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The Evolving, Flowless Drainhole: A Nongravitating-Particle Model in General Relativity Theory[1]

HOMER G. ELLIS

Department of Mathematics, University of Colorado, Boulder, Colorado 80309

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Abstract

Nonstationary, spherically symmetric solutions of the coupled field equations $R_{\mu\nu} = 2\phi_{,\mu}\phi_{,\nu}$ and $\Box\phi = 0$, in which the coupling polarity is opposite to the orthodox, are derived. The basic solution, termed the evolving, flowless drainhole manifold, has these properties: (1) geodesic completeness; (2) a topological hole that shrinks to a point at a singular event and immediately begins to expand back to infinite size; (3) multiple branching of geodesics that arrive at the singular event; (4) asymptotic flatness at spatial infinity, luminal infinity, and temporal infinity; (5) isometric symmetry under time reversal and under space reflection through the drainhole; (6) conformal symmetry under space-time dilatations that leave the singular event fixed, and also under space–time inversions that interchange the singular event and a point at infinity. An earlier, static drainhole solution of the same equations was able to represent an ordinary star's external field or to serve as a model of a simple gravitating or nongravitating particle, replacing in these capacities the Kruskal–Fronsdal–Schwarzschild black-hole manifolds. The evolving, flowless drainhole can be thought of as modeling the death and rebirth of a scalar particle that is infinitely large in the infinite past and the infinite future. This particle does not gravitate, for the "ether flow" whose spatial variations in the static drainhole solution are identified with gravitation is removed from consideration in the evolving, flowless drainhole solution by being turned off at the outset. What is left is space alone, evolving dynamically in accordance with the field equations.

§(1): *Introduction*

Many are the ways to couple a scalar field to the geometry of space–time, and various are the motivations that have precipitated the use of this one, that

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one, or another one. Among the couplings that have found favor, perhaps the simplest are those expressed by the variational principle

$$0 = \delta \int (R^{\kappa}{}_{\kappa} - K\phi^{,\kappa}\phi_{,\kappa})\, dv \tag{1.1}$$

in which ϕ is the scalar field, $R_{\kappa\lambda}$ are Ricci tensor field components,[2] dv is the metric volume element, and K is a constant. Because a factor $|K|^{1/2}$ can be absorbed into ϕ, there are essentially just two distinct couplings involved in Equation (1.1), one corresponding to $K > 0$, the other to $K < 0$. In one application Hoyle and Narlikar, following up a suggestion of M. H. L. Pryce, introduced the $K > 0$ coupling into the steady-state cosmological theory as a replacement for the more involved coupling previously used by Hoyle [1]; they identified ϕ with the matter creation field (the C-field) [2]. In another application Jordan, in his theory aimed at formalizing Dirac's cosmological hypothesis that the gravitational scalar κ varies with the age of the universe instead of remaining constant, adopted a variational principle which was afterward shown to be equivalent to Equation (1.1) under a conformal transformation of the metric, with conformal factor κ ($\equiv e^{\phi}$); both $K > 0$ and $K < 0$ were admitted, with perhaps some preference indicated for $K > 0$ [3]. Others have used one or both of these simple couplings for less cosmical ends [4–7]. In this vein can be included an application by Ellis of the $K > 0$ coupling in a search for manifolds better adapted than black-hole manifolds, with their inscrutable, probe-destroying singularities, to the modeling of collapsed stars, on the one hand, and elementary particles on the other [8].

The manifolds resulting from the latter application and put forward as an improvement over black-hole manifolds are static, spherically symmetric space-time manifolds (termed "drainholes") that are geodesically complete and are free of the curvature singularities and associated horizons found in black-hole manifolds. For representing an ordinary star's external gravitational influences, these drain-hole solutions of the variational principle of Equation (1.1) are as serviceable as the Schwarzschild solutions of the Einstein vacuum field equations. In addition they have properties peculiar to themselves that make them in many respects more rewarding of analysis than the Kruskal–Fronsdal–Schwarzschild black-hole manifolds. A brief description of a static drainhole manifold is that it consists of two unchanging three-spaces of asymptotically Euclidean topology, connected to one another by a central topological hole through which drains an "ether," whose stationary but nonuniform flow into the "high" side of the hole and out the "low" side produces a gravitational field that is attractive on the high side and repulsive on the low side. The role of the scalar field is to allow

[2] The notational conventions and the computational framework used here are those given in the appendix of [8]. In particular, $R_{\kappa\lambda} = R_{\kappa}{}^{\mu}{}_{\lambda\mu}$, where $R_{\kappa}{}^{\mu}{}_{\lambda\nu} = \{_{\kappa}{}^{\mu}{}_{\nu}\}_{,\lambda} - \{_{\kappa}{}^{\mu}{}_{\lambda}\}_{,\nu} + \{_{\kappa}{}^{\rho}{}_{\nu}\}\{_{\rho}{}^{\mu}{}_{\lambda}\} - \{_{\kappa}{}^{\rho}{}_{\lambda}\}\{_{\rho}{}^{\mu}{}_{\nu}\}$, the $\{_{\kappa}{}^{\mu}{}_{\lambda}\}$ being Christoffel symbols of the metric.

(more accurately, to signify the presence of) the spatial curvatures that hold the hole open so that the ether may migrate from the one side to the other.

The Euler–Lagrange equations of the variational principle of Equation (1.1) are

$$R_{\mu\nu} - \tfrac{1}{2} R^{\kappa}{}_{\kappa} g_{\mu\nu} = K(\phi_{,\mu}\phi_{,\nu} - \tfrac{1}{2}\phi^{,\kappa}\phi_{,\kappa} g_{\mu\nu}) \tag{1.2}$$

which come from varying the $g_{\mu\nu}$ and are equivalent to

$$R_{\mu\nu} = K\phi_{,\mu}\phi_{,\nu} \tag{1.3}$$

and the scalar wave equation

$$0 = \Box\phi \equiv \phi^{,\kappa}{}_{;\kappa} \tag{1.4}$$

which arises from varying ϕ. To ensure that these equations have solutions that are static, spherically symmetric, and nonflat but lack curvature singularities, it is compulsory to take $K > 0$. In that case the Ricci tensor field is positive definite at every point at which the gradient of ϕ does not vanish, in the sense that at each such point $u^{\mu}R_{\mu\nu}u^{\nu} \geqslant 0$ for *every* tangent vector u, and $u^{\mu}R_{\mu\nu}u^{\nu} > 0$ for *some* tangent vector u. Because of this one can safely ignore the singularity theorems of Penrose and Hawking [9], for these theorems have as essential hypotheses the so-called null and timelike convergence conditions, which say that at every point $u^{\mu}R_{\mu\nu}u^{\nu} \leqslant 0$ for every tangent vector u that is, respectively, null or timelike.[3]

The supposition that $K > 0$ causes concern in many quarters; it is thought that taking K positive is tantamount to forcing the energy density of the scalar field to be negative wherever it does not vanish, and this, it is said, causes difficulties in the quantum domain, if not in the classical. Although that line of reasoning is open to question, the issue is subsidiary here. Whatever one's views about the "physicality" of choosing $K > 0$ in Equations (1.1)–(1.3), these equations do have in that case solutions that are in many ways more interesting than the corresponding solutions for $K \leqslant 0$ and that can hardly be ignored in any serious attempt to assess the relative merits of the two coupling polarities. It is the purpose of this paper to present a derivation of a class of such solutions that is supplementary to the class discussed in [8] and to mention briefly some of their salient properties. A fuller treatment of these properties is reserved for a later article.

