1. **(16 points)** Find a parametrization of the surface given by the intersection of the plane $x + 2y + 3z = 12$ and the solid cylinder $x^2 + y^2 \leq 1$.

Since $x^2+y^2 \leq 1$, we let $x = r\cos\theta$, $y = r\sin\theta$.

Since $x+2y+3z=12$, we have

$$z = \frac{12-x-2y}{3} = 4 - \frac{1}{3}x - \frac{2}{3}y = 4 - \frac{r\cos\theta}{3} - \frac{2r\sin\theta}{3}.$$

Therefore, a parameterization of our surface is

$x = r\cos\theta$, $y = r\sin\theta$, $z = 4 - \frac{r\cos\theta}{3} - \frac{2r\sin\theta}{3}$,

$0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$.

2. **(17 points)** Consider the function $f(x, y) = (x^2 + y^2)^{\frac{3}{2}}$ and the point $P = \left(\sqrt{\frac{2}{3}}, 0\right)$.

(a) **(5 points)** Find the directional derivative of $f(x, y)$ at P in the direction towards the origin.

A vector from P in the direction towards the origin is $\vec{v} = \langle -\sqrt{\frac{2}{3}}, 0 \rangle$. A unit vector in the same direction is $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \langle -1, 0 \rangle$.

We have $\vec{\nabla} f = \langle f_x, f_y \rangle = \langle 3x\sqrt{x^2+y^2}, 3y\sqrt{x^2+y^2} \rangle$

Hence, $D_{\vec{u}} f(P) = \vec{\nabla} f(P) \cdot \vec{u}$

$= \langle 2, 0 \rangle \cdot \langle -1, 0 \rangle = -2$

(b) **(4 points)** In what (unit) direction does $f(x, y)$ have its maximum rate of change at P?

The unit direction of maximum rate of change of f at P is given by

$$\frac{\vec{\nabla} f(P)}{\|\vec{\nabla} f(P)\|} = \frac{\langle 2, 0 \rangle}{\|\langle 2, 0 \rangle\|} = \langle 1, 0 \rangle$$

(c) **(4 points)** What is the maximum rate of change in the direction from part (b)?

$\|\vec{\nabla} f(P)\| = \|\langle 2, 0 \rangle\| = 2$

(d) **(4 points)** Find and sketch the set of all points Q at which the maximum rate of change of $f(x, y)$ is equal to the maximum rate of change at P from part (c).

Let $Q = (x, y)$. Then we need

$\|\vec{\nabla} f(Q)\| = 2$

$\|\vec{\nabla} f(x, y)\| = 2$

$\|\langle 3x\sqrt{x^2+y^2}, 3y\sqrt{x^2+y^2} \rangle\| = 2$

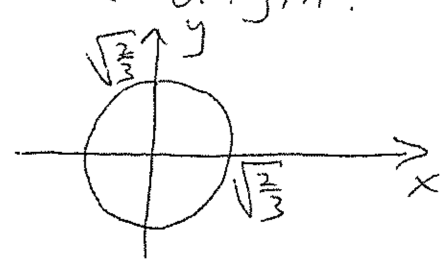
$\sqrt{9x^2(x^2+y^2) + 9y^2(x^2+y^2)} = 2$

$\sqrt{9(x^2+y^2)^2} = 2$

$3(x^2+y^2) = 2$

$x^2+y^2 = \frac{2}{3} = \left(\sqrt{\frac{2}{3}}\right)^2$

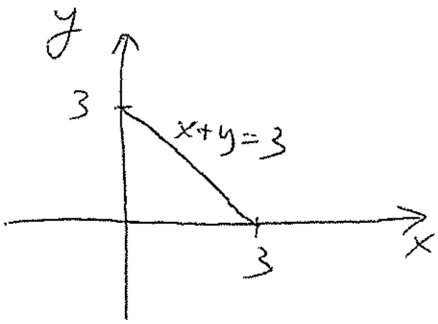
This is a circle of radius $\sqrt{\frac{2}{3}}$ centered at the origin:



3. (17 points) Find the absolute maximum and absolute minimum values of

$$f(x, y) = xy - 8x - y^2 + y + 2$$

over the (closed) triangular region with vertices $(0, 0)$, $(3, 0)$, and $(0, 3)$.



