Beta ensembles and random differential operators: Edelman-Sutton conjectures

β-Hermite ensembles

Consider the measure on $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n$: for any $\beta > 0$,

$$\mathbb{P}_\beta(\lambda_1, \lambda_2, \ldots, \lambda_n) = \frac{1}{Z_{n,\beta}} e^{-\beta/4 \sum_{k=1}^n \lambda_k^2} \times \prod_{j<k} |\lambda_j - \lambda_k|^\beta.$$

At $\beta = 1, 2$ or 4, these are the joint density of eigenvalues for the Gaussian Orthogonal, Unitary, and Symplectic ensembles.

The latter may be built-up entry-wise in terms of independent real, complex, or quaternion Gaussians. They, or at least their eigenvalues, define "solvable" models.

As you have begun and will see, all finite dimensional correlation functions (or "intensities") of the $\beta = 1, 2,$ and 4 points have explicit expressions in terms of Hermite polynomials. This allows for a startling number of very refined, local limit theorems for the spectrum.

Tracy Widom laws

The basic result is: with λ_{max} the largest eigenvalue for G(O/U/S)E with the normalization just introduced,

$$\mathbb{P}\Big(n^{1/6}(\lambda_{max} - 2\sqrt{n}) \le \lambda\Big) \to \begin{cases} \exp\Big(-\frac{1}{2}\int_\lambda^\infty (s-\lambda)u^2(s)\,ds\Big)\exp\Big(-\frac{1}{2}\int_\lambda^\infty u(s)\,ds\Big) \\ \exp\Big(-\int_\lambda^\infty (s-\lambda)u^2(s)\,ds\Big) \\ \exp\Big(-\frac{1}{2}\int_{\lambda'}^\infty (s-\lambda')u^2(s)\,ds\Big)\cosh\big(\int_{\lambda'}^\infty u(s)\,ds\big). \end{cases}$$

Here, $u(s)$ is the solution of $u'' = su + 2u^3$ (Painlevé II) subject to $u(s) \sim Ai(s)$ at infinity. ($\lambda' = 2^{2/3}\lambda$ in the GSE law.)

For other $\beta \neq 1, 2, 4$ there is certainly a $\lambda_{max}(\beta)$ under $\mathbb{P}_\beta$ and presumably some limit law

$$n^\star\left(\lambda_{\max}(\beta) - 2\sqrt{n}\right) \Rightarrow TW_\beta$$

defining a general Tracy-Widom(β) law, interpolating amongst the classic triple. (Try guessing yourself!)

Note that the fact that the centering is still $2\sqrt{n}$ is obvious; presumably $\star = 1/6$, but this is not obvious.

Log-gas viewpoint

Exponentiating the interaction term,

$$\mathbb{P}_\beta(\lambda_1, \lambda_2, \ldots, \lambda_n) = \frac{1}{Z_{n,\beta}} \exp\left[-\beta\left(\frac{1}{4}\sum_{k=1}^{n} \lambda_k^2 - \sum_{j<k} \log|\lambda_j - \lambda_k|\right)\right],$$

makes our measure(s) look like a 1-d caricature of a coulomb gas.

(It also is one way to see that you still have semi-circle law – view the above as a measure on measures – in particular, on $n^{-1}\sum_{1\le k\le n}\delta_{\lambda_k/\sqrt{n}}$.)

This is a point of view taken seriously, see for instance the forthcoming book 'Log-gases and Random Matrices' by Peter Forrester which explains among other things the connections of the above to Calogero-Sutherland quantum systems and also to Jack polynomials.

Deal is: There is as of yet no forms of the general β correlations that seem amenable to asymptotics.

Matrix Models

It was therefore a big deal when Dumitriu-Edelman (*J. Math. Phys.* **43**, no. 11, 5830-5847 (2002)) discovered a corresponding (simple) family of matrix models for all $\beta > 0$.

In particular, the eigenvalues of

$$H_n^\beta = \frac{1}{\sqrt{\beta}} \begin{bmatrix} \sqrt{2}g_1 & \chi_{\beta(n-1)} & & & \\ \chi_{\beta(n-1)} & \sqrt{2}g_2 & \chi_{\beta(n-2)} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{\beta 2} & \sqrt{2}g_{n-1} & \chi_\beta \\ & & & \chi_\beta & \sqrt{2}g_n \end{bmatrix}$$

have joint density $\mathbb{P}_\beta$.