The solutions to be derived are spherically symmetric but, unlike the earlier ones, not static, or even stationary. They represent the evolution in time of a three-space consisting, as in the static case, of two asymptotic regions connected through a topological hole which, in contrast to its fixedness in the static case,

[3]Because the $R_{\mu\nu}$ used here are the negatives of those of [9], these inequalities involving $R_{\mu\nu}$ must be reversed when comparisons with those of [9] are made.

shrinks from infinite size down to point size, then immediately reverses the process to expand back to infinite size. These space–time manifolds are, in a sense to be made explicit, asymptotically flat, in the temporal and the null directions as well as in the spatial: go where you will, faster than light, slower than light, or neither, forward in time, backward, or sideways, your surroundings will ultimately flatten out. What one has, in effect, is a (nonlinear) superposition of a symmetric pair of diffuse curvature waves imploding from the two asymptotic regions, meeting in the middle of space and time, then exploding outward, each into the asymptotic region from which the other came. A more graphic description is that a three-cylinder of infinite radius begins to contract in the middle, develops an hourglass waist which becomes for an instant totally pinched, then relaxes into its old, obesely cylindrical shape.

Such a manifold can be thought of as modeling the death and rebirth of a scalar particle, albeit one that tends to giantism in the infinite past and the infinite future. This particle is nongravitating: at every point of space a test particle can sit permanently at rest, even as the space itself evolves in the manner described. The ether flow whose spatial variations in the static case are identified with gravitation is here turned off at the outset by design. It is this inactivity of the ether that causes gravity to be absent even though space is curved.[4]

Ahead one will find a more thorough description of the evolving, flowless drainhole (Section 2), formulas for its curvature tensor fields and the field equations specialized to the case in question (Section 3), a four-stage procedure that solves these equations (Section 4), and a brief enumeration of interesting aspects of the family of evolving, flowless drainhole solutions (Section 5). Apart from the solutions themselves, the geometric method used in Section 4 to elicit them from the field equations is thought to be novel and of some independent interest.

§(2): *The Basic Evolving, Flowless Drainhole Line Element*

The line element of the family of solutions sought is required to have the spherically symmetric form

$$\begin{aligned} d\tau^2 &= dt^2 - d\rho^2 - r^2(t,\rho)[d\vartheta^2 + (\sin\vartheta)^2\, d\varphi^2] \\ &\equiv dt^2 - d\rho^2 - r^2(t,\rho)\, d\Omega^2 \end{aligned} \tag{2.1}$$

in which r is a nonnegative function to be determined by the field equations. The coordinate ranges are supposed given by

$$-\infty < t < \infty, \qquad -\infty < \rho < \infty, \qquad 0 < \vartheta < \pi, \qquad -\pi < \varphi < \pi \tag{2.2}$$

[4]The terms in the metric and curvature tensor fields that correspond to Newtonian gravitational potentials and forces are all zero when the ether flow velocity vanishes everywhere.

The portion of the underlying manifold not covered by these coordinates is presumed to consist of just those points at which lim $\vartheta = 0$ or π or at which lim $\varphi = \pm\pi$. The metric is nondegenerate everywhere except at events $r(t, \rho) = 0$.

The shape of the manifold $\mathfrak{M}$ that the line element (2.1) measures is most easily visualized by examination of its cross sections. Consider first the spacelike cross section Σ_t of $\mathfrak{M}$ on which the time coordinate has the fixed value t.[5] This three-dimensional submanifold inherits from $\mathfrak{M}$ the Riemannian line element given by

$$d\sigma^2 = d\rho^2 + r^2(t, \rho)\, d\Omega^2 \tag{2.3}$$

The cross section $S_{t,\rho}$ of Σ_t on which the radial coordinate has the constant value ρ is a geometrical two-sphere of radius $r(t, \rho)$; it is therefore the behavior of the radius function r that determines the shape of Σ_t, hence the shape of $\mathfrak{M}$. If, for example, $r(t, \rho)$ has a positive minimum value as ρ varies with t fixed, then Σ_t has a central hole whose radius is that minimum value. If, on the other hand, $r(t, \rho)$ vanishes for some value of ρ, then Σ_t has a central hole of radius zero, a pinhole. The hole in Σ_t is unchanging in size if the minimum value of $r(t, \rho)$ is independent of t; otherwise it evolves.

The evolving, flowless drainhole will be found (Section 4) to have its radius function r of the simple, homogeneous form expressed by

$$r(t, \rho) = (a^2 t^2 + b^2 \rho^2)^{1/2} \tag{2.4}$$

with $a > 0$ and $b > 0$. For this r let the temporal cross section Σ_t be identified by the more explicit symbol $\Sigma_t(a, b)$. Then the radius of the central hole in $\Sigma_t(a, b)$ is $a|t|$; so, as time goes on, the drainhole shrinks from infinite size to pinhole size (at time 0), then grows back to infinite size. If $t \neq 0$, then $\Sigma_t(a, 1)$ consists of two asymptotic regions joined through the central hole, these regions being both geometrically and topologically Euclidean as $\rho \to \pm\infty$. The equatorial ($\vartheta = \pi/2$) cross section of $\Sigma_t(a, 1)$, as well as every other of its great-circle cross sections, is a catenoid indistinguishable but for size from the corresponding cross section of the static, flowless drainhole [8] ; a drawing of it is exhibited in Figure 1. On the other hand $\Sigma_0(a, 1)$ consists of back-to-back copies of Euclidean three-space E^3, joined at a common point. The equatorial cross section of $\Sigma_0(a, 1)$ consists of two copies of E^2, joined at the same point; it is the double-sheeted, degenerate limit, as $t \to 0$, of the catenoid of Figure 1.

If $b < 1$, the equatorial cross section of $\Sigma_0(a, b)$ is isometric to a right circular cone of two nappes. This cone can be obtained by cutting out of the joined

[5] Thus, as a matter of convenience, t (and soon ρ) will represent in some places a coordinate function and in others a fixed value of that coordinate function. The contexts will make clear which is intended.

