Problem 4. Let $X = [\![x, y, z]\!]$, a rectangular cartesian coordinate system for $\mathbb{E}^3$. Let $\mathcal{E}$ be the ellipsoid in $\mathbb{E}^3$ specified by the equation $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$. Let H be the metric induced on $\mathcal{E}$ by the euclidean metric of $\mathbb{E}^3$. Let $\mathbf{d}$ be the torsion free covariant differentiation that is compatible with H. Let $\Delta = [\![\vartheta, \varphi]\!]$, a coordinate system for $\mathcal{E}$ related to X by

$$
\begin{aligned}
x &= a\,(\sin\vartheta)(\cos\varphi),\\
y &= b\,(\sin\vartheta)(\sin\varphi),\\
z &= c\,(\cos\vartheta),
\end{aligned}
$$

with $\operatorname{ran}\Delta = \{[\![\vartheta, \varphi]\!] \mid 0 < \vartheta < \pi \text{ and } -\pi < \varphi < \pi\}$.

a. Compute the representation of H in the coordinate system Δ.

b. Let $b = a$. The ellipsoid $\mathcal{E}$ is now an ellipsoid of revolution, called also a **spheroid**, **oblate** if $a > c$, **prolate** if $a < c$. The frame system E defined by $e_1 := \partial_\vartheta / |\partial_\vartheta|_H$ and $e_2 := \partial_\varphi / |\partial_\varphi|$ is orthonormal. Compute the matrix $[\omega_k{}^m]$ of 1-forms of $\mathbf{d}$ in E. Keep $b = a$ in what follows.

c. Compute the matrix $[\Theta_k{}^m]$ of curvature 2-forms of $\mathbf{d}$ in E.

d. Compute the representation in E of the curvature tensor field Θ of $\mathbf{d}$.

e. Compute the representation in E of the contracted curvature tensor field Φ of $\mathbf{d}$.

f. Compute the curvature scalar field Ψ of $\mathbf{d}$.

g. Compute the gaussian curvature κ of $\mathcal{E}$ at a generic point of $\operatorname{dom}\Delta$ by computing

$$
\frac{(H\Theta)uvuv}{\langle u, u\rangle_H \langle v, v\rangle_H - \langle u, v\rangle_H^2}
$$

for a convenient choice of the tangent vectors u and v.