$f_x = y - 8 = 0 \Rightarrow y = 8$

$f_y = x - 2y + 1 = 0 \Rightarrow x = 2y - 1 = 2(8) - 1 = 15$

However, the point $(15, 8)$ is not in our triangular region (as $15 + 8 = 23 > 3$), so we will ignore it.

We will consider the 3 corner boundary points $(0,0)$, $(3,0)$, $(0,3)$.

Next:

On the boundary $y = 0$, $0 \le x \le 3$: $f(x, 0) = -8x + 2 = f_1(x)$

$f_1'(x) = -8 \ne 0$ for any x, so no critical points.

On the boundary $x = 0$, $0 \le y \le 3$: $f(0, y) = -y^2 + y + 2 = f_2(y)$

$f_2'(y) = -2y + 1 = 0 \Leftrightarrow y = \frac{1}{2}$. A critical point is $(0, \frac{1}{2})$

On the boundary $x + y = 3$, we have $x = 3 - y$, and so

$f(x, y) = (3-y)y - 8(3-y) - y^2 + y + 2 = -2y^2 + 12y - 22 = f_3(y)$

$f_3'(y) = -4y + 12 = 0 \Leftrightarrow y = 3$. This leads to the corner point $(0, 3)$.

We have

(x,y)	$f(x,y)$
$(0,0)$	2
$(3,0)$	-22
$(0,3)$	-4
$(0,\frac{1}{2})$	$9/4$

Hence, the absolute maximum value of f is $\frac{9}{4}$ at the point $(0, \frac{1}{2})$, and the absolute minimum value of f is -22 at the point $(3, 0)$

4. (17 points) Find all points on the hyperboloid of one sheet $x^2+y^2-z^2=1$ where the tangent plane is parallel to the plane $x+y-z=0$.

Let $F(x,y,z)=x^2+y^2-z^2$. The hyperboloid of one sheet $x^2+y^2-z^2=1$ is the level surface $F(x,y,z)=1$.

A normal vector to our hyperboloid of one sheet at the point (x,y,z) is given by

$$\vec{\nabla}F(x,y,z)=\langle F_x, F_y, F_z\rangle=\langle 2x,2y,-2z\rangle.$$

A normal vector to the plane $x+y-z=0$ is $\vec{n}=\langle 1,1,-1\rangle$.

We need $\vec{\nabla}F(x,y,z)$ and $\vec{n}$ to be parallel, i.e.

$$\vec{\nabla}F(x,y,z)=\lambda\vec{n} \text{ for some } \lambda\neq 0.$$

$$\langle 2x,2y,-2z\rangle=\lambda\langle 1,1,-1\rangle$$

$$2x=\lambda,\ 2y=\lambda,\ 2z=\lambda$$

$$x=\frac{\lambda}{2},\ y=\frac{\lambda}{2},\ z=\frac{\lambda}{2}$$

Since the point (x,y,z) lies on $x^2+y^2-z^2=1$, we have

$$\left(\frac{\lambda}{2}\right)^2+\left(\frac{\lambda}{2}\right)^2-\left(\frac{\lambda}{2}\right)^2=1 \iff \left(\frac{\lambda}{2}\right)^2=1 \iff \lambda^2=4 \iff \lambda=2 \text{ or } \lambda=-2$$

Since our points (x,y,z) are given by $\left(\frac{\lambda}{2},\frac{\lambda}{2},\frac{\lambda}{2}\right)$, we have 2 such points: $(1,1,1)$ and $(-1,-1,-1)$.