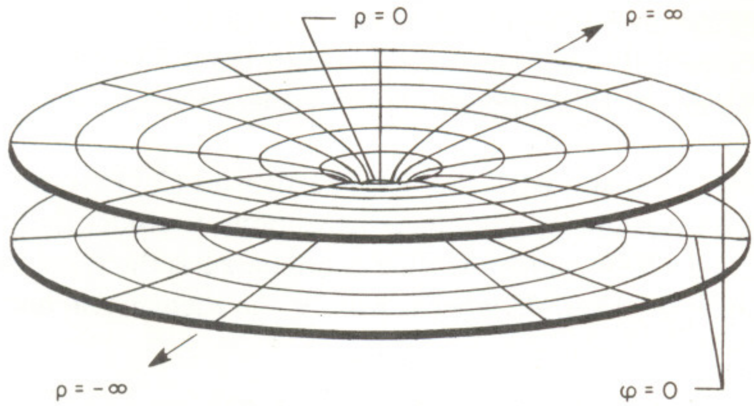

Fig. 1. The equatorial cross section of $\Sigma_t(a, 1)$ for $t \neq 0$. The line element of this surface is given by

$$d\sigma^2 = d\rho^2 + (\rho^2 + a^2 t^2)\, d\varphi^2.$$

The surface is isometric to the catenoid

$$\{[x, y, z] \mid (x^2 + y^2)^{1/2} = a|t| \cosh (z/a|t|)\}$$

in E^3. The radius of the central hole, where $\rho = 0$, is $a|t|$.

copies of E^2 mentioned before, imagined now as made of a flexible material, back-to-back wedges of angular width $2\pi(1 - b)$ and then pulling each remaining sector's edges together without stretching or shrinking the material. The tangent of the angle between axis and generators of the resulting cone is $b/(1 - b^2)^{1/2}$. When $b < 1$ and $t \neq 0$, the equatorial cross section of $\Sigma_t(a, b)$ can be similarly constructed. One snips from the catenoidal equatorial cross section of $\Sigma_{t/b}(a, 1)$ (cf. Figure 1, with t/b in place of t) a wedge of angular width $2\pi(1 - b)$, and pulls together the two edges of the remainder, again without stretching or shrinking the flexible fabric.[6] The result of this alteration is a transcendental surface that looks like (but is not) a hyperboloid of one sheet asymptotic to the conical equatorial cross section of $\Sigma_0(a, b)$ constructed before.

When $b > 1$, to construct as before the equatorial cross section of $\Sigma_t(a, b)$ from that of $\Sigma_{t/b}(a, 1)$ it is necessary to *insert* in the latter an extra wedge, of angular width $2\pi(b - 1)$. Though this operation is easy enough to imagine, it is difficult to visualize, for it cannot be performed in Euclidean three-space while the symmetry of revolution about an axis is maintained: only the central portion of $\Sigma_t(a, b)$ on which $(dr/d\rho)^2 \leqslant 1$ (equivalently, on which $(b^2 - 1)\, b^2 \rho^2 \leqslant a^2 t^2$) can be embedded in E^3 as a surface of revolution; this is a single point if $t = 0$,

[6]Technically, there is an isometry between the equatorial cross section of $\Sigma_t(a, b)$ and a wedge of angular width $2\pi b$ (with its edges identified) in the equatorial cross section of $\Sigma_{t/b}(a, 1)$. One such isometry matches the point with coordinates $[t, \rho, \pi/2, \varphi]$ in $\Sigma_t(a, b)$ with the point in $\Sigma_{t/b}(a, 1)$ whose coordinates are $[t/b, \rho, \pi/2, b\varphi]$. This holds for $t = 0$ as well as for $t \neq 0$.

but it encompasses more and more of $\Sigma_t(a, b)$ as $|t|$ grows larger. An isometry between the equatorial cross section of $\Sigma_t(a, b)$ and the augmented equatorial cross section of $\Sigma_{t/b}(a, 1)$ is given by the same formulas as for the previous case (cf. footnote 6).

As will be seen (Section 4), in the evolving, flowless drainhole $b^2 = 1 + a^2$. Thus only the latter construction, where $b > 1$, applies to it; but the construction when $b < 1$, because it takes place in Euclidean three-space, helps in the visualization. In either case one sees the evolution of the equatorial cross section by juxtaposing in the mind's eye the results of the construction for the various values of t.

§(3): *The Curvature Tensor Fields and the Field Equations*

The space-time manifold $\mathfrak{M}$ bearing the metric of Equation (2.1) possesses an orthonormal frame system $\{e_\mu\}$ defined in this way:

$$e_0 = \frac{\partial}{\partial t}, \quad e_1 = \frac{\partial}{\partial \rho}, \quad e_2 = \frac{1}{r(t,\rho)} \frac{\partial}{\partial \vartheta}, \quad e_3 = \frac{1}{r(t,\rho)\sin\vartheta} \frac{\partial}{\partial \varphi} \tag{3.1}$$

The coframe system $\{\omega^\mu\}$ dual to $\{e_\mu\}$ is given by

$$\omega^0 = dt, \quad \omega^1 = d\rho, \quad \omega^2 = r(t,\rho)\, d\vartheta, \quad \omega^3 = r(t,\rho)\,(\sin\vartheta)\, d\varphi \tag{3.2}$$

The unique torsion-free covariant differentiation $\boldsymbol{d}$ that is consistent with the metric has connection one-forms $\omega_\kappa{}^\mu$ that satisfy $\boldsymbol{d}e_\kappa = \omega_\kappa{}^\mu \otimes e_\mu$ and $\boldsymbol{d}\omega^\mu = -\omega_\kappa{}^\mu \otimes \omega^\kappa$; these are given by

$$[\omega_\kappa{}^\mu] = \begin{bmatrix} 0 & 0 & (r_t/r)\,\omega^2 & (r_t/r)\,\omega^3 \\ 0 & 0 & (r_\rho/r)\,\omega^2 & (r_\rho/r)\,\omega^3 \\ (r_t/r)\,\omega^2 & -(r_\rho/r)\,\omega^2 & 0 & [(\operatorname{ctn}\vartheta)/r]\,\omega^3 \\ (r_t/r)\,\omega^3 & -(r_\rho/r)\,\omega^3 & -[(\operatorname{ctn}\vartheta)/r]\,\omega^3 & 0 \end{bmatrix} \quad (\mu \rightarrow,\ \kappa \downarrow) \tag{3.3}$$

where $(\ )_t \equiv \partial(\ )/\partial t$, $(\ )_\rho \equiv \partial(\ )/\partial\rho$, and the omitted arguments t and ρ are to be treated as if present.

The curvature two-forms $\Theta_\kappa{}^\mu$ $(\equiv d\omega_\kappa{}^\mu - \omega_\kappa{}^\lambda \wedge \omega_\lambda{}^\mu)$ are readily calculated from Equations (3.3) and (3.2), with the result that

$$[\Theta_\kappa{}^\mu] = \begin{array}{c} \mu \rightarrow \\ \kappa \downarrow \end{array} \left[\begin{array}{c|c|c|c} 0 & + & + & + \\ \hline 0 & 0 & - & - \\ \hline \begin{array}{l}(r_{tt}/r)\,(\omega^0 \wedge \omega^2) \\ \quad +(r_{t\rho}/r)\,(\omega^1 \wedge \omega^2)\end{array} & \begin{array}{l}-(r_{\rho t}/r)\,(\omega^0 \wedge \omega^2) \\ \quad -(r_{\rho\rho}/r)\,(\omega^1 \wedge \omega^2)\end{array} & 0 & - \\ \hline \begin{array}{l}(r_{tt}/r)\,(\omega^0 \wedge \omega^3) \\ \quad +(r_{t\rho}/r)\,(\omega^1 \wedge \omega^3)\end{array} & \begin{array}{l}-(r_{\rho t}/r)\,(\omega^0 \wedge \omega^3) \\ \quad -(r_{\rho\rho}/r)\,(\omega^1 \wedge \omega^3)\end{array} & \begin{array}{c}\dfrac{1 + r_t^2 - r_\rho^2}{r^2} \\ \cdot(\omega^2 \wedge \omega^3)\end{array} & 0 \end{array}\right] \tag{3.4}$$

The entries represented by the + and − signs can be read off from those shown by use of the symmetry $\Theta_0{}^m = \Theta_m{}^0$ and the antisymmetry $\Theta_k{}^m = -\Theta_m{}^k$, which hold for $k, m = 1, 2, 3$.

