The general $\beta > 0$ Soft Edge

Based on:

“Beta ensembles, stochastic Airy spectrum, and a diffusion,”
Ramírez, J., Rider, B., Virág, B. (*arXiv:math.PR/0607331*) (2007).

Also see Chapter 4 (section 4.5) of “An Introduction to Random Matrices” by Anderson, Guionnet, and Zeitouni.

A re-cap of the goal

Let

$$\lambda_1(\beta) > \lambda_2(\beta) > \cdots$$

be the ordered points of the β-Hermite ensemble

$$P_\beta(\lambda_1, \lambda_2, \ldots, \lambda_n) = \frac{1}{Z_{n,\beta}} e^{-\beta/4 \sum_{k=1}^{n} \lambda_k^2} \times \prod_{j<k} |\lambda_j - \lambda_k|^\beta.$$

Then, for any fixed k as $n \uparrow \infty$,

$$\left\{ n^{1/6}(\sqrt{2n} - \lambda_k(\beta)) \right\}_{\ell=1,\ldots,k} \Rightarrow \left\{ \Lambda_0(\mathcal{H}_\beta), \Lambda_1(\mathcal{H}_\beta), \cdots, \Lambda_{k-1}(\mathcal{H}_\beta) \right\}$$

where $\mathcal{H}_\beta$ is the Stochastic Airy Operator,

$$\mathcal{H}_\beta = -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} b'(x),$$

understood as acting on the positive half-line with Dirichlet conditions at the origin.

Making sense of $\mathcal{H}_\beta$

Let D be the usual Schwartz distributions.

Let also H^1_{loc} be the space of functions $f : \mathbb{R}^+ \to \mathbb{R}$ for which $f'\mathbf{1}_I \in L^2$ for any compact set I.

$\mathcal{H}_\beta$ is then well defined as a random linear map $H^1_{\text{loc}} \to D$, sending f to the distribution

$$\mathcal{H}_\beta f = -f'' + xf + \frac{2}{\sqrt{\beta}} fb'.$$

As D is only closed under multiplication by smooth functions, one must make sense fb' as an element of that space. Stieltjes integration by parts prompts

$$\int_0^y fb'\,dx := -\int_0^y bf'\,dx + f(y)b_y - f(0)b_0.$$

We define fb' as the derivative of this a.s. continuous function.

Weak formulation

Said differently, $(\lambda, f) \in (\mathbb{R}, H^1_{\mathrm{loc}} \cap L^2)$ is an eigenvalue/eigenfunction pair for $\mathcal{H}_\beta$ if, for all $\phi \in C_0^\infty$,

$$\lambda \int_0^\infty \phi(x) f(x) dx = \int_0^\infty [\phi''(x) f(x) - x\phi(x) f(x)] dx$$
$$-\frac{2}{\sqrt{\beta}} \Big[\int_0^\infty \phi'(x) f(x) b_x dx + \int_0^\infty \phi(x) b_x f'(x) dx \Big].$$

Of course, f is also subject to the normalizations $f(0) = 0$ and $\int_0^\infty f^2(x) dx = 1$.

The natural space for eigenfunctions turns out to be

$$L^* = \Big\{ f : f(0) = 0, \int_0^\infty [f'(x)^2 + x f^2(x)]\, dx < \infty \Big\},$$

as may be seen upon considering the (equivalent) variational form of the eigenvalue problem, which we do next.

Raleigh-Ritz

One would like to make sense of the inner product

$$< f, \mathcal{H}_\beta f >= \int_0^\infty [f'(x)^2 + x f^2(x)]\, dx + \frac{2}{\sqrt{\beta}} \int_0^\infty f^2(x) db_x,$$

and define $\Lambda_0(\mathcal{H}_\beta)$ as the appropriate infimum over (some subspace) of L^2.

The problem is of course that $\frac{2}{\sqrt{\beta}} \int_0^\infty f^2(x) db_x$ is not well behaved on its own for $f \in L^2$.

The only hope is to control this random potential in terms of the positive part of the potential: with $\mathcal{A} = -\frac{d^2}{dx^2} + x$ the usual Airy operator,

$$< f, \mathcal{A} f >= \int_0^\infty [f'(x)^2 + x f^2(x)]\, dx.$$

That is to say, in terms of the L^* norm.

The integration-by-parts trick

For say $f \in C_0^\infty$, $\int_0^\infty f(x)^2 db_x = 2\int_0^\infty f'(x)f(x)b_x dx$ is nothing mysterious.

However, since $|b_x| > \sqrt{x \log\log x}$ infinitely often, this is not enough when taking infimums in $< f, \mathcal{H}_\beta f >$.