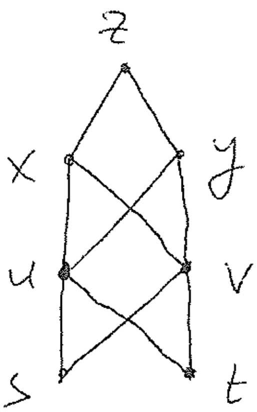
5. (16 points) Let

$$z = f(x, y), \quad x = u^2v^2 + 3, \quad y = -\cos(u) - v, \quad u = \frac{s}{t}, \quad v = e^{st}.$$

Suppose that f is a differentiable function of x and y and that

$$\frac{\partial z}{\partial x} = \frac{1}{x^2 + 2xy + y^2 + 1} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{1}{x^2 + 2xy + y^2 + 1}.$$

Find $\left.\frac{\partial z}{\partial s}\right|_{(s,t)=(0,1)}$



By the Chain Rule,

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial z}{\partial x}\frac{\partial x}{\partial v}\frac{\partial v}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\frac{\partial v}{\partial s}$$

We compute $\frac{\partial x}{\partial u} = 2uv^2$, $\frac{\partial x}{\partial v} = 2u^2v$, $\frac{\partial y}{\partial u} = \sin u$, $\frac{\partial y}{\partial v} = -1$, $\frac{\partial u}{\partial s} = \frac{1}{t}$, $\frac{\partial v}{\partial s} = te^{st}$

We have $u(0,1) = 0$ and $v(0,1) = 1$. From here, we note that

$$\left.\frac{\partial x}{\partial u}\right|_{(s,t)=(0,1)} = \left.\frac{\partial x}{\partial u}\right|_{(u,v)=(0,1)} = 0, \quad \left.\frac{\partial x}{\partial v}\right|_{(s,t)=(0,1)} = \left.\frac{\partial x}{\partial v}\right|_{(u,v)=(0,1)} = 0,$$

and $\left.\frac{\partial y}{\partial u}\right|_{(s,t)=(0,1)} = \left.\frac{\partial y}{\partial u}\right|_{(u,v)=(0,1)} = \sin(0) = 0$. Hence

$$\left.\frac{\partial z}{\partial s}\right|_{(s,t)=(0,1)} = \left(\left.\frac{\partial z}{\partial y}\right|_{(s,t)=(0,1)}\right)\left(\left.\frac{\partial y}{\partial v}\right|_{(s,t)=(0,1)}\right)\left(\left.\frac{\partial v}{\partial s}\right|_{(s,t)=(0,1)}\right)$$

We have $\left.\frac{\partial v}{\partial s}\right|_{(s,t)=(0,1)} = 1$ and $\left.\frac{\partial y}{\partial v}\right|_{(s,t)=(0,1)} = \left.\frac{\partial y}{\partial v}\right|_{(u,v)=(0,1)} = -1$.

Finally, when $(u,v) = (0,1)$, $x(0,1) = 3$ and $y(0,1) = -2$. Thus

$\left.\frac{\partial z}{\partial y}\right|_{(s,t)=(0,1)} = \left.\frac{\partial z}{\partial y}\right|_{(x,y)=(3,-2)} = \frac{1}{2}$. We obtain $\left.\frac{\partial z}{\partial s}\right|_{(s,t)=(0,1)} = \left(\frac{1}{2}\right)(-1)(1) = -\frac{1}{2}$

6. **(17 points)** Let $\mathbf{r}(t) = \langle \frac{1}{2}e^t(\cos(t) + \sin(t)),\ \frac{1}{2}e^t(\cos(t) - \sin(t)) \rangle$.

(a) **(7 points)** What is the arclength of $\mathbf{r}(t)$ between $t = 0$ and $t = 1$?

$$\vec{r}\,'(t) = \left\langle \frac{1}{2}\left(e^t(\cos t + \sin t) + e^t(-\sin t + \cos t)\right), \frac{1}{2}\left(e^t(\cos t - \sin t) + e^t(-\sin t - \cos t)\right) \right\rangle$$

$$= \langle e^t \cos t, -e^t \sin t \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{e^{2t}\cos^2 t + e^{2t}\sin^2 t} = \sqrt{e^{2t}(\cos^2 t + \sin^2 t)} = \sqrt{e^{2t}} = e^t$$

$$\text{Arc length} = \int_0^1 \|\vec{r}\,'(t)\|\,dt = \int_0^1 e^t\,dt = e^t\Big|_0^1 = e - 1$$

(b) **(7 points)** What is the curvature of $\mathbf{r}(t)$?