The Ricci tensor field, defined as -2Φ, where Φ is the contracted curvature tensor field $\omega^\kappa \otimes \Theta_\kappa{}^\mu e_\mu$, is now computed from Equation (3.4); the result is that

$$\begin{aligned} -2\Phi = (2/r)[r_{tt}(\omega^0 \otimes \omega^0) + r_{t\rho}(\omega^0 \otimes \omega^1) + r_{\rho t}(\omega^1 \otimes \omega^0) + r_{\rho\rho}(\omega^1 \otimes \omega^1)] \\ - (1/2r^2)[2 + (r^2)_{tt} - (r^2)_{\rho\rho}](\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3) \end{aligned} \tag{3.5}$$

Having this, we can specialize the field equations (1.3) and (1.4) to the case in question.

First let us observe that in terms of the coframe system $\{\omega^\mu\}$ the scalar field ϕ has the representation

$$d\phi = \phi_t \omega^0 + \phi_\rho \omega^1 + (1/r)\,\phi_\vartheta \omega^2 + (1/r \sin \vartheta)\,\phi_\varphi \omega^3 \tag{3.6}$$

From this and Equation (3.5) it follows that the 22, 23, and 33 component equations of Equation (1.3) with respect to the system $\{\omega^\mu\}$ imply that $\phi_\vartheta = \phi_\varphi = 0$. Hence ϕ may be expressed by $\phi = \alpha(t, \rho)$, and then $d\phi = \alpha_t \omega^0 + \alpha_\rho \omega^1$. The remaining content of Equation (1.3) is incorporated in the equations

$$r_{tt}/r = \alpha_t^2, \qquad r_{t\rho}/r = \alpha_t \alpha_\rho = r_{\rho t}/r, \qquad r_{\rho\rho}/r = \alpha_\rho^2 \tag{3.7}$$

and

$$(r^2)_{tt} - (r^2)_{\rho\rho} = -2 \tag{3.8}$$

(Here, in accordance with remarks made in Section 1, it is assumed that $K > 0$ and that ϕ is normalized so that in fact $K = 2$.) A straightforward calculation shows that Equation (1.4) is equivalent to

$$(r^2\alpha_t)_t - (r^2\alpha_\rho)_\rho = 0 \tag{3.9}$$

Equations (3.7), (3.8), and (3.9) are to be solved for r and α, with $r \geqslant 0$ and with each of r^2 and α "maximally analytic" in the sense that it is analytic, its domain is connected, and it cannot be analytically continued to a more extensive (connected) domain.

It will suffice, actually, to solve Equations (3.7) and (3.8), for as Schücking noticed earlier [3, p. 208], Equation (1.4) is essentially redundant in the presence of the field equation (1.2). Indeed, if ϕ is twice- and the metric thrice-continuously differentiable, then after equating the divergences of the two sides of Equation (1.2) and dividing by K one has that $0 = \phi_{,\mu}\Box\phi$, in other words that $0 = (\Box\phi)\,d\phi$. If there is a point P such that $\Box\phi(P) \neq 0$, then there is a neighborhood of P on which $\Box\phi \neq 0$, hence on which, according to this equation, $d\phi = 0$; but this makes $\Box\phi(P) = 0$, contrary to hypothesis. Thus $\Box\phi = 0$ everywhere if Equation (1.2) holds and the smoothness requirements are met, as they will be here, where the search is for maximally analytic solutions, out of which less smooth ones can later be built.

§(4): *Solution of the Field Equations*

The process of finding all maximally analytic solutions of the field equation equivalents (3.7), (3.8), and, redundantly, (3.9) will be divided into four stages. The first stage will establish the Minkowskian nature of those solutions for which $d\phi \equiv 0$. The second stage will take up the solutions for which $d\phi$ vanishes nowhere. From the four equations (3.7) will be extracted the following geometrical facts about the functions α and r: (1) The two-dimensional surface that is the graph of r has Gaussian curvature 0, is therefore developable, and hence is a ruled surface. (2) The level curves of α are portions of straight lines in $\mathbb{R}^2$; hence the graph of α is horizontally ruled. (3) Along the (connected) components of α's (straight) level curves the gradient of r is constant and r itself is a linear function of (Euclidean) arc length, so every horizontal ruling on the graph of α projects vertically to a ruling on the graph of r. (4) The gradient curves of α (which are also the orthogonal trajectories of the level curves of α) all have constant curvature, hence are portions of straight lines or of circles. The third stage will utilize the information gathered in the second stage to split the problem into several cases and, in each case, to simplify the representations of α and r by the introduction of new variables in place of t and ρ. The fourth and final stage will employ the representations provided by the third stage to solve Equation (3.8) for r (or show that there is no solution) in each of the cases distinguished in the third stage, whereupon α can be calculated by an elementary integration.

The outcome of stages 2, 3, and 4 will be, for each case that yields a solution, and after isometric equivalences are factored out, a one-parameter family of maximally analytic solutions of the field equation equivalents (3.7), (3.8), and (3.9). In each case r^2 will be analytic on all of $\mathbb{R}^2$, α will be either analytic on all of $\mathbb{R}^2$ or analytic (in the multivalued sense) on $\mathbb{R}^2$ minus a single point, at which α will have an irremovable singularity, and $d\phi$ will vanish nowhere. These solutions, together with the Minkowskian solutions from stage 1, will exhaust the list of maximally analytic solutions. A solution not in either group cannot be maximally analytic, even if it is analytic and not analytically continuable. Such a solution either has $d\phi \equiv 0$ and thus is, if maximally analytic, a Minkowskian solution from stage 1, or else comes with a point P for which $d\phi(P) \neq 0$. In the latter event there is a neighborhood of P on which $d\phi$ vanishes nowhere, hence on which the solution agrees with a solution from stages 2, 3, and 4; if maximally analytic, it must *be* that solution. There do exist hybrid, nonmaximally analytic solutions, flat in some regions and curved in others, but their treatment will be deferred to a future date. Now let us begin.

