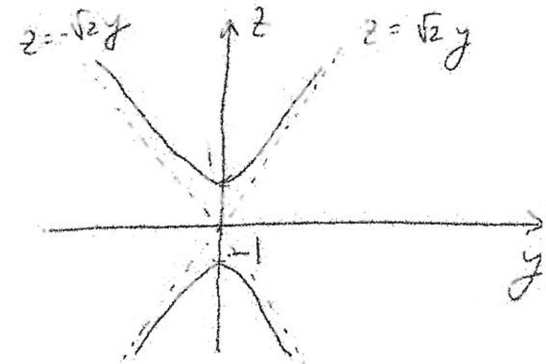
i. $x = 0$
ii. $y = 0$
iii. $z = 0$
iv. $z = 2$
v. $x = 1$
vi. $x = 2$

(If the intersection is a hyperbola, the asymptotes are drawn by a dashed line and are also labeled.)

Remember to label your axes!

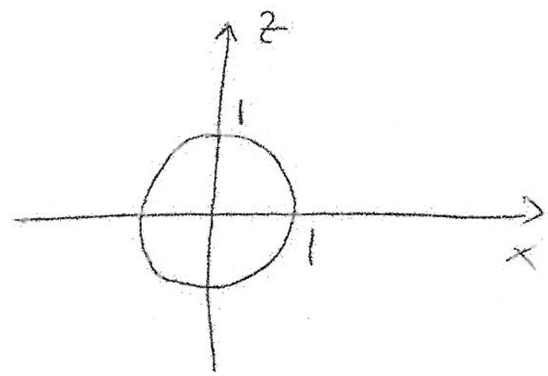
i)

$x=0 \Rightarrow -2y^2+z^2 = 1$, This is a hyperbola,



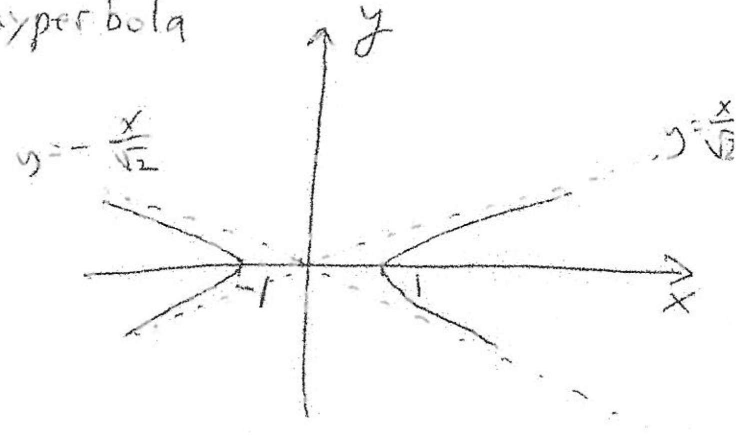
ii)

$y=0 \Rightarrow x^2+z^2=1$, This is a circle



iii)

$z=0 \Rightarrow x^2-2y^2=1$, This is a hyperbola

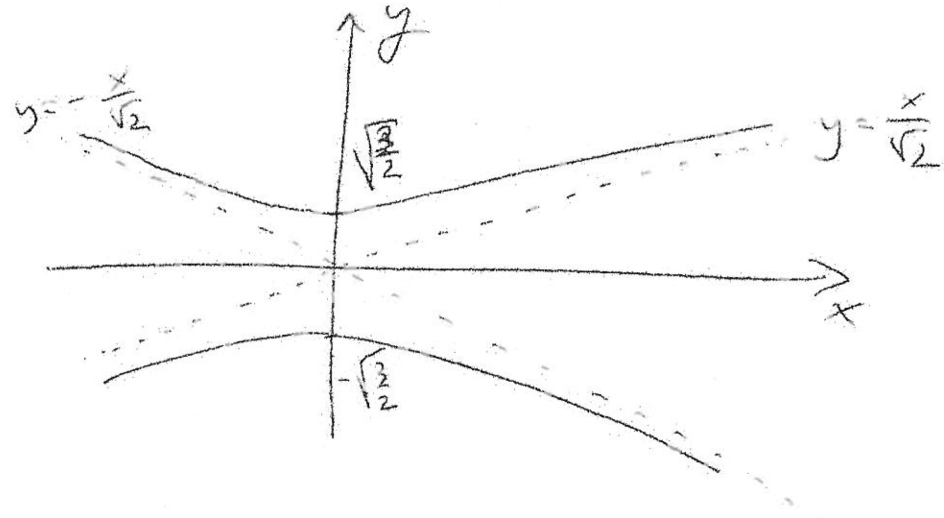


(over $\longrightarrow$)