Instead write,

$$b_x = \int_x^{x+1} b_y dy - \int_x^{x+1} (b_y - b_x) dy$$

and notice, for similar f's,

$$\int_0^\infty f(x)^2 db_x = \int_0^\infty f(x)^2 (b_{x+1} - b_x) dx + 2\int_0^\infty f(x)f'(x)\left[\int_x^{x+1} (b_y - b_x) dy\right] dx.$$

The point being

$$\sup_{x>0} \frac{|b_{x+1} - b_x|}{\log(2+x)} < \infty \quad \text{with probability one.}$$

Homework problem: Prove that.

Lower bound on $< f, \mathcal{H}_\beta f >$

The above implies: for an a.s. finite constant $C(b)$,

$$\frac{2}{\sqrt{\beta}} \int_0^\infty f^2(x) db_x \le \frac{1}{2} \int_0^\infty [f'(x)^2 + x f^2(x)]\, dx + C(b) ||f||_2^2$$

from which it follows that

$$\Lambda_0 := \inf_{f \in L^*, ||f||_2 = 1} \left\{ \int_0^\infty [f'(x)^2 + x f^2(x)]\, dx + \frac{2}{\sqrt{\beta}} \int_0^\infty f^2(x) db_x \right\}$$

not only exists, but is achieved at some $f_0 \in L^*$ and (Λ_0, f_0) is an eigenvalue/eigenvector pair for $\mathcal{H}_\beta$.

Homework problem: Check all that. Notice that the a.s. bound below provides control of any minimizing sequence in H^1, as well as control of the moment $\int_0^\infty x f^2(x) dx$.

Starting the convergence proof

Recalling the matrix model,

$$H_n^\beta = \frac{1}{\sqrt{\beta}} \begin{bmatrix} \sqrt{2}g_1 & \chi_{\beta(n-1)} & & & \\ \chi_{\beta(n-1)} & \sqrt{2}g_2 & \chi_{\beta(n-2)} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{\beta 2} & \sqrt{2}g_{n-1} & \chi_\beta \\ & & & \chi_\beta & \sqrt{2}g_n \end{bmatrix}$$

we write: for $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$,

$$\begin{aligned} \langle v, n^{1/6}\Big(2\sqrt{n}I - H_n^\beta\Big)v\rangle \quad &= \quad n^{2/3}\sum_{k=0}^{n}(v_{k+1} - v_k)^2 + 2n^{1/6}\sum_{k=1}^{n}\gamma_k v_k v_{k+1} \\ &\quad + n^{1/6}\sum_{k=1}^{n}\zeta_k v_k^2 + 2n^{1/6}\sum_{k=1}^{n}\xi_k v_k v_{k+1} \end{aligned}$$

with the understanding that $v_0 = v_{n+1} = 0$. Here

$$\gamma_k = \sqrt{n} - \frac{1}{\sqrt{\beta}}\mathbb{E}\chi_{\beta(n-k)}, \ \zeta_k = -\sqrt{\frac{2}{\beta}}g_k, \ \xi_k = \frac{1}{\sqrt{\beta}}(\mathbb{E}\chi_{\beta(n-k)} - \chi_{\beta(n-k)}).$$

Embedding into $C \cap L^2$

Again, the spacial scaling is identified as $\triangle x = n^{-1/3}$, and so would like to view $v_k = n^{-1/6} f(kn^{-1/3})$ for some nice $f \in L^2$.

With $\tilde{H}_n^\beta = n^{1/6}(2\sqrt{n}I - H_n^\beta)$, the form $\langle \cdot, \tilde{H}_\beta \cdot \rangle$ can be made to live on L^2 by:

i. Projecting onto the finite dimensional subspace consisting of continuous polygonal paths of step size $n^{-1/3}$ which are identically zero for $x \geq n^{2/3}$.

ii. Sample the function produced at the $\{kn^{-1/3}\}$ nodes.

That is, with P_n the projection operator, and S the sampler,

$$\langle \cdot, \tilde{H}_n^\beta \cdot \rangle = \langle SP_n \cdot, \tilde{H}_n^\beta SP_n \cdot \rangle,$$

the former acting on ℓ^2, the latter acting on L^2. In particular they produce the same minimizers.

Really, I've done nothing here but change $v_k \mapsto v_k n^{-1/6}$ throughout.

Rewriting...