Stage 1. If $d\phi = 0$, then $\alpha_t = \alpha_\rho = 0$, so Equations (3.7) imply that $r_{tt} = r_{t\rho} = r_{\rho t} = r_{\rho\rho} = 0$ except where r vanishes. It follows that r depends linearly on t and ρ, hence that, if r^2 is maximally analytic, there exist constants c, c', and d such that $r(t, \rho) = |ct + c'\rho + d|$. Equation (3.8) is satisfied by such an r only if $c'^2 = 1 + c^2$, so the family of solutions is given by

$$r(t, \rho) = |ct \pm (1 + c^2)^{1/2}\rho + d| \qquad (4.1)$$

and $\alpha(t, \rho) =$ constant. The line in the $t\rho$ plane along which r vanishes is timelike. A Lorentz boost in this plane through the angle $\sinh^{-1}(\pm c)$ can be followed by a translation of the origin to make this line the new time axis. The effect of such an isometry is more easily produced by taking $c = d = 0$ in Equation (4.1), with the result that $r(t, \rho) = |\rho|$. Thus each of the space-time manifolds $\mathfrak{M}$ measured by the line element (2.1) with r given by Equation (4.1) consists of two copies of Minkowski's flat space-time (when $c = d = 0$, one copy has all the points at which $\rho > 0$, the other has those at which $\rho < 0$). One may imagine these two Minkowski space-times to be joined along their world-lines on which $r(t, \rho) = 0$, but to do so is not forced by any previous supposition about coordinate ranges and the like.

Stage 2. Here $d\phi$ is presumed to vanish nowhere. The initiating insight is this: Equations (3.7) imply that $r_{tt}r_{\rho\rho} - r_{t\rho}r_{\rho t} = 0$; the geometrical significance of this equation is that the graph of r (as a surface in E^3) has Gaussian curvature

0, hence is developable; because it is developable, this surface is ruled;[7] Equations (3.7) obviously relate the rulings of the graph of r to the gradient curves of α. The job is to determine this relationship precisely.

Because the field equations lose their sense wherever r vanishes, it will be convenient to restrict attention for the present to the set $r^{-1}(\mathbb{R}^+)$, and to consider the domain of α to be this set. Also, it will suffice for now to assume that α is C^1-smooth and that r^2, hence r, is C^3-smooth on this set; from Equations (3.7) it then follows that α is C^2-smooth. Upon differentiating the first of Equations (3.7) with respect to ρ and the second with respect to t, and eliminating between them the equal terms $r_{tt\rho}/r$ and $r_{t\rho t}/r$, one obtains

$$\alpha_\rho \alpha_{tt} - \alpha_t \alpha_{t\rho} = \alpha_t(\alpha_t r_\rho - \alpha_\rho r_t)/r \tag{4.2}$$

The symmetric equation, derived from the third and the fourth of Equations (3.7), is

$$\alpha_t \alpha_{\rho\rho} - \alpha_\rho \alpha_{\rho t} = \alpha_\rho(\alpha_\rho r_t - \alpha_t r_\rho)/r \tag{4.3}$$

From these follows

$$\alpha_\rho^2 \alpha_{tt} - \alpha_\rho \alpha_t(\alpha_{t\rho} + \alpha_{\rho t}) + \alpha_t^2 \alpha_{\rho\rho} = 0 \tag{4.4}$$

Consider now a level curve of α, that is, a curve in $\mathbb{R}^2$ along which α is constant. Let $t(\lambda)\,\mathbf{i} + \rho(\lambda)\,\mathbf{j}$ be a parametrization of some component of it by a sensed Euclidean arc length parameter λ. Then the velocity $\mathbf{v}$ of this parametrization is everywhere of unit length and normal to $\nabla\alpha$, so

$$\begin{aligned} \mathbf{v} &\equiv \dot{t}\mathbf{i} + \dot{\rho}\mathbf{j} \\ &= [\alpha_\rho(t,\rho)\,\mathbf{i} - \alpha_t(t,\rho)\,\mathbf{j}]\,/(\alpha_t^2 + \alpha_\rho^2)^{1/2}(t,\rho) \end{aligned} \tag{4.5}$$

(or else $\mathbf{v}$ is the negative of this, in which event reparametrization by $-\lambda$ allows us to work with Equation (4.5) anyway). A straightforward computation gives as the acceleration

$$\begin{aligned} \dot{\mathbf{v}} &= -\frac{(\alpha_\rho \alpha_{tt} - \alpha_t \alpha_{\rho t})(t,\rho)\,\dot{t} - (\alpha_t \alpha_{\rho\rho} - \alpha_\rho \alpha_{t\rho})(t,\rho)\,\dot{\rho}}{(\alpha_t^2 + \alpha_\rho^2)^{3/2}(t,\rho)} \nabla\alpha(t,\rho) \\ &= -\frac{\alpha_\rho^2 \alpha_{tt} - \alpha_\rho \alpha_t(\alpha_{t\rho} + \alpha_{\rho t}) + \alpha_t^2 \alpha_{\rho\rho}}{(\alpha_t^2 + \alpha_\rho^2)^2}(t,\rho)\,\nabla\alpha(t,\rho) \\ &= \mathbf{0} \end{aligned} \tag{4.6}$$

[7]Discussions of developable surfaces and ruled surfaces can be found in [10] and [11]. To accommodate surfaces that are graphs of functions we must here admit, as developable, surfaces that are only portions of envelopes of one-parameter families of planes. A ruling on such a surface might be only a piece of a line. An example would be one nappe of a cone, which is ruled by half-lines; the envelope of its one-parameter family of tangent planes is the whole two-napped cone.

in light of Equation (4.4). Consequently, this (and every other) component of a level curve of α is a portion of a straight line in $\mathbb{R}^2$; the graph of α is ruled, therefore, by (perhaps unbounded) segments of horizontal straight lines, which are the components of its contour curves.

Next let us differentiate r_t and r_ρ along this same, straight, parametrized component of a level curve of α. We have that

$$\begin{aligned} [r_t(t,\rho)]^{\cdot} &= \nabla r_t(t,\rho) \cdot \mathbf{v} \\ &= [r_{tt}(t,\rho)\,\mathbf{i} + r_{t\rho}(t,\rho)\,\mathbf{j}] \cdot \mathbf{v} \\ &= (r\alpha_t)\,(t,\rho)[\nabla\alpha(t,\rho)\cdot\mathbf{v}] \\ &= 0 \end{aligned} \tag{4.7}$$

by application of Equations (3.7) and the orthogonality of $\mathbf{v}$ and $\nabla\alpha$. Similarly,

$$[r_\rho(t,\rho)]^{\cdot} = 0 \tag{4.8}$$

It follows that $[\nabla r(t,\rho)]^{\cdot} = \mathbf{0}$, hence that ∇r is constant along the component. This in turn, in conjunction with $\dot{\mathbf{v}} = \mathbf{0}$, entails that

$$[r(t,\rho)]^{\cdot\cdot} = [\nabla r(t,\rho)\cdot\mathbf{v}]^{\cdot} = 0 \tag{4.9}$$

Thus r varies linearly with distance along each component of a level curve of α. Geometrically, this means that every horizontal ruling on the graph of α is a vertical projection of a ruling on the graph of r.