(iv)

$z = 2 \Rightarrow x^2 - 2y^2 = -3 \Rightarrow -\frac{x^2}{3} + \frac{2}{3}y^2 = 1$

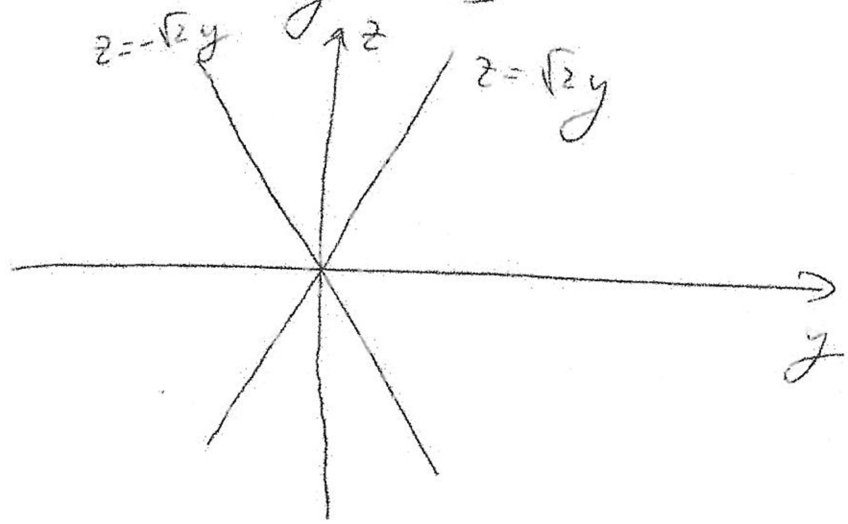
This is a hyperbola



(v)

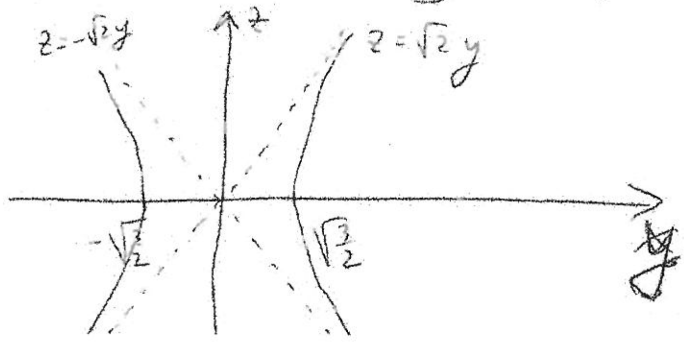
$x = 1 \Rightarrow -2y^2 + z^2 = 0 \Rightarrow z^2 = 2y^2 \Rightarrow z = \pm\sqrt{2}\,y$

This is a pair of intersecting lines



(vi) $x = 2 \Rightarrow -2y^2 + z^2 = -3 \Rightarrow \frac{2}{3}y^2 - \frac{1}{3}z^2 = 1$

This is a hyperbola



(b) **5 points** Write down the equation of the paraboloid with apex at $(0, 0, 0)$ opening in the positive x-direction which intersects the plane $x = 4$ in a circle of radius 3.

The equation will have the form

$$x = ay^2 + bz^2 \quad \text{for some } a > 0, b > 0$$

Since the intersection with the plane $x = 4$ is a circle, we must have $b = a$.

So, now the equation is

$$x = ay^2 + az^2$$

$$\frac{x}{a} = y^2 + z^2$$

Since the intersection with the plane $x = 4$ is a circle of radius 3, we must have

$$\frac{4}{a} = 3^2$$

$$a = \frac{4}{9}$$

Therefore, the equation of the paraboloid is

$$x = \frac{4}{9}(y^2 + z^2)$$

3. **(16 points)** Let ρ be the plane given by the equation $x + 2y + 3z = 6$ and let L be the line passing through the points $P(1, 0, 0)$ and$Q(-1, 3, 1)$.

(a) **(8 points)** What is the intersection of L and ρ?