Here $g_1, g_2, \ldots$ are independent unit Gaussians and each χ_r stands for the $\sqrt{\ }$ of an independent χ^2 with parameter r. In other words

$$\mathbb{P}\left(\chi_r \in dx\right) = \frac{1}{2^{(r-2)/2}\Gamma(r/2)} x^{r-1} e^{-x^2/2} dx$$

for $x > 0$.

Householder

When $\beta = 1, 2$ these Jacobi models may be derived from the "full" G(O/U)E via appropriate conjugations, as was already observed and used in an RMT context by H. Trotter (*Adv. in Math.* **54** no. 1, 67-82 (1984)).

For instance, take $X_n \sim GOE$ ($\beta = 1$). Set

$$\zeta_n = X_n(1,1)/\sqrt{2} \sim N(0,1) \text{ and } Z_{n-1}^T = (X_n(1,2), \ldots X_n(1,n)),$$

which is independent of both ζ_n and the lower $(n-1)$-minor. Next, find an orthogonal $\tilde{U}_n$ measurable $\sigma(Z_{n-1})$ such that

$$\tilde{U}_n Z_{n-1} = (||Z_{n-1}||_2, 0, \ldots 0). \text{ If now } U_n = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{U}_n \end{bmatrix},$$

then

$$U_n X_n U_n^T = \begin{bmatrix} \sqrt{2}\zeta_n & ||Z_{n-1}||_2 & 0_{n-2} \\ ||Z_{n-1}||_2 & & \\ & X_{n-1} & \\ 0_{n-2} & & \end{bmatrix}$$

where $X_{n-1} \sim GOE$ and is independent of ζ_n and $||Z_{n-1}||_2 \sim \chi_{n-1}$.

Homework Problem: Find a choice for $\tilde{U}_n$ and iterate the above to finish the job.

Sketch for general beta

Indexing the tridiagonal as

$$X_n(k,k) = a_{n-k+1}, k = 1, \ldots n, \text{ and } X_n(k,k+1) = b_{n-k}, k = 1, \ldots n-1,$$

we have for the map to eigenvalues/first eigenvector components

$$(a,b) \mapsto (\lambda, v) : \mathbb{R}^n \times \mathbb{R}_+^{n-1} \mapsto \triangle_n \times S_+^{n-1} \text{ has } Jac = \frac{\triangle(\lambda)}{\prod_{i=1}^{n-1} b_i^{i-1}}.$$

This can be read off from the GOE case: full density on (λ, U) is proportional to

$$\triangle(\lambda) e^{-\sum_{i=1}^{n} \lambda_i^2/4} = J e^{-\sum_{i=1}^{n} a_i^2/4 - \sum_{i=1}^{n-1} b_i^2/2} \prod_{i=1}^{n-1} b_i^{i-1}.$$

Then use a general fact for Jacobi matrices

$$\triangle(\lambda) = \frac{\prod_{i=1}^{n-1} b_i^i}{\prod_{i=1}^{n} v_i},$$

a proof of which you can find in Ch. 4 of "Introduction to Random Matrices" by Anderson, Guoinnet, and Zeitouni.

Scaling the operator: first Edelman-Sutton conjecture

With H_n^β the random tridiagonal, view the $n \uparrow \infty$ limit as a continuum limit for the sequence of operators

$$\tilde{H}_n^\beta = n^{1/6}\Big(2\sqrt{n} I_n - H_n^\beta\Big).$$

Formally employing $\chi_{\beta(n-k)} \approx \sqrt{n\beta} - \frac{\sqrt{\beta}}{2}\frac{k}{n} + \frac{1}{\sqrt{2}}\tilde{g}_k$ entry-wise yields:

$$\tilde{H}_n^\beta \quad \approx \quad n^{2/3} \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \ddots \end{bmatrix} + \frac{1}{2n^{1/3}} \begin{bmatrix} & 1 & & \\ 1 & & 2 & \\ & \ddots & \ddots & \ddots \end{bmatrix}$$
$$+ \frac{n^{1/6}}{(2\beta)^{1/2}} \begin{bmatrix} g_1 & \tilde{g}_1 & & \\ \tilde{g}_1 & g_2 & \tilde{g}_2 & \\ & \ddots & \ddots & \ddots \end{bmatrix}.$$

Prompting Edelman-Sutton to conjecture

$$\tilde{H}_n^\beta \to \mathcal{H}_\beta = -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} b'(x)$$

at the level of the corresponding spectra.