The next step is to show that the gradient curves of α all have constant curvature. Toward this end let $\bar{t}(\bar{\lambda})\,\mathbf{i} + \bar{\rho}(\bar{\lambda})\,\mathbf{j}$ be a parametrization of a gradient curve of α by a sensed Euclidean arc length parameter $\bar{\lambda}$. Then, after a reparametrization by $-\bar{\lambda}$, if needed, we have

$$\begin{aligned} \bar{\mathbf{v}} \equiv \dot{\bar{t}}\,\mathbf{i} + \dot{\bar{\rho}}\,\mathbf{j} &= \nabla\alpha(\bar{t},\bar{\rho})/|\nabla\alpha(\bar{t},\bar{\rho})| \\ &= [\alpha_t(\bar{t},\bar{\rho})\,\mathbf{i} + \alpha_\rho(\bar{t},\bar{\rho})\,\mathbf{j}]\,/(\alpha_t^2 + \alpha_\rho^2)^{1/2}(\bar{t},\bar{\rho}) \end{aligned} \tag{4.10}$$

A calculation like that for Equation (4.6) shows that $\dot{\bar{\mathbf{v}}} = \mathrm{K}\mathbf{n}$, where $\mathbf{n}$ is the unit normal $[\alpha_\rho(\bar{t},\bar{\rho})\,\mathbf{i} - \alpha_t(\bar{t},\bar{\rho})\,\mathbf{j}]\,/(\alpha_t^2 + \alpha_\rho^2)^{1/2}(\bar{t},\bar{\rho})$, and K, whose absolute value is the curvature of the gradient curve, is given by

$$\begin{aligned} \mathrm{K} &= \frac{\alpha_t(\alpha_\rho\alpha_{tt} - \alpha_t\alpha_{\rho t}) - \alpha_\rho(\alpha_t\alpha_{\rho\rho} - \alpha_\rho\alpha_{t\rho})}{(\alpha_t^2 + \alpha_\rho^2)^{3/2}}(\bar{t},\bar{\rho}) \\ &= [(\alpha_t r_\rho - \alpha_\rho r_t)/r(\alpha_t^2 + \alpha_\rho^2)^{1/2}](\bar{t},\bar{\rho}) \end{aligned} \tag{4.11}$$

according to Equations (4.2) and (4.3) and because $\alpha_{t\rho} = \alpha_{\rho t}$.

If one now computes $\dot{\mathrm{K}}$ from the second expression in Equation (4.11) and applies Equations (3.7), (4.2), and (4.3), plus $\alpha_{t\rho} = \alpha_{\rho t}$, one learns that

$$\dot{\mathrm{K}} = [(\alpha_t r_\rho - \alpha_\rho r_t)^2/r^2(\alpha_t^2 + \alpha_\rho^2)^{3/2}](\bar{t},\bar{\rho})[\alpha_\rho(\bar{t},\bar{\rho})\,\dot{\bar{t}} - \alpha_t(\bar{t},\bar{\rho})\,\dot{\bar{\rho}}] \tag{4.12}$$

Substituting for $\dot{\bar{t}}$ and $\dot{\bar{\rho}}$ from Equation (4.10), one finds that $\dot{\mathrm{K}} = 0$, hence that the curvature $|\mathrm{K}|$ is constant. If $|\mathrm{K}| = 0$, then the gradient curve in question is straight; those components of level curves of α that intersect it do so orthogonally, hence are parallel to one another. If $|K| > 0$, then the gradient curve must be part of a circle. In this case the level curve components that meet it are portions of the radii of the circle; extended, they would all meet at the center of the circle. The gradient curves of α nearby, being perpendicular to the same radii, necessarily lie on the circles concentric with this one. A mixture of these two gradient curve configurations, the straight and the circular, can occur, but only under the hybrid solutions to be examined at a future time, not under the maximally analytic solutions presently sought.

Stage 3. The results of stage 2 make it possible to distinguish two situations: the "ladder case," in which components of level curves of α are parallel to one another, like rungs of a ladder, and the "wagon wheel case," in which components of level curves of α, if extended far enough, would all meet in a point, like spokes of a wheel. Additionally, the ladder case splits into three subcases, according to whether the parallel components (the "rungs") are timelike, lightlike, or spacelike in the $t\rho$ plane with the Minkowski metric $dt^2 - d\rho^2$ (by present convention $\partial/\partial t$, $\partial/\partial t \pm \partial/\partial\rho$, and $\partial/\partial\rho$ are timelike, lightlike, and spacelike, respectively). To get maximally analytic solutions it will suffice to suppose the level curves of α connected, hence to study the wagon wheel case with open half-lines for spokes, and the three ladder cases with whole lines for rungs. This is true also of solutions for which r^2 is only required to be maximally C^3-smooth, for C^3-smoothness of r^2 entails not only C^3-smoothness of r (on $r^{-1}(\mathbb{R}^+)$) and C^2-smoothness of α, but also analyticity of r^2 and of α, as will become apparent.

In the case of timelike rungs a Lorentz boost in the $t\rho$ plane can be carried out (leaving the content of the field equations intact) to make the time axis parallel to the level curves of α. The same effect is achieved by going back to the beginning and supposing α to be a function of ρ alone, independent of t; Equations (3.7) then imply that r, also, depends only on ρ.

When the rungs are lightlike, either the null line given by $t + \rho = 0$ or the null line given by $t - \rho = 0$ is parallel to every level curve of α. In the first instance α depends only on $t + \rho$, and, as Equations (3.7) imply, so does r. In the second instance the dependence is on $t - \rho$. Thus $r(t, \rho) = f(t + \rho)$ or $r(t, \rho) = f(t - \rho)$ for some C^3-smooth function f of one variable.

If the rungs are spacelike, there is a Lorentz boost in the $t\rho$ plane whose effect is equivalent to that of the supposition that α is a function of t alone. Then, much as before, one finds that $r(t, \rho) = g(t)$, g being C^3-smooth.

Turn now to the wagon wheel case. Without affecting the geometry of $\mathfrak{M}$, we can translate the origin of the $t\rho$ plane to the center of the wheel. Suppose that to have been done already, and introduce polar coordinates $[s, \xi]$ such that $[t, \rho] = [s \cos \xi, s \sin \xi]$. Then α, whose level curves are the spokes of the wheel, has the representation

$$\alpha(t, \rho) = \gamma(\xi) \tag{4.13}$$

γ being C^2-smooth; on the other hand r, as a linear function of Euclidean arc length along each spoke, has the representation

$$r(t, \rho) = sh(\xi) + k(\xi) \tag{4.14}$$

each of h and k being C^3-smooth.

In terms of the mutually orthogonal unit vectors $\hat{s}$ and $\hat{\boldsymbol{\xi}}$ associated with the polar coordinates s and ξ we have that (in abbreviated notation)

$$\begin{aligned} \nabla r &= r_s \hat{s} + (1/s)\, r_\xi \hat{\boldsymbol{\xi}} \\ &= h(\xi)\, \hat{s} + [h'(\xi) + (1/s)\, k'(\xi)]\, \hat{\boldsymbol{\xi}} \end{aligned} \tag{4.15}$$

Because ∇r is constant along each level curve of α, this expression must be independent of s; therefore, $k' = 0$.