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \langle \cos t, -\sin t \rangle$$

$$\vec{T}\,'(t) = \langle -\sin t, -\cos t \rangle$$

$$\|\vec{T}\,'(t)\| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

The curvature of $\vec{r}(t)$ is

$$\kappa(t) = \frac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|} = \frac{1}{e^t}$$

1. (16 points) Suppose that

$$f(x,y) = x^3 + 6x^2y + axy^2 + by^3,$$

for some constants a and b. Then find a and b such that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

for every (x, y).

(Note that this equation can also be equivalently written as $f_{xx} + f_{yy} = 0$.)

Solution. Look at

$$\frac{\partial f}{\partial x} = 3x^2 + 12xy + ay^2 \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2} = 6x + 12y;$$
$$\frac{\partial f}{\partial y} = 6x^2 + 2axy + 3by^2 \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2} = 2ax + 6by.$$

So

$$0 = (6x + 12y) + (2ax + 6by) = 2(3 + a)x + 6(2 + b)y$$

for all x and y.

Therefore we have $a = -3$ and $b = -2$.

2. (17 points) Consider the hyperbolic paraboloid surface given by the equation

$$z = 2x^2 - 3y^2.$$

(a) **(12 points)** In what (unit) direction does z have its maximum rate of change at the point $(2, 1)$?

Solution. Let $f(x, y) = 2x^2 - 3y^2$. Then we have $\nabla f(x, y) = \langle 4x, -6y \rangle$ and $\nabla f(2, 1) = \langle 8, -6 \rangle$. This vector has length $\sqrt{8^2 + (-6)^2} = 10$. So the desired unit direction is $\langle \frac{8}{10}, \frac{-6}{10} \rangle = \langle \frac{4}{5}, \frac{-3}{5} \rangle$.

(b) **(5 points)** What is the maximum rate of change in the direction in (a) ?

Solution. It is equal to $|\nabla f(2, 1)| = \sqrt{8^2 + (-6)^2} = 10$.

3. (17 points) Find and classify the critical points (local maxima, local minima, or saddle points) of

$$f(x, y) = x^3 + y^3 - 3xy.$$

Solution. Look at

$$\frac{\partial f}{\partial x} = 3x^2 - 3y, \qquad \frac{\partial^2 f}{\partial x^2} = 6x, \qquad \frac{\partial^2 f}{\partial y \partial x} = -3;$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3x, \qquad \frac{\partial^2 f}{\partial y^2} = 6y.$$

To find the critical points, solve

$$3x^2 - 3y = 3y^2 - 3x = 0;$$

$y = x^2$ and $0 = x^4 - x = x(x^3 - 1) = x(x - 1)(x^2 + x + 1)$; and get $(x, y) = (0, 0)$ or $(1, 1)$. Note

$$D(x, y) = (6x)(6y) - (-3)^2 = 36xy - 9.$$

Thus we have

$$D(0, 0) = -9 < 0, \qquad D(1, 1) = 27 > 0, \qquad \text{and} \qquad f_{xx}(1, 1) = 6 > 0.$$

Therefore f has local minimum $f(1, 1) = -2$ at $(1, 1)$ and a saddle point at $(0, 0)$.

4. (17 points) Find the tangent plane to the surface defined by the equation

$$x^2 z + yz = 1$$

at the point $(1, 1, \frac{1}{2})$.

