Bulk scaling limits, open questions

Based on:

"Continuum limits of random matrices and the Brownian carousel"
B. Valkó, B. Virág. *Inventiones* (2009).

"Eigenvalue statistics for CMV matrices: from Poisson to Clock via random matrix ensembles"
R. Killip, M. Stoiciu. *Duke Math. Journal* (2009).

Bulk scaling

Back to the β-Hermite ensembles

$$\mathbb{P}_\beta(\lambda_1, \lambda_2, \ldots, \lambda_n) = \frac{1}{Z_{n,\beta}} e^{-\beta/4 \sum_{k=1}^n \lambda_k^2} \times \prod_{j<k} |\lambda_j - \lambda_k|^\beta.$$

For GUE we have the bulk scaling limit

$$\frac{1}{(s_t\sqrt{n})^m} \rho^m \left(t + \frac{x_1}{s_t\sqrt{n}}, \ldots, t + \frac{x_m}{s_t\sqrt{n}}\right) \to \det\left(\frac{\sin \pi(x_i - x_j)}{\pi(x_i - x_j)}\right)_{1 \le i,j \le m},$$

and similarly for GOE/GSE.

Here, $s_t = \frac{1}{2\pi}\sqrt{4 - t^2}$ is the density of the limiting semi-circle law.

Diffusion representation for general $\beta > 0$

Valkó and Virág prove the existence of a point process Sine_β such that for following holds.

Let Λ_n be the point process corresponding to $\mathbb{P}_\beta$ and take any sequence μ_n with

$$n^{1/6}(2\sqrt{n} - |\mu_n|) \to \infty.$$

Then,

$$\sqrt{4n - \mu_n^2}\,(\Lambda_n - \mu_n) \Rightarrow \mathrm{Sine}_\beta.$$

The Sine_β process has a full description as a functional of Brownian motion on the hyperbolic plane, I'll just describe its "marginals".

Counting points of Sine_β

Introduce the one-dimensional process $t \mapsto \alpha_t$

$$d\alpha_t = \lambda f(t) dt + 2 \sin(\alpha_t/2) db_t$$

for $f(t) = (\beta/4)\, e^{-\beta t/4}$ and $\alpha_0 = 0$.

As $t \to \infty$,

$$\alpha_t \;\to\; \text{a (random) integer multiple of } 2\pi.$$

Then:

Let $N(\lambda)$ *be the number of points of* Sine_β in $[0, \lambda]$. *There is the equality in law*

$$N(\lambda) = \alpha(\infty)/2\pi.$$

This is the $n \to \infty$ description for the asymptotic number of eigenvalues for the β-Hermite ensemble in a window of length $\lambda/\sqrt{n}$.

Phase equations

The idea is to study the limiting recursion equations (this is like doing the Riccati directly on the discrete model). Conjugate the β-Hermite model to produce an equivalent random tridiaonal

$$H_n = \begin{bmatrix} X_n & s_n + Y_{n-1} & & & \\ s_{n-1} & X_{n-1} & s_{n-1} + Y_{n-1} & & \\ & & s_{n-2} & X_{n-2} & \ddots \\ & & & \ddots & \ddots \end{bmatrix}$$

where

$$X_j = \frac{g_j}{\sqrt{\beta}}, \quad s_j = \sqrt{j - 1/2}, \quad Y_j = \frac{\chi^2_{\beta(j-1)}}{\beta s_{j-1}} - s_j.$$

Fix λ and let $u_\ell = u_{\ell,\lambda}$ be a solution of

$$s_{n-\ell} u_\ell + X_{n-\ell} u_{\ell+1} + (Y_{n-\ell} + s_{n-\ell}) u_{\ell+2} = \lambda u_{\ell+1}, \quad 0 \le \ell \le n-2$$

with $u_0 = 0, u_1 = 1$.

Fact: λ is an eigenvalue only if $u_{n+1} = 0$.

Centered phase

Put $r_\ell = u_{\ell+1}/u_\ell$ which solves

$$r_{\ell+1} = \left(-\frac{1}{r_\ell} + \frac{\lambda}{s_{n-\ell}} - \frac{X_{n-\ell}}{s_{n-\ell}} \right) \left(1 + \frac{Y_{n-\ell}}{s_{n-\ell}} \right)^{-1},$$

starting at $r_0 = \infty$ (feels a bit more Riccati). Now, λ is an eigenvalue if $r_n = 0$.

What they show is that a mollified version of the phase

$$\theta_\ell = \arctan(r_\ell)$$

converges to the process defined by the stochastic differential equation introduced above.