A routine calculation shows that

$$\begin{aligned} r_{\xi\xi} &= s^2\, [(\sin \xi)^2 r_{tt} - (\sin \xi)\, (\cos \xi)\, (r_{t\rho} + r_{\rho t}) + (\cos \xi)^2 r_{\rho\rho}] - s r_s \\ &= r\alpha_\xi^2 - s r_s \end{aligned} \tag{4.16}$$

in which the second step uses Equations (3.7). Substituting for α and r from Equations (4.13) and (4.14), and remembering that $k' = 0$, we find that

$$sh''(\xi) = [sh(\xi) + k(\xi)]\, \gamma'^2(\xi) - sh(\xi) \tag{4.17}$$

Because this is an identity in s and ξ, and $\gamma'(\xi) \neq 0$ (a consequence of the supposition that $d\phi$ vanishes nowhere), it follows that $k(\xi) = 0$ and that

$$\gamma'^2(\xi) = (h'' + h)\, (\xi)/h(\xi) \tag{4.18}$$

wherever $h(\xi) \neq 0$.

To summarize in the wagon wheel case: $\alpha(t, \rho) = \gamma(\xi)$ and $r(t, \rho) = sh(\xi)$; γ is C^2-smooth and h is C^3-smooth; γ and h are related by Equation (4.18). No more information is to be had from Equations (3.7), so it is time to go on to the final stage.

Stage 4. In this stage the field equation (3.8) is to be applied. Let us continue with the wagon wheel case. Starting from $r^2(t, \rho) = s^2 h^2(\xi)$, one computes $(r^2)_{tt} - (r^2)_{\rho\rho}$ and then finds that Equation (3.8) is equivalent to

$$(\cos 2\xi)\, (h^2)''(\xi) + 2\,(\sin 2\xi)\, (h^2)'(\xi) = 2 \tag{4.19}$$

This equation is easily integrated to yield

$$h^2(\xi) = A + B \sin 2\xi - \tfrac{1}{2} \cos 2\xi \tag{4.20}$$

where each of A and B is a disposable constant. Equation (4.18) reduces to

$$\gamma'^2(\xi) = (A^2 - B^2 - \tfrac{1}{4})/h^4(\xi) \tag{4.21}$$

This equation implies that $A^2 - B^2 > \frac{1}{4}$, which in turn implies that $|A| > \frac{1}{2}$ and $|B/A| < 1$.

From Equation (4.20) it follows that

$$\begin{aligned} r^2(t, \rho) &= s^2 h^2(\xi) \\ &= As^2 + 2B(s \cos \xi)(s \sin \xi) - \tfrac{1}{2}[(s \cos \xi)^2 - (s \sin \xi)^2] \\ &= (A - \tfrac{1}{2})t^2 + 2Bt\rho + (A + \tfrac{1}{2})\rho^2 \end{aligned} \tag{4.22}$$

The discriminant $4B^2 - 4A^2 + 1$ of this quadratic form in t and ρ is negative, so the form is definite; to have it be positive definite, it is necessary and sufficient to take $A > \frac{1}{2}$. Suppose this is done. Then a Lorentz boost in the $t\rho$ plane through the angle $\frac{1}{2}\tanh^{-1}(B/A)$ will serve to reduce the expression for $r^2(t, \rho)$ to a sum of squares. Knowing this, one can take a short cut by setting $B = 0$ in Equation (4.22); the result is that

$$r^2(t, \rho) = a^2 t^2 + (1 + a^2)\rho^2 \tag{4.23}$$

where $a \equiv (A - \frac{1}{2})^{1/2} > 0$. Setting $B = 0$ in Equation (4.21) and integrating it gives $\gamma(\xi) = \gamma_0 \pm \tan^{-1}[(b/a)\tan \xi]$; hence

$$\alpha(t, \rho) = \gamma_0 \pm \tan^{-1}(b\rho/at) \tag{4.24}$$

where $b \equiv (1 + a^2)^{1/2}$ and γ_0 is an integration constant. The graphs of the solutions r and α thus arrived at are, respectively, a right elliptical cone and a helicoid of variable pitch; they are depicted in Figure 2. Whereas r^2 is analytic on all of $\mathbb{R}^2$, r itself fails to be differentiable at the origin. Making allowance for the multivaluedness of $\tan^{-1}$, one finds that r and α satisfy Equations (3.7), (3.8), and (3.9) at all points of the $t\rho$ plane except the origin.

Let us return now to the ladder case, taking up first the ladder with lightlike rungs. But having picked it up we can immediately lay it down, for when $r(t, \rho) = f(t + \rho)$ or $r(t, \rho) = f(t - \rho)$, and f is C^3-smooth, the field equation $(r^2)_{tt} - (r^2)_{\rho\rho} = -2$ reduces to $0 = -2$; thus there is no ladder solution with lightlike rungs.

When the rungs are timelike, then, as noted in stage 3, both α and r may be taken to depend on ρ alone. The solution in this case has already been found and thoroughly examined [8] (see also [12]); the formula for r^2 is

$$r^2(t, \rho) = \rho^2 + n^2 \tag{4.25}$$

where n is a positive constant. The space–time manifold that goes with the line element (2.1) for this r^2 is geodesically complete and represents a static, flowless drainhole, which neither evolves nor gravitates.

If the rungs are spacelike, what then? As we have seen, we may suppose that $r(t, \rho) = g(t)$, g being C^3-smooth, and that α also depends only on t. Equation

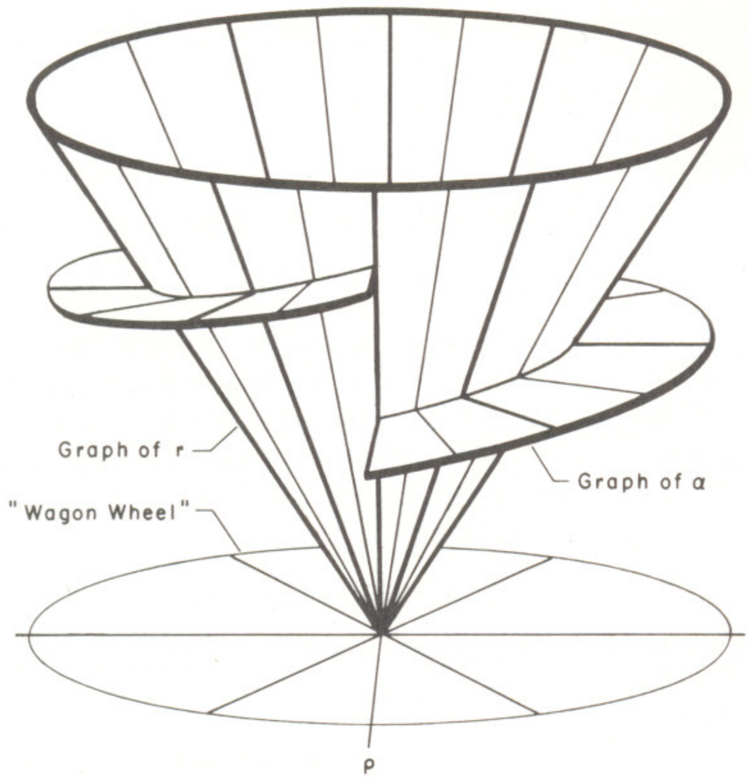

Fig. 2. The graphs of r and α for the solution of the field equations in the wagon wheel case; here $r^2(t, \rho) = a^2t^2 + b^2\rho^2$ and $\alpha(t, \rho) = \gamma_0 + \tan^{-1}(b\rho/at)$, with $b = (1 + a^2)^{1/2}$. The graph of α is a helicoid of variable pitch and that of r is a right elliptical cone. Each "spoke" of the "wagon wheel" in the $t\rho$ plane lies directly under a generator of the helicoid and a generator of the cone.