The form is

$$\begin{aligned}\langle v, \tilde{H}_n^\beta v\rangle &= n^{1/3}\sum_{k=0}^{n}(v_{k+1}-v_k)^2 + 2n^{-1/6}\sum_{k=1}^{n}\gamma_k v_k v_{k+1} \\ &\quad + n^{-1/6}\sum_{k=1}^{n}\zeta_k v_k^2 + 2n^{-1/6}\sum_{k=1}^{n}\xi_k v_k v_{k+1},\end{aligned}$$

keeping the identification $v_{[n^{1/3}x]} = f_v(x)$ in the back of one's mind.

We introduce

$$\begin{aligned} w_k^{(1)} &= n^{-1/6}\sum_{i=1}^{k}\zeta_i = n^{-1/6}\sum_{i=1}^{k}\sqrt{\frac{2}{\beta}}g_i \\ w_k^{(2)} &= 2n^{-1/6}\sum_{i=1}^{k}\xi_i = n^{-1/6}\sum_{i=1}^{k}\frac{2}{\sqrt{\beta}}(\mathbb{E}\chi_{\beta(n-i)} - \chi_{\beta(n-i)}) \\ \eta_k &= 2n^{-1/6}\left(\sqrt{n} - \frac{1}{\sqrt{\beta}}\mathbb{E}\chi_{\beta(n-k)}\right)\end{aligned}$$

Behind the discrete integration by parts

There is a probability space on which: for $j = 1, 2$ and $B_j(x)$ two independent Brownian Motions,

$$w^{(j)}_{[n^{1/3}x]} \Rightarrow \sqrt{\frac{2}{\beta}} B_j(x)$$

with probability one. (This is with respect to the usual Skorohod topology on paths.)

On that same space, there is a sequence of tight random variables c_n so that

$$\sup_{k \le i \le k+n^{1/3}} |w^{(j)}_k - w^{(j)}_i|^2 \le c_n \Big(1 + \phi(kn^{-1/3}) \Big)$$

for a $\phi(x) = o(x)$ at infinity.

This is to say that the "integrated" noise does the right thing and satisfies the same range bound as the continuum noise.

Also, we easily have a (nonrandom) constant C with

$$\frac{1}{C} kn^{-1/3} - C \le \eta_k \le Ckn^{-1/3} + C, \quad \text{and} \quad \sum_{k=1}^{[n^{1/3}x]} \eta_k \to x^2/2.$$

Tightness

Bring in the "discrete L^* norm

$$||v||_*^2 = n^{1/3} \sum_{k=0}^{n} (v_{k+1} - v_k)^2 + n^{-2/3} \sum_{k=1}^{n} k v_k^2 + n^{-1/3} \sum_{k=1}^{n} v_k^2.$$

The upshot of the preceding discussion is that: there exist tight random constants $c_i = c_i(n), i = 1, 2, 3$ such that

$$c_1 ||v||_*^2 - c_2 ||v||_2^2 \leq \langle v, \tilde{H}_\beta^n v \rangle \leq c_3 ||v||_*^2.$$

This is an exercise in summation by parts.

And now you're off to the races, basically.

Any sequence $n \to \infty$ contains a further subsequence along which we can take $c_1, c_2,$ and c_3 as fixed, or independent on n. Along this subsequence you have a.s. uniform control of the n-level eigenvalue sequence as well as uniform control in $|| \cdot ||_*$-norm of the corresponding sequence of eigenvectors/eigenfunctions.

Finishing up

Let (λ_n, v_n) denote the minimal eigenvalue/eigenvector pair for $\tilde{H}_n^\beta$. Take a subsequence along which (a.s.)

$$\lambda_n' \to \liminf \lambda_n := \lambda \quad \text{and} \quad ||v_{n'}||_* \le C.$$

This control is enough to show that: in L^2, $v_{n'} \to f$ for some $f \in L^*$ (after possibly a further choice of susequence), and

$$\lambda\langle \phi, f\rangle = \lim_{n'\to\infty} \langle \phi, \lambda_{n'} v_{n'}\rangle = \lim_{n'\to\infty} \langle \phi, \tilde{H}_n^\beta v_{n'}\rangle = \langle \phi, \mathcal{H}_\beta f\rangle.$$

for any $\phi \in C_0^\infty$.

This shows (λ, f) is an eigenvalue/eigenfunction for $\mathcal{H}_\beta$, but why is it not the case that $\lambda > \Lambda_0(\mathcal{H}_\beta)$. Trick is...

You know that $\mathcal{H}_\beta$ has ground state $f_0 \in L^*$. Feed a discretized version of this into the $\tilde{H}_\beta^n$ form:

$$\limsup_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \frac{\langle f_0^n, \tilde{H}_\beta^n f_0^n\rangle}{\langle f_0^n, f_0^n\rangle} \to \langle f_0, \mathcal{H}_\beta f_0\rangle.$$

Final equality uses ideas from above.