$\vec{PQ} = \langle -2, 3, 1 \rangle$ is parallel to L. Using the point $P(1,0,0)$, parametric equations for L are

$x = 1-2t,\ y = 3t,\ z = t$

Now $x+2y+3z=6$ leads to

$1-2t+2(3t)+3t=6$

$1-2t+6t+3t=6$

$7t=5$

$t=5/7$

From here, $x\left(\frac{5}{7}\right) = -\frac{3}{7}$, $y\left(\frac{5}{7}\right)=\frac{15}{7}$, $z\left(\frac{5}{7}\right)=\frac{5}{7}$. The intersection of L and ρ is the point $\left(-\frac{3}{7}, \frac{15}{7}, \frac{5}{7}\right)$

(b) **(8 points)** Compute the angle between L and ρ.

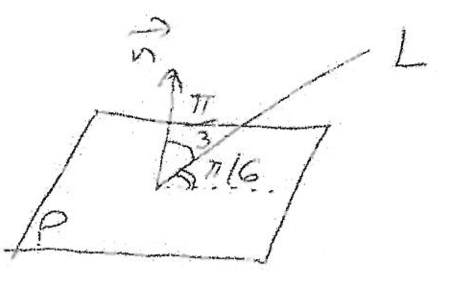
$\vec{PQ} = \langle -2, 3, 1 \rangle$ is parallel to L

$\vec{n} = \langle 1, 2, 3 \rangle$ is perpendicular to ρ

The angle between $\vec{PQ}$ and $\vec{n}$ is $\arccos\left(\frac{\vec{PQ}\cdot\vec{n}}{\|\vec{PQ}\|\,\|\vec{n}\|}\right)$

$= \arccos\left(\frac{7}{\sqrt{14}\sqrt{14}}\right) = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$

Hence, the angle between L and ρ is $\frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$.



4. **(17 points)**

(a) **(9 points)** Two particles travel along the space curves $\vec{r}_1 = \langle 2t, -t^3, 2t^2 \rangle$ and $\vec{r}_2 = \langle 1+t, -t^2, 1-t \rangle$. Do the particles collide? If so, find the coordinates of the point of collision. If not, explain why they do not collide.

If the particles collide, there is t such that

$$\begin{cases} 2t = 1+t \\ -t^3 = -t^2 \\ 2t^2 = 1-t \end{cases}$$

$2t = 1+t$ gives $t = 1$

$t=1$ satisfies $-t^3 = -t^2$

However, $t=1$ does not satisfy $2t^2 = 1-t$. Therefore, the particles do not collide.

(b) **(8 points)** Find parametric equations for the tangent line to the curve with parametric equations

$$x = t + \sin t, \quad y = t - \cos t, \quad z = te^{-t}$$

at the point when $t = 0$.

Let $\vec{r}(t) = \langle t+\sin t, t-\cos t, te^{-t} \rangle$

Then $\vec{r}(0) = \langle 0, -1, 0 \rangle$

Also, $\vec{r}\,'(t) = \langle 1+\cos t, 1+\sin t, e^{-t} - te^{-t} \rangle$, so that

$\vec{r}\,'(0) = \langle 2, 1, 1 \rangle$

Therefore, the tangent line is given parametrically by

$x = 2t, \ y = -1+t, \ z = t$

5. (16 points)

(a) (6 points) Find the cylindrical coordinates of the point P given by spherical coordinates $(4,\ 3\pi/4,\ 2\pi/3)$.

We have $\rho = 4$, $\theta = \frac{3\pi}{4}$, $\phi = \frac{2\pi}{3}$. From here

