Problem 5. Let $\{\mathcal{M}, G\}$ be a doubly smooth metric manifold whose maximal atlas has in it a coordinate system $\Pi = [\![\rho, \vartheta, \varphi]\!]$ with respect to which G has the representation

$$G = d\rho \otimes d\rho + r^2(\rho)(d\vartheta \otimes d\vartheta) + r^2(\rho)(\sin\vartheta)^2(d\varphi \otimes d\varphi),$$

where r is a C^2 real function whose domain is an open interval. Take for the range of Π the set $\{[\![\rho, \vartheta, \varphi]\!] \mid \rho \in \operatorname{dom} r,\ 0 < \vartheta < \pi, \text{ and } -\pi < \varphi < \pi\}$. Let $\mathbf{d}$ be the torsion free covariant differentiation that is compatible with G. Let E be the frame system whose dual coframe system Ω is given by

$$\omega^1 = d\rho, \quad \omega^2 = r(\rho)\, d\vartheta, \quad \omega^3 = r(\rho)\,(\sin\vartheta)\, d\varphi.$$

a. Compute the matrix $[\omega_k{}^m]$ of 1-forms of $\mathbf{d}$ in E.

b. Compute the matrix $[\Theta_k{}^m]$ of curvature 2-forms of $\mathbf{d}$ in E.

c. Compute the representation in E of the curvature tensor field Θ of $\mathbf{d}$.

d. Compute the representation in E of the contracted curvature tensor field Φ of $\mathbf{d}$.

e. Compute the curvature scalar field Ψ of $\mathbf{d}$.

f. For $(m, n) = (1, 2)$, $(1, 3)$, and $(2, 3)$, compute the sectional curvatures $\kappa(m, n)$ of G at a generic point of $\operatorname{dom}\Pi$ from the formula

$$\kappa(m, n) := \frac{(G\Theta)e_m e_n e_m e_n}{\langle e_m, e_m\rangle_G \langle e_n, e_n\rangle_G - \langle e_m, e_n\rangle_G^2}.$$

Problem 6. Specialize the results of Problem 5 to the case where $r(\rho) = a\rho$.

Problem 7. Specialize the results of Problem 5 to the case where $r(\rho) = \sqrt{\rho^2 + a^2}$.

Problem 8. Specialize the results of Problem 5 to the case where $r(\rho) = a\sin(\rho/a)$. Can you identify the manifold $\mathcal{M}$ in this case?