Homework Problem: How would the argument work for the 2^{nd}, 3^{rd} etc. eigenvalues?

What just happened - Universality

Let H be the form

$$H = -\frac{d^2}{dx^2} + y'(x) \text{ for } y(x) = \int^x \kappa(x') + \int^x \sigma(x')db(x')$$

with $\kappa(x) \uparrow \infty$ sufficiently fast as $x \uparrow \infty$ and $||\sigma||_\infty < \infty$.

Assume that for a given family

$$H_n = -\triangle_n + \text{tridiag}\Big(b_n(\omega),\, a_n(\omega),\, b_n(\omega)\Big),$$

in which $\triangle_n$ is a second-difference on the space-scale δ_n.

Gist of the Theorem: *If, along with a certain technical tightness conditions, the interpolated process satisfies*

$$\sum_{k=1}^{[x/\delta_n]} (a_{n,k} + 2b_{n,k}) \Rightarrow y(x)$$

(locally uniform convergence in law). Then the bottom k *eigenvalues of* H_n *tend to those of* H *in distribution.*

Application: Shape Theorem

With $TW_\beta = -\Lambda_0(\beta)$ well defined by

$$\Lambda_0 = \inf_{f \in L^*} \left\{ \int_0^\infty [f'^2(x) + x f^2(x)] dx + \frac{2}{\sqrt{\beta}} \int_0^\infty f^2(x) db(x) \right\}$$

for $L^* = \{||f||_2 = 1\} \cap \{\int_0^\infty [(f')^2 + x f^2] < \infty\}$, along with the Riccati correspondence we can show:

Theorem *It holds*

$$P\Big(TW_\beta \geq a\Big) = \exp\Big(-\frac{2}{3}\beta\, a^{3/2}(1 + o(1))\Big)$$

and

$$P\Big(TW_\beta \leq -a\Big) = \exp\Big(-\frac{1}{24}\beta\, |a|^3(1 + o(1))\Big).$$

for all large enough $a > 0$.

Cheap proof of the right tail, lower bound

Plainly,

$$P(TW_\beta > a) = P(\Lambda_0(\beta) < -a) \geq P\Big(\langle f, \mathcal{H}_\beta f\rangle < -a\langle f, f\rangle\Big)$$

for any admissible f.

Choose f to optimize the probability of:

$$\frac{2}{\sqrt{\beta}}\left(\int_0^\infty f^4(x)dx\right)^{1/2} \times \mathcal{N} < -a\int_0^\infty f^2 - \int_0^\infty [f'^2(x) + xf^2(x)]dx$$

with a standard normal $\mathcal{N}$.

Intuition is that for a small eigenvalue the potential and so the eigenfunction f should be localized.

So drop the $\int_0^\infty xf^2$ term, do the above optimization, and find

$$f_a(x) = \text{sech}(\sqrt{a}x)$$

shifted and modified at the origin. Substitution yields the anticipated $-\frac{2}{3}\beta a^{3/2}$ exponent.

The slightly less cheap proof of the left tail, lower bound

This uses the Riccati diffusion:

$$\begin{aligned}
\mathbb{P}(TW_\beta < -a) &= \mathbb{P}_\infty(p(x,a,\beta) \text{ never explodes}) \\
&\geq \mathbb{P}_1(p(x,a,\beta) \text{ never explodes }) \\
&\geq \mathbb{P}_1(p(x,a,\beta) \in [0,2] \text{ for } x \leq a) P_0(p(x,0,\beta) \text{ never explodes}).
\end{aligned}$$

The last step is the Markov property. The second factor is just some number, the tail comes from the first factor.

Intuition in potential space is that asymptotics should stem from

$$\frac{2}{\sqrt{\beta}} b'(x) + x = a \text{ for } x < a,$$

and

$$\mathbb{P}\left(\left| \frac{2}{\sqrt{\beta}} b(x) - (ax - x^2/2) \right| \leq 1 \text{ for } x < a \right) \sim e^{-\frac{\beta}{24} a^3}.$$

Last two bounds

The estimate

$$\mathbb{P}\Big(TW_\beta \ge a\Big) \le \exp\left(-\frac{2}{3}\beta a^{3/2}(1+o(1))\right)$$

is the most involved. (I'll skip it.)

On the other hand...

Homework problem: The left tail, upper bound, or

$$\mathbb{P}\Big(TW_\beta < -a\Big) \le e^{-\frac{\beta}{24}a^3(1+o(1))},$$

will also follow from a choice of test function in the variational formula. Prove it.