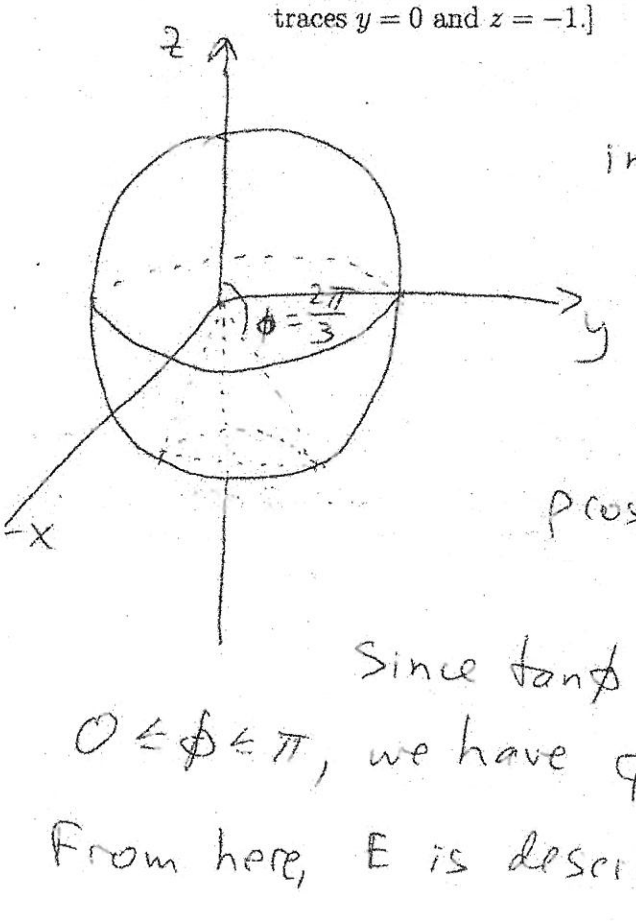
$r = \rho \sin\phi = (4)\sin\left(\frac{2\pi}{3}\right) = (4)\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$,

$\theta = \frac{3\pi}{4}$,

and $z = \rho\cos\phi = (4)\cos\left(\frac{2\pi}{3}\right) = (4)\left(-\frac{1}{2}\right) = -2$.

The cylindrical coordinates of P are $(2\sqrt{3}, \frac{3\pi}{4}, -2)$.

(b) (10 points) The solid E lies strictly inside the sphere $x^2+y^2+z^2=4$ and strictly below the cone $z = -\sqrt{\frac{1}{3}(x^2+y^2)}$. Describe E using inequalities in spherical coordinates, and simplify your answer as much as possible. [Hint: you may find it helpful to consider the traces $y=0$ and $z=-1$.]



The equation of the sphere $x^2+y^2+z^2=4$ in spherical coordinates is $\rho^2 = 4$, and from here, $\rho = 2$.

Next, we convert the cone equation $z = -\sqrt{\frac{1}{3}(x^2+y^2)}$ to spherical coordinates:

$\rho\cos\phi = -\sqrt{\frac{r^2}{3}} = -\frac{r}{\sqrt{3}} = -\frac{\rho\sin\phi}{\sqrt{3}}$

$\tan\phi = -\sqrt{3}$

Since $\tan\phi$ is a negative number and we need $0 \le \phi \le \pi$, we have $\phi = \pi + \arctan(-\sqrt{3}) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$.

From here, E is described in spherical coordinates by

$0 \le \rho \le 2$, $0 \le \theta \le 2\pi$, $\frac{2\pi}{3} \le \phi \le \pi$

6. **(17 points)** Let $\mathbf{r}(t) = \langle \frac{1}{2}e^t(\cos(t) + \sin(t)),\ \frac{1}{2}e^t(\cos(t) - \sin(t)) \rangle$.

(a) **(7 points)** What is the arclength of $\mathbf{r}(t)$ between $t = 0$ and $t = 1$?

$\vec{r}'(t) = \langle \frac{1}{2}(e^t(\cos t + \sin t) + e^t(-\sin t + \cos t)), \frac{1}{2}(e^t(\cos t - \sin t) + e^t(-\sin t - \cos t)) \rangle$

$= \langle e^t \cos t, -e^t \sin t \rangle$

$\|\vec{r}'(t)\| = \sqrt{e^{2t}\cos^2 t + e^{2t}\sin^2 t} = \sqrt{e^{2t}(\cos^2 t + \sin^2 t)} = \sqrt{e^{2t}} = e^t$

Arc length $= \int_0^1 \|\vec{r}'(t)\| dt = \int_0^1 e^t dt = e^t \Big|_0^1 = e - 1$

(b) **(7 points)** What is the curvature of $\mathbf{r}(t)$?

$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \langle \cos t, -\sin t \rangle$

$\vec{T}'(t) = \langle -\sin t, -\cos t \rangle$

$\|\vec{T}'(t)\| = \sqrt{\sin^2 t + \cos^2 t} = 1$

The curvature of $\vec{r}(t)$ is

$\kappa(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \frac{1}{e^t}$