Problem 3. Let $X = [\![x, y, z]\!]$, a rectangular cartesian coordinate system for $\mathbb{E}^3$. Let $\Pi = [\![\rho, \vartheta, \varphi]\!]$, a spherical polar coordinate system for $\mathbb{E}^3$ related to X by

$$
\begin{aligned}
x &= \rho\,(\sin\vartheta)(\cos\varphi), & & & \rho &= \sqrt{x^2+y^2+z^2}, \\
y &= \rho\,(\sin\vartheta)(\sin\varphi), & &\text{and} & \cos\vartheta &= \frac{z}{\sqrt{x^2+y^2+z^2}}, \\
z &= \rho\,(\cos\vartheta), & & & \cos\varphi &= \frac{x}{\sqrt{x^2+y^2}} \quad \text{and} \quad \sin\varphi = \frac{y}{\sqrt{x^2+y^2}},
\end{aligned}
$$

with $\operatorname{ran}\Pi = \{[\![\rho, \vartheta, \varphi]\!] \mid 0 < \rho,\ 0 < \vartheta < \pi,\ \text{and} \ -\pi < \varphi < \pi\}$ and $\operatorname{dom}\Pi = \mathbb{E}^3 - \{P \mid y(P) = 0 \text{ and } x(P) \le 0\}$. One can show that Π and X are C^1-compatible (in fact, analytically compatible). Let $\mathcal{M}$ be the C^1 manifold for which a minimal C^1 atlas is $\{X\}$, and whose maximal C^1 atlas therefore has Π as well as X in it. Let $E = \{e_m\} := \{\partial_\rho, (1/\rho)\partial_\vartheta, (1/\rho\sin\vartheta)\partial_\varphi\}$, and $\Omega = \{\omega^m\} := \{d\rho, \rho\, d\vartheta, (\rho\sin\vartheta)\, d\varphi\}$, the coframe system dual to the frame system E.

Let

$$
\overset{k\ m\longrightarrow}{\downarrow}[\omega_k{}^m] := \begin{bmatrix} 0 & \frac{1}{\rho}\omega^2 & \frac{1}{\rho}\omega^3 \\ -\frac{1}{\rho}\omega^2 & 0 & \frac{\operatorname{ctn}\vartheta}{\rho}\omega^3 \\ -\frac{1}{\rho}\omega^3 & -\frac{\operatorname{ctn}\vartheta}{\rho}\omega^3 & 0 \end{bmatrix},
$$

and let $\mathbf{d}$ be the covariant differentiation whose 2-forms in the frame system E are the $\omega_k{}^m$, so that $\mathbf{d}e_k = \omega_k{}^m \otimes e_m$ and $\mathbf{d}\omega^m = -\omega_k{}^m \otimes \omega^k$.

Let G be the euclidean metric on $\mathcal{M}$. The representation of G in the coordinate system X is $G = dx \otimes dx + dy \otimes dy + dz \otimes dz$.

Let ϕ be a C^2 scalar field of $\mathcal{M}$. Let $u = u^m e_m$, a smooth vector field of $\mathcal{M}$.

a. Show that $G = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3$.

b. Show that $\mathbf{d}G = 0$.

c. Show that $\mathbf{T} = 0$.

d. Let $\mathbf{Div}\ u := \operatorname{Tr}\mathbf{d}u := (\mathbf{d}u)\,{}^2_1$. Compute $\mathbf{Div}\ u$, in terms of Π, E, and Ω.

e. Let $\mathbf{Grad}\ \phi := G^{-1}d\phi$. Compute $\mathbf{Grad}\ \phi$, in terms of Π, E, and Ω.

f. Let $\mathbf{Curl}\ u := G^{-1}(*\mathbf{d}_\wedge(Gu))$, where the 'duality operator (field)' $*$ is defined by $*(\omega^2 \wedge \omega^3) = \omega^1$, $*(\omega^3 \wedge \omega^1) = \omega^2$, and $*(\omega^1 \wedge \omega^2) = \omega^3$, extended linearly to the other 2-forms of $\mathcal{M}$. Compute $\mathbf{Curl}\ u$, in terms of Π, E, and Ω.

g. Let $\boldsymbol{\nabla}^2\phi := \mathbf{Div}(\mathbf{Grad}\ \phi)$. Compute $\boldsymbol{\nabla}^2\phi$, in terms of Π, E, and Ω.