Aside on Brownian Motion

The stochastic process

$$x \mapsto b(x)$$

normalized so that $b(0) = 0$ is a Brownian motion if, for any finite collection of time co-ordinates

$$b(x_1), b(x_2), \ldots, b(x_n) \qquad x_1 < x_2 < \cdots < x_n$$

is multivariate Gaussian specified by: the increments $b(x_{k+1}) - b(x_k)$ are independent, and

$$\mathbb{E}(b(x_{k+1}) - b(x_k)) = 0, \quad \mathbb{E}(b(x_{k+1}) - b(x_k))^2 = x_{k+1} - x_k.$$

Brownian motion may be taken a.s. continuous, but is nowhere differentiable. (It is $C^{1/2-}$). On the other hand $b'(x)$ may be viewed as a random distribution (understood via an integration by parts procedure) and that part of the Edelman-Sutton conjecture may understood by noting we have

$$c(\beta) n^{1/6} \mathfrak{g} = c(\beta)(\Delta x)^{-1/2} \left(\frac{b(x + \Delta x) - b(x)}{\sqrt{\Delta x}} \right),$$

using the scaling properties of the Gaussian (the equality is in law).

General $\beta > 0$ soft-edge theorem

With J. Ramírez and B. Virág we prove:

The (suitably interpreted) random operator

$$\mathcal{H}_\beta = -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} b'(x),$$

acting of $L^2(\mathbb{R}^+)$ *with Dirichlet conditions at the origin, has a.s. discrete spectrum* $\Lambda_0(\mathcal{H}_\beta) < \Lambda_1(\mathcal{H}_\beta) < \cdots$, and, with $\lambda_1(\beta) < \lambda_2(\beta) < \cdots$ *the ordered* β*-Hermite eigenvalues,*

$$\{n^{1/6}(2\sqrt{n} - \lambda_j(\beta))\}_{1 \le j \le k} \Rightarrow \{\Lambda_0(\beta), \ldots, \Lambda_{k-1}(\beta)\}$$

for any fixed k *as the dimension goes to infinity.*

In particular we know have: over an appropriate Sobolev space,

$$TW_\beta = -\Lambda_0(\beta) = \sup \left\{ \frac{2}{\sqrt{\beta}} \int_0^\infty f^2(x) db(x) - \int_0^\infty [f'(x)^2 + x f^2(x)] dx \right\},$$

which defined the "general Tracy-Widom law", independently of any random matrix developments.

Wishart ensembles, minimal eigenvalues

Switching gears, consider next a random matrix ensemble of type XX^* where all entries of the $n \times m$ matrix X are independent complex Gaussians of mean zero and mean-square one

All eigenvalues are (obviously) non-negative and the shape of the density of states depends on the limit of the ratio $\frac{m}{n} \geq 1$: say $\frac{m}{n} \to c$, then

$$\frac{1}{n}\sum_{k=1}^{n} \delta_{\lambda_k/n}(\lambda) \to \frac{1}{2\pi\lambda}\sqrt{(\lambda-\ell)^+(r-\lambda)^+}\,d\lambda$$

where $\ell = (1-\sqrt{c})^2$ and $r = (1+\sqrt{c})^2$. This is the Marchenko-Pastur law.

When $c > 1$ both ℓ and r are soft edges, and see $O(n^{-1/6})$ fluctuations with Tracy-Widom limits at both points.

If $c = 1$, then $\ell = 0$ and sees a different type of phenomenon. The eigenvalues now feel the "hard edge" of the origin and exhibit $O(n^{-1})$ fluctuations, with a separate family of limit laws after scaling.

Classical hard edge laws

In fact, if $m = n + a$ these limit laws feel the deviation from square (that is, they depend on a).

The deal is: the full joint density of eigenvalues is now

$$P(\lambda_1, \lambda_2, \dots, \lambda_n) = \frac{1}{Z_n} \times \prod_{j<k} |\lambda_j - \lambda_k|^2 \prod_{1 \le k \le n} \lambda_k^a e^{-\lambda_k},$$

with explicit correlation functions written in terms of Laguerre polynomials. Thus the alternate tag "Laguerre Unitary Ensemble"

Using the Fredholm determinant gap formulas, Tracy-Widom have also shown:

$$\lim_{n\to\infty} P_{LUE}\Big(n\lambda_{min} \ge t\Big) = \exp\left(-\frac{1}{4}\int_0^t \log(t/s) r^2(s,a)\, ds\right)$$

where $r(s,a)$ is a certain solution of Painlevé V.