Solution. Use implicit differentiation to find

$$2xz + x^2\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial x} = 0 \qquad \text{or} \qquad \frac{\partial z}{\partial x} = -\frac{2xz}{x^2+y}$$

$$x^2\frac{\partial z}{\partial y} + z + y\frac{\partial z}{\partial y} = 0 \qquad \text{or} \qquad \frac{\partial z}{\partial y} = -\frac{z}{x^2+y},$$

where $x^2 + y \neq 0$. Evaluate these two partial derivatives at $(1, 1, \frac{1}{2})$ to get $-\frac{1}{2}$ and $-\frac{1}{4}$, respectively. So the desired tangent plane is given by

$$z - \frac{1}{2} = -\frac{1}{2}(x-1) - \frac{1}{4}(y-1) \qquad \text{or} \qquad 2x + y + 4z - 5 = 0.$$

(Alternatively, you could use $z = \frac{1}{x^2+y}$ to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ and evaluate these at $(x, y) = (1, 1)$ above.)

5. **(16 points)** Let

$$z = f(x, y), \quad x = u^2 - v^3, \quad y = u + 2v^2.$$

Suppose that f is a differentiable function of x and y, and that

$$\left.\frac{\partial z}{\partial x}\right|_{(x,y)=(-7,9)} = -2 \qquad \text{and} \qquad \left.\frac{\partial z}{\partial y}\right|_{(x,y)=(-7,9)} = 3.$$

Then find

$$\left.\frac{\partial z}{\partial v}\right|_{(u,v)=(1,2)}.$$

(Note that, for example, $\left.\frac{\partial z}{\partial x}\right|_{(x,y)=(-7,9)}$ (respectively $\left.\frac{\partial z}{\partial v}\right|_{(u,v)=(1,2)}$) means the value of $\frac{\partial z}{\partial x}$ at $(x, y) = (-7, 9)$ (respectively the value of $\frac{\partial z}{\partial v}$ at $(u, v) = (1, 2)$).)

Solution. Use the chain rule to find

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}. \tag{1}$$

Also note that

$$\frac{\partial x}{\partial v} = -3v^2 \qquad \text{and} \qquad \frac{\partial y}{\partial v} = 4v; \qquad \text{so}$$

$$\left.\frac{\partial x}{\partial v}\right|_{(u,v)=(1,2)} = -12 \qquad \text{and} \qquad \left.\frac{\partial y}{\partial v}\right|_{(u,v)=(1,2)} = 8. \tag{2}$$

Look at the formulas for x and y in terms of u and v and note that $(u, v) = (1, 2)$ implies $(x, y) = (-7, 9)$. Then use the results in (2) and the given data in the problem to evaluate the partial derivatives in (1) at $(u, v) = (1, 2)$ and get

$$\left.\frac{\partial z}{\partial v}\right|_{(u,v)=(1,2)} = (-2)(-12) + 3 \cdot 8 = 48.$$

6. **(17 points)** Let C be the curve given by $\boldsymbol{r}(t) = \langle 2t, \ln t, t^2 \rangle$, where ln stands for the natural logarithm.

(a) **(9 points)** Find the arc length of the curve C for $1 \leq t \leq 4$.

Solution: Note that $\boldsymbol{r}(t)$ traces points only once when t runs over the interval $[1, 4]$ and that $\boldsymbol{r}$ has its derivative $\boldsymbol{r}'(t) = \langle 2, \frac{1}{t}, 2t \rangle$. Now we have the desired arc length

$$\begin{aligned}
\int_1^4 \sqrt{|\boldsymbol{r}'(t)|}\, dt &= \int_1^4 \sqrt{2^2 + \left(\frac{1}{t}\right)^2 + (2t)^2}\, dt \\
&= \int_1^4 \sqrt{4 + \frac{1}{t^2} + 4t^2}\, dt \\
&= \int_1^4 \sqrt{\left(\frac{1}{t} + 2t\right)^2}\, dt \\
&= \int_1^4 \left(\frac{1}{t} + 2t\right) dt \\
&= [\ln t + t^2]_1^4 \\
&= \{\ln 4 + 4^2\} - \{\ln(1) + 1^2\} \\
&= \ln 4 + 15.
\end{aligned}$$

(b) **(8 points)** Find the curvature of the curve C at $t = 1$.