β-Circular ensembles

Turn to the measure on $(\theta_1, \theta_2, \ldots, \theta_n) \in [0, 2\pi)^n$ defined by the density

$$\mathbb{P}_\beta(\theta_1, \ldots, \theta_n) = \frac{1}{Z_\beta} \prod_{j<k} \left| e^{i\theta_j} - e^{i\theta_k} \right|^\beta$$

These should be considered as the general β versions of the eigenvalue ensembles of the classical compact groups (O(n), U(n)...) introduced in P. Diaconis' talks.

For these objects, it is all bulk.

Rather than tridiagonal matrix models, one has a family of five-diagonal matrices which produce these laws.

CMV Matrices

Given a sequence $\alpha_0, \alpha_1, \dots \in D$ (the open unit disk in the complex plane), define the 2×2 matrices

$$\Xi_k = \begin{bmatrix} \bar{\alpha}_k & \rho_k \\ \rho_k & -\alpha_k \end{bmatrix}$$

where $\rho_k = \sqrt{1 - |\alpha_k|^2}$. From these build the block diagonals

$$L = \text{diag}\Big(\Xi_0, \Xi_2, \Xi_4, \dots\Big) \text{ and } M = \text{diag}\Big(\Xi_{-1}, \Xi_1, \Xi_3, \dots\Big),$$

where $\Xi_{-1} = [1]$

Then, the CMV matrix associated to $\alpha_0, \alpha_1, \dots$ is

$$C(\alpha) = LM.$$

An important discovery of Killip and I. Nenciu (IMRN 2004) is that for a certain choice of *random* α's, the eigenvalues of $C(\alpha)$ are distributed according to circular beta ensemble.

Identifying the limit

CMV matrices are connected to Orthogonal Polynomials on the unit circle in a way that Jacobi (tridiagonal) matrices are linked to OPs on the line. Following the "phase" Killip-Stoiciu prove:

For any x consider

$$d\psi(x,t) = x\,dt + \frac{2}{\sqrt{\beta t}}\mathrm{Im}\left\{[e^{i\psi(x,t)} - 1][db_1(t) + i\,db_2(t)]\right\}.$$

At $t = 1$ this is an increasing function of x.

Then, with $\Psi(x) = \psi^{-1}(x, t = 1)$ and any $f \in C_0^\infty$:

$$\lim_{n\to\infty} \mathbb{E}_\beta\left[e^{\sum_{k=1}^{n} f(n\theta_k)}\right] = \mathbb{E}\left[\int_0^{2\pi} \exp\left(-\sum_{m\in Z} f \circ \Psi(2\pi m + \omega)\right) d\omega\right].$$

Problem #1: Equivalence of bulk limits

Show that the limiting beta bulk statistics as described by Valko-Virág and Killip-Stoiciu are the same.

Granted these start with different types of β-ensembles but for $\beta = 1, 2, 4$ the bulk limits are the same for $G\{O/U, S\}E$ and $C\{O/U/S\}E$. Namely you see sine kernel.

Problem #2: Compute anything

Does the general TW_β distribution function have description in terms of Painlevé II?

Maybe a hint: Want to compute

$$\lim_{a\uparrow\infty}\lim_{L\uparrow\infty} P_a\Big(p(x,\lambda,\beta) \text{ does not explode for } x \le L\Big).$$

By the Cameron-Martin formula, can write this probability as in

$$\int_{p(0)=a} e^{\frac{\beta}{4}\int_0^L [\lambda+x-p^2(x)]dp(x)-\frac{\beta}{8}\int_0^L[\lambda+x-p^2(x)]^2dx} \times e^{-\frac{\beta}{8}\int_0^L[p'(x)]^2dx} \frac{d^\infty p}{(2\pi 0^+)^{\infty/2}}$$

By Itô, the first exponent only gives a boundary contribution. Thus we expect the path to reside (in a neighborhood) of where the functional

$$p \mapsto \int_0^L \Big[(\lambda+x-p^2(x))^2 + (p'(x))^2\Big]\,dx$$

is minimized. The associated Euler-Lagrange equation is Painlevé II.

Problem #2: Compute anything, con't

The Hard Edge is even more frustrating. Just from the density:

$$P_{\beta,a}(\lambda_1,\ldots,\lambda_n) = \frac{1}{Z_{\beta,a}} \prod_{j<k} |\lambda_j - \lambda_k|^\beta \times \prod_{k=0}^{n-1} \lambda_k^{\frac{\beta}{2}(a+1)-1} e^{-\frac{\beta}{2}\lambda_k}$$

one can see:

$$\text{Whenever } \frac{\beta}{2}(a+1) = 1 \text{ one has } n\lambda_{min} \Rightarrow \exp(\beta/2).$$

Where is this in the limiting operator description? Recall there we have

$$speed(dx) = \exp\left[-(a+1)x - \frac{2}{\sqrt{\beta}} b(x)\right] dx, \quad scale(dx) = \exp\left[ax + \frac{2}{\sqrt{\beta}} b(x)\right] dx.$$

The only thing "special" I see when $\frac{\beta}{2}(a+1) = 1$ is that the speed measure (invariant measure) is a martingale.