(3.8) reduces to $(g^2)'' = -2$, which yields $r^2(t, \rho) = -t^2 + C$ (the disposable integration constant other than C has been eliminated by an immaterial adjustment of the time origin). The first of Equations (3.7) now becomes $-C/r^4 = \alpha_t^2$; because α_t does not vanish ($d\phi = \alpha_t\, dt$, and $d\phi$ by hypothesis does not vanish), it follows that $C < 0$, hence that

$$r^2(t, \rho) = -(t^2 + n^2) \tag{4.26}$$

where $n \equiv (-C)^{1/2}$.

At first sight the presence of the minus sign in this formula for $r^2(t, \rho)$ seems inconsistent. But it is not, for only by notational fiat has it been made to appear that the last two diagonal coefficients in the line element (2.1) are definitely negative—they could as easily be positive and the field equations would not care. Let us then substitute the r^2 we have into Equation (2.1). The resulting line element is $dt^2 - d\rho^2 + (t^2 + n^2)\, d\Omega^2$. Despite appearances there is nothing new here. The manifold that this line element measures represents a static, flowless drainhole: switch the coordinate names t and ρ, make an overall change of signs (neither action alters geometry), and the result is the line element (2.1) with r^2 given by Equation (4.25) instead of Equation (4.26). The "new" turns out to be old. Nor should this come as a surprise, for the formal invariance of the line element (2.1) under interchange of t and ρ and an overall change of signs is manifest, whatever the choice of $r^2(t, \rho)$. For example, under this symmetry operation the metric of the wagon wheel solution, which is given by

$$d\tau^2 = dt^2 - d\rho^2 - [a^2 t^2 + (1 + a^2)\rho^2]\, d\Omega^2 \tag{4.27}$$

goes over into $dt^2 - d\rho^2 + [(1 + a^2)t^2 + a^2\rho^2]\, d\Omega^2$. This is just the solution one could obtain by taking the quadratic form for $r^2(t, \rho)$ in Equation (4.22) to be negative definite rather than positive and letting $a \equiv (-A - \frac{1}{2})^{1/2}$.

The maximally analytic solution manifolds sought have now all been found. They consist of the doubled Minkowski manifolds from stage 1, together with all space-time manifolds isometric (for some value of n) to the static, flowless drainhole of the ladder case or isometric (for some value of a) to the evolving, flowless drainhole of the wagon wheel case. There remains here only to list some of the more salient properties of the evolving, flowless drainhole. Let it be said in passing, however, that the partial interchangeability exhibited by these solutions of the roles of the temporal and the radial coordinates suggests an answer to the age-old question, What is time? That answer and some of its consequences appear in [13].

§(5): *Properties of the Evolving, Flowless Drainhole*

The metric of the evolving, flowless drainhole in the form (4.27) is degenerate where $t = \rho = 0$. The two-dimensional cross section of $\mathfrak{M}$ on which this occurs is metrically a two-sphere of radius 0 and can therefore be treated as a single point. At all other points of $\mathfrak{M}$ the metric is nondegenerate and the curvature tensor field is regular, but this one event is a point of curvature singularity, as is evidenced, for example, by the trace of the Ricci tensor field, which is $2a^2b^2(\rho^2 - t^2)(a^2t^2 + b^2\rho^2)^{-2}$, with $b^2 = 1 + a^2$.

There are geodesics that arrive at the singular event without having exhausted their affine parameters. Although the geodesic equation breaks down at this event, each such geodesic can be continued beyond it—not uniquely, but in uncountably many ways.[8] The proverbial (pointlike) bug, approaching geodesically, from above, the collapsing central hole on the time-dependent catenoid of Figure 1, passes uneventfully through to the lower sheet on its predestined

[8] A simple surface that exhibits a similar phenomenon is the graph in E^3 of the equation $z = (xy)^{1/3}$. This surface is smooth at each of its points except the origin. Most geodesics that arrive there have uncountably many extensions.

course, unless it arrives at the hole at the instant the hole pinches off. For arriving at just that moment it will be rewarded with an uncountable selection of paths from which to choose a way into the future; some of these will take it to the lower sheet, others will leave it in the upper sheet, and others will keep it forever in the hole, stuck between the sheets, so to speak.

The evolving, flowless drainhole manifold $\mathfrak{M}$ is thus geodesically complete, in that every geodesic that approaches the singular event has such extensions, each of which will serve to complete it, whereas every geodesic that misses the singular event is complete in itself. The latter conclusion is a ready consequence of the geodesic equation. It is also suggested by the fact that $\mathfrak{M}$ is asymptotically flat in every direction of recession from the singular event, whether the path of recession be spatial, luminal, temporal, or a mixture of these. Specifically, with respect to the orthonormal frame system $\{e_\mu\}$ defined in Section 3, every component of the curvature tensor field Θ tends to 0 as $r^2(t, \rho) \to \infty$, hence as $t^2 + \rho^2 \to \infty$. This implies that every scalar invariant that is polynomial in the components of Θ and algebraic in the metric components tends asymptotically to 0, and that, for every distribution of two-planes that asymptotically do not approach the light cones too rapidly, the corresponding field of sectional curvatures tends to 0, as $t^2 + \rho^2 \to \infty$.

Besides the usual isometries associated with spherical symmetry and time reversal, there is an isometry of reflection through the drainhole, the point with coordinates $[t, \rho, \vartheta, \varphi]$ exchanging places with the point whose coordinates are $[t, -\rho, \vartheta, \varphi]$. In addition the evolving, flowless drainhole has some conformal self-mappings that are not isometries. The more obvious of these are the dilatations in which, formally, $[t', \rho', \vartheta', \varphi'] = [kt, k\rho, \vartheta, \varphi]$ and $d\tau'^2 = k^2\, d\tau^2$, where k is a positive number (distinct from 1). Less obvious are the inversions in "spheres" (hyperboloids, really) represented by $[t', \rho', \vartheta', \varphi'] = [kt\,|t^2 - \rho^2|^{-2}, k\rho\,|t^2 - \rho^2|^{-2}, \vartheta, \varphi]$, for which $d\tau'^2 = k^2(t^2 - \rho^2)^{-2}\, d\tau^2$, where again k is a positive number. A full treatment of these inversions requires that $\mathfrak{M}$ be augmented with points at infinity, but that is a story for another time.

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