Solution: The curvature is given by

$$\kappa = \frac{|\boldsymbol{r}'(1) \times \boldsymbol{r}''(1)|}{|\boldsymbol{r}'(1)|^3}.$$

Now recall the formula for $\boldsymbol{r}'(t)$ in (a) above and look at

$$\boldsymbol{r}'(1) = \langle 2, 1, 2 \rangle \text{ and } |\boldsymbol{r}'(1)| = \sqrt{2^2 + 1^2 + 2^2} = 3.$$

Since $\boldsymbol{r}''(t) = \langle 0, -\frac{1}{t^2}, 2 \rangle$, we have

$$\boldsymbol{r}''(1) = \langle 0, -1, 2 \rangle.$$

Observe

$$\boldsymbol{r}'(1) \times \boldsymbol{r}''(1) = \langle 2, 1, 2 \rangle \times \langle 0, -1, 2 \rangle = \langle 4, -4, -2 \rangle.$$

(You need to show detailed work for this in the exam.) Therefore we have

$$\kappa = \frac{\sqrt{4^2 + (-4)^2 + (-2)^2}}{3^3} = \frac{6}{27} = \frac{2}{9}.$$

1. (15 points) Find the length of the curve $\underline{r}(t) = \langle 2t^{3/2}, \cos(2t), \sin(2t)\rangle, 0 \le t \le 1$.

Solution

First, we need the derivative:

$$\underline{r}'(t) = \langle 3t^{1/2}, -2\sin(2t), 2\cos(2t)\rangle.$$

Next, we integrate the magnitude:

$$\begin{aligned} L &= \int_0^1 |\underline{r}'(t)|\, dt, \\ &= \int_0^1 \sqrt{9t + 4\sin^2(2t) + 4\cos^2(2t)}\, dt, \\ &= \int_0^1 \sqrt{9t+4}\, dt, \\ &= \left[\frac{2}{27}(9t+4)^{3/2}\right]_0^1, \\ &= \frac{2}{27}\left((13)^{3/2} - 4^{3/2}\right). \end{aligned}$$

2. (10 points) (a) Let C be the curve in the xz-plane given by $z = \frac{1}{x}, 2 \leq x \leq 5$. Find parametric equations for the surface S obtained by rotating the curve C around the z-axis.

One possible solution

Let

$$\begin{aligned} x &= u\cos(v), \\ y &= u\sin(v), \\ z &= \frac{1}{u}, \end{aligned}$$

where $u \in [2, 5]$ and $v \in [0, 2\pi]$.

(b) Find parametric equations of the upper half of the sphere centered at $(0, 0, 1)$ and with radius $R = 3$.

One possible solution

Let

$$\begin{aligned} x &= 3\cos(u)\sin(v), \\ y &= 3\sin(u)\sin(v), \\ z &= 3\cos(v) + 1, \end{aligned}$$

where $u \in [0, 2\pi]$ and $v \in [0, \frac{\pi}{2}]$.

3. (20 points) Let $z = f(x, y) = e^{-(x^2+y^2)}$ model a mountain.

(a) If a hiker standing at $(\frac{1}{2}, \frac{1}{3})$ wishes to descend as quickly as possible, in what direction must she walk?

Solution

The gradient of f will point in the direction of quickest increase, so we want

$$-\nabla f = \langle 2xe^{-(x^2+y^2)}, 2ye^{-(x^2+y^2)} \rangle$$

evaluated at $(\frac{1}{2}, \frac{1}{3})$:

$$\left\langle e^{-(\frac{1}{4}+\frac{1}{9})}, \frac{2}{3}e^{-(\frac{1}{4}+\frac{1}{9})} \right\rangle.$$

(b) How steep is the slope from $(\frac{1}{2}, \frac{1}{3})$ in the direction of $\underline{u} = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$. This means that you have to find the directional derivative $z = f(x, y)$ in the $\underline{u}$ direction.

Solution

We may use the gradient to calculate the directional derivative:

$$D_{\underline{u}} f = \nabla f \cdot \underline{u} = \frac{5\sqrt{2}}{6} e^{-(\frac{1}{4}+\frac{1}{9})}.$$

(c) Find an equation of the tangent plane to $z = f(x, y) = e^{-(x^2+y^2)}$ at the point $(\frac{1}{2}, \frac{1}{3})$ where the hiker is standing.