(Various "special cases": when $a = 0$, $r \equiv constant$ and the limiting minimal eigenvalue is simply exponentially distributed.)

Beta versions

Introduce the (β, a)-Laguerre ensembles

$$P_{L\beta E}(\lambda_1, \ldots, \lambda_n) = \frac{1}{Z_{\beta,a}} \prod_{j<k} |\lambda_j - \lambda_k|^\beta \times \prod_{k=1}^{n} \lambda_k^{\frac{\beta}{2}(a+1)-1} e^{-\frac{\beta}{2}\lambda_k},$$

which are sensible for all $\beta > 0$ and $a > -1$, and agree with the eigenvalue laws for the classical Wishart ensembles when $a = 0, 1, 2, \ldots$ and $\beta = 2$ (or $1, 4$).

There is again a tridiagonal matrix model due to Dumitriu-Edleman. Let

$$L_{\beta,a} = \frac{1}{\sqrt{\beta}} \begin{bmatrix} \chi_{(a+n)\beta} & \chi_{(n-1)\beta} & & & \\ & \chi_{(a+n-1)\beta} & \chi_{(n-2)\beta} & & \\ & & \ddots & \ddots & \\ & & & \chi_{(a+2)\beta} & \chi_\beta \\ & & & & \chi_{(a+1)\beta} \end{bmatrix}$$

where all variables are independent. Then, the eigenvalues of $L_{\beta,a} L_{\beta,a}^T$ have $P_{L\beta E}$ as their joint density function.

Edelman-Sutton hard-edge conjecture

Using the same "random difference operator" heuristic, Edelman-Sutton suggested:

Let s_k *denote the* k-th *smallest singular value of the bidiagonal operator* $L_{\beta,a}$. *Then, as* $n \uparrow \infty$ *the family* $\{\sqrt{n} s_k\}$ *converges in law to the corresponding singular values of*

$$\mathcal{L}_{\beta,a} = -\sqrt{x}\frac{d}{dx} + \frac{a}{2\sqrt{x}} + \frac{1}{\sqrt{\beta}}b'(x).$$

Here, $\mathcal{L}_{\beta,a}$ *is understood to act on functions* $f \in L^2[0,1]$ *subject to* $f(1) = 0$ *and* $(\mathcal{L}_{\beta,a}f)(0) = 0$.

I don't know how to make this rigorous in the present form. Specifically, the needed compactness for convergence is an issue. Also the boundary condition at zero is a bit odd – though it does produce the correct $\beta = \infty$ answer.

Homework Problem: Derive the form of $\mathcal{L}_{\beta,a}$ from the tridiagonal $L_{\beta,a}$ using the same CLT argument entry-wise on the chi-variables.

A second-order operator for the hard edge

What we prove is: with J. Ramírez,

With probability one, when restricted to the positive half-line with Dirichlet conditions at the origin,

$$\mathfrak{G}_{\beta,a} = -e^x \Big(\frac{d^2}{dx^2} - (a + \frac{2}{\sqrt{\beta}} b'(x)) \frac{d}{dx} \Big),$$

has discrete spectrum comprised of simple eigenvalues $0 < \Lambda_0(\beta,a) < \Lambda_1(\beta,a) < \cdots \uparrow \infty$. Moreover, with now $0 < \lambda_0 < \lambda_1 < \cdots < \lambda_n$ the ordered (β,a)-Laguerre eigenvalues,

$$\{n\lambda_0, n\lambda_1, \dots, n\lambda_k\} \Rightarrow \{\Lambda_0(\beta,a), \Lambda_1(\beta,a), \dots, \Lambda_k(\beta,a)\}$$

as $n \rightarrow \infty$.

Here still the definition of $\mathfrak{G}_{\beta,a}$ must be taken in quotes. We will see that the eigenfunctions will only be $C^{3/2-}$, so the white noise cannot be integrated by parts away. (It is actually much simpler to put this on firm ground than the Stochastic Airy Operator.)

Hitting time descriptions of the edge laws

Use $\Lambda(\mathcal{H}_\beta)$ or $\Lambda(\mathfrak{G}_{\beta,a})$ to denote the limiting soft or hard edge law. (The limiting maximal/minimal eigenvalue in each set-up, and the ground state eigenvalue for each of the two continuum operators introduced.)