Problem #3: Putting in time

There is a dynamical version of GUE (Dyson's Brownian motion). Namely, replace each independent real component with a Brownian motion:

$$M_{jk} = m^r_{jk} + \sqrt{-1} m^c_{jk} \mapsto b^r_{jk}(t) + \sqrt{-1} b^c_{jk}(t)$$

This gives reproduces GUE at $t = 1$ (alternatively could make a stationary version with Ornstein Uhlenbeck rather than Brownian entries.)

More interestingly, the n-eigenvalues $t \mapsto (\lambda_1(t), \ldots, \lambda_n(t))$ performs a diffusion (is Markov) and

$$a_t n^{1/6} \Big(\lambda_{max}(t) - b_t \sqrt{n} \Big) \Rightarrow Airy(t),$$

a continuous-time process with Tracy-Widom marginals.

Should correspond to an evolution of Random Airy Operators. My conjecture is

$$\mathcal{H}_\beta(t) = -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} B(x,t)$$

where $B(x,t)$ is White in space and Ornstein Uhlenbeck (!?) in time.

Problem #4: Proving the other half

At the edges we get a limit operator and then a Riccati substitution gives a diffusion that counts eigenvalues. So....

Can you get the edge diffusions directly from a limit theorem on the tridiagonal recursions?

More interesting. In the bulk they went directly to the recursions (or phase function). Is there a continuum random operator description of the bulk?

Problem #5: Full hard to soft transition

We proved

$$\frac{a^2 - \Lambda_{hard}(2a, \beta)}{a^{4/3}} \Rightarrow \Lambda_{soft}(\beta)$$

via the Riccati diffusions.

Should be the case that the full hard-edge point processes converge to the soft edge process.

Appears painful (and a bit besides the point) to prove this via the Riccati business. More satisfying would be a direct understanding of why (the rescaled)

$$spec\Big(-e^{x}\left[\frac{d^2}{dx^2} - (a + \frac{2}{\sqrt{\beta}}b'(x))\frac{d}{dx}\right]\Big)$$

goes over to

$$spec\Big(\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}}b'(x)\Big)$$

as $a \uparrow \infty$. In particular, can we get convergence of the resolvents?

Problem #6: Connections to RWRE

Our hard-edge operator

$$-\mathfrak{G}_{\beta,a} = e^x \left[\frac{d^2}{dx^2} - (a + \frac{2}{\sqrt{\beta}} b'(x)) \frac{d}{dx} \right]$$

is intimately connected to the study of Random Walks in Random Environment – it is practically Brox's example of a continuum version of Sinai's walk.

It is a basic fact that: if X_t is the process generated by $-\mathfrak{G}_{\beta,a}$

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}(T_0 > t) = \Lambda_0(-\mathfrak{G}_{\beta,a}).$$

Hard to soft transition implies

$$\lim_{a\to\infty} \frac{1}{a^{4/3}} \lim_{t\to\infty} \frac{1}{t} \log\left(e^{a^2 t} \mathbb{P}(T_0 > t) \right) = TW_\beta.$$

Prove it this way.

More interesting: Is there a qualitatively connected family of RWRE's that satisfy such a thing.

Problem #7: As a mechanism to prove Universality

Wigner matrices: Can you understand enough about the tridiagonal process for "general" Wigner matrices to invoke our CLT $\rightarrow$ spectral convergence result to get TW?

Non-central Wisharts: The largest eigenvalues of sample covariance matrices of type XX^T for i.i.d. X are known to have TW limits. If you take the non-central case

$X\Sigma X^T$ for even diagonal but non-iid Σ

interesting things can happen. Let $\Sigma = (r, \ldots, r, 1, 1, \ldots, 1)$: Then see TW if $r < r_c$, a $k-$fold GUE if $r > r_c$ and a new distribution at the critical r_c. (Baik, Ben Arou, Peche.) Only can do this for $\beta = 2$. The conjecture is that what you get here is the Stochastic Airy Operator with different boundary conditions.

TASEP, LPP, etc: Can you find Stochastic Airy or the attached diffusion in any of these other models??