Solution

We may represent the surface as the level curve

$$F(x, y, z) = e^{-(x^2+y^2)} - z = 0.$$

The gradient of F evaluated at $(\frac{1}{2}, \frac{1}{3}, e^{-(\frac{1}{4}+\frac{1}{9})})$ is the normal vector for the tangent plane:

$$\nabla F = \left\langle e^{-(\frac{1}{4}+\frac{1}{9})}, \frac{2}{3}e^{-(\frac{1}{4}+\frac{1}{9})}, -1 \right\rangle.$$

An equation for the plane is therefore

$$e^{-(\frac{1}{4}+\frac{1}{9})}\left(x - \frac{1}{2}\right) + \frac{2}{3}e^{-(\frac{1}{4}+\frac{1}{9})}\left(y - \frac{1}{3}\right) - \left(z - e^{-(\frac{1}{4}+\frac{1}{9})}\right) = 0.$$

4. (20 points) (a) Determine whether the following limit exists. If the limit exists, find it. Explain your answer.

$$\lim_{(x,y)\to(0,0)} \left(\frac{xy}{x^2 - 2y^3} \right)$$

Solution
Using polar coordinates, we see that

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \left(\frac{xy}{x^2 - 2y^3} \right) &= \lim_{r\to 0} \left(\frac{r^2\cos(\theta)\sin(\theta)}{r^2\cos^2(\theta) - 2r^3\sin^3(\theta)} \right), \\ &= \lim_{r\to 0} \left(\frac{\cos(\theta)\sin(\theta)}{\cos^2(\theta) - 2r\sin^3(\theta)} \right), \\ &= \tan(\theta).\end{aligned}$$

The limit depends on θ and therefore does not exist.

(b) Is the following function continuous at $(0,0)$? Use limits to explain your answer.

$$f(x,y) = \begin{cases} \frac{\sin(x^2+y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 1 & \text{if } (x,y) = (0,0), \end{cases}$$

Solution
The function is continuous at $(0,0)$ if

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0).$$

Again using polar coordinates, we have

$$\begin{aligned}\lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{r\to 0} \frac{\sin(r^2)}{r^2}, \\ &= \lim_{r\to 0} \frac{2r\cos(r^2)}{2r}, \text{ by L'Hôpital's Rule} \\ &= \lim_{r\to 0} \cos(r^2) = 1.\end{aligned}$$

(c) Calculate $\frac{dz}{dt}$ at $t = \pi$ for $z = f(x, y) = x^2 - xy - 4y^2$, $x(t) = \cos(2t)$, $y(t) = \sin(2t)$.
Solution
First, note that when $t = \pi$, $x = 1$ and $y = 0$. The chain rule tells us that

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt},\\ &= (2x(t) - y(t)) \cdot (-2\sin(2t)) + (-x(t) - 8y(t)) \cdot (2\cos(2t)),\\ &= 0 + (-1 - 0) \cdot 2,\\ &= -2.\end{aligned}$$

(d) Calculate $\frac{\partial z}{\partial u}$ for $z = f(x, y) = \ln(\frac{x}{y+1})$, $x(u, v) = uv$, $y(u, v) = \frac{u}{v}$.
Solution

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u},\\ &= \frac{1}{x(u, v)} \cdot v + \frac{-1}{y(u, v) + 1} \cdot \frac{1}{v},\\ &= \frac{1}{u} - \frac{1}{u + v}.\end{aligned}$$

5. (20 points) Consider the surface given by the equation

$$x^2 - y^2 + z^2 = 4.$$

(a) Find a vector normal to the surface at $(-1, 1, 2)$.
Solution
Because the surface is a level surface:

$$f(x, y, z) = x^2 - y^2 + z^2 = 4,$$

the gradient vector suffices:

$$\nabla f = \langle 2x, -2y, 2z \rangle.$$

Evaluating at $(-1, 1, 2)$, we have our normal vector:

$$\langle -2, -2, 4 \rangle.$$

(b) Find an equation for the tangent plane to the surface at $(-1, 1, 2)$.
Solution
We use the normal vector previously calculated:

$$-2(x + 1) - 2(y - 1) + 4(z - 2) = 0.$$

6. (15 points) Consider the function $f(x, y) = x^3 + y^2 - 3x - 2y$.

   (a) Find and classify all critical points for the function
   Solution
   First, the relevant partial derivatives:

   $$f_x(x,y) = 3x^2 - 3, \quad f_y(x,y) = 2y - 2, \quad f_{xx}(x,y) = 6x, \quad f_{yy}(x,y) = 2, \quad f_{xy}(x,y) = f_{yx}(x,y) = 0.$$

   Next, the determinant:
   $$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 12x.$$
   Now we find our critical points. The condition that $f_y = 0$ implies
   $$y = 1.$$
   The condition that $f_x = 0$ implies
   $$x = \pm 1.$$
   Thus, we need to classify two points: $(-1, 1)$ and $(1, 1)$. Evaluating D:
   $$D(-1,1) = -12 < 0, \quad D(1,1) = 12 > 0,$$
   and so $(-1, 1)$ is a saddle point. To complete the classification of $(1, 1)$, we check the sign of f_{xx}:
   $$f_{xx}(1,1) = 6 > 0,$$
   and so $(1, 1)$ is a minimum.

(b) Find the absolute maximum and minimum of $f(x,y)$ in the rectangle with vertices $(0,0)$, $(0,1)$, $(1,1)$, $(1,0)$.

Solution

We must check the value of f at the corners of the box, check for extrema on the boundary, and finally check for local extrema in the center.

$$
\begin{aligned}
f(x,0) &= x^3 - 3x, & f'(x,0) &= 3x^2 - 3,\\
f(x,1) &= x^3 - 3x - 1, & f'(x,1) &= 3x^2 - 3,\\
f(0,y) &= y^2 - 2y, & f'(0,y) &= 2y - 2,\\
f(1,y) &= -2 + y^2 - 2y, & f'(1,y) &= 2y - 2.
\end{aligned}
$$

The list of values we must check is therefore $(0,0),(0,1),(1,1),(1,0)$. As

$$
\begin{aligned}
f(0,0) &= 0,\\
f(1,0) &= -2,\\
f(1,1) &= -3,\\
f(0,1) &= -1,
\end{aligned}
$$

we conclude that the maximum is 0 at $(0,0)$ and the minimum is -3 at $(1,1)$.

2. Suppose that $z = F(x, y)$ and that $x = X(u, w)$ and $y = Y(u, w)$, where F, X and Y all have continuous partial derivatives at all points.

Caution: one can view z as a function of x and y, and one can view z as a function of u and w.

Suppose that the following facts are given:

$$\frac{\partial z}{\partial x}(3,4) = q \quad \frac{\partial z}{\partial x}(a,b) = 5 \quad \frac{\partial z}{\partial y}(3,4) = 11 \quad \frac{\partial z}{\partial y}(a,b) = 12$$

$$\frac{\partial x}{\partial u}(a,b) = 10 \quad \frac{\partial x}{\partial w}(a,b) = 7 \quad \frac{\partial y}{\partial u}(a,b) = p \quad \frac{\partial z}{\partial w}(a,b) = 0$$

$$X(p,q) = -1 \quad Y(p,q) = -7 \quad X(a,b) = 3 \quad Y(a,b) = 4$$

(a) (10 points) Find $\frac{\partial z}{\partial u}(a,b)$

(b) (10 points) Find $\frac{\partial y}{\partial w}(a,b)$

5. Consider the function $f(x, y) = 2x^3 + 3y^2 - 6xy + 7$.

   (a) (8 points) Find all the critical points of f.

   (b) (8 points) Classify each critical point as a local maximum, local minimum, or saddle point.

   (c) (4 points) Does f have a global maximum on $\mathbb{R}^2$ (i.e. the plane)? How about a global minimum?