Associate with $\Lambda(\mathcal{H}_\beta)$ the random process $p(t)$ defined by

$$dp(t) = \frac{2}{\sqrt{\beta}} db(t) + (t - \lambda - p^2(t))dt,$$

and with $\Lambda(\mathfrak{G}_{\beta,a})$,

$$dp(t) = \frac{2}{\sqrt{\beta}} p(t) db(t) + ((a + 2/\beta)p(t) - p^2(t) - \lambda e^{-t})dt.$$

These are Itô equations. The presence of $-p^2$ in each "drift" term means either process may be started at $+\infty$ and it is possible for either process to hit $-\infty$ in finite time.

The result is

$$\mathbb{P}(\Lambda > \lambda) = \mathbb{P}\Big(t \mapsto p(t) \text{ never hits } -\infty \,\Big|\, p(0) = +\infty \Big).$$

A brief aside on Itô equations

One wants to interpret the random differential equation: thinking time-homogeneous,

$$dX_t = \sigma(X_t, t)db_t + f(X_t, t)dt,$$

as forming the process $t \mapsto X_t$ by piecing together infinitesimal Normals, of mean $f(x,t)\Delta t$ and variance $\sigma^2(x,t)\Delta t$, (x,t) being the present coordinate.

One can build a theory based on the integrated version

$$X_t = X_0 + \int_0^t \sigma(X_s, s)db_s + \int_0^t f(X_s, s)ds,$$

which can be solved by a familiar Picard scheme, with familiar sounding results: the solution exists and is unique, up to a possible random "explosion" time, if the coefficients are Lipschitz. The solution is continuous and Markov.

The novelty here is the Brownian integral. One most easily develops an L^2 theory, hinging on the "Itô isometry" via Itô's $(db)^2 = dt$ rule:

$$\mathbb{E}\left(\int_0^t f(s,\omega)db_s\right)^2 = \int_0^t \mathbb{E}f^2(s,\omega)ds.$$

Riccati's map

In the soft-edge case, we turn the equation

$$\mathcal{H}_\beta \psi(x) = -\psi''(x) + x\psi(x) + \frac{2}{\sqrt{\beta}} b'(x)\psi(x) = \lambda\psi(x)$$

(linear, second-order) into

$$p'(x) = \frac{2}{\sqrt{\beta}} b'(x) + x - \lambda - p^2(x)$$

(quadratic, first-order).

This is what Riccati's transformation is for: begin with a non-vanishing solution to

$$-\psi''(x) + V(x)\psi(x) = \lambda\psi(x)$$

with "traditional" potential $V(x)$. Then,

$$p(x) = \frac{\psi'(x)}{\psi(x)} \text{ solves } V(x) = \lambda + p'(x) + p^2(x).$$

We only need make sense of this for $V(x) = x + \frac{2}{\sqrt{\beta}} b'(x)$.

Sturm-Oscillation

With λ fixed, the sine-like solution $\psi_0(x,\lambda)$ of $\mathcal{H}_\beta \psi = \lambda\psi$ with

$$\psi_0(0) = 0 \text{ and } \psi_0'(0) = 1.$$

solves the Itô system:

$$d\psi_0'(x) = (x-\lambda)\psi_0(x) + \frac{2}{\sqrt{\beta}}\psi_0(x)db(x), \quad d\psi_0(x) = \psi_0'(x)dx.$$

(It is Markov, as is $p(x) = \psi_0'(x)/\psi_0(x)$ which solves the advertised SDE.)

The standard "shooting" method dictates that: with $\Lambda_0(L)$ the minimal Dirichlet eigenvalue on $[0,L]$,

$$\Big\{\Lambda_0(L) > \lambda\Big\} \Leftrightarrow \Big\{\psi_0(x,\lambda) \text{ has no root before } x = L\Big\}.$$

Further, the event that $\psi_0(x,\lambda)$ does not vanish during $x \le L$ is the p-event that

$$\left\{ \begin{array}{c} p(x,\lambda) \text{ begun at } \frac{\psi_0'(0,\lambda)}{\psi_0(0,\lambda)} = +\infty \text{ at } x = 0 \\ \text{fails to hit } -\infty \text{ before } x = L. \end{array} \right\}.$$

Remark: Counting multiple explosions gives description of the distribution function for higher eigenvalues.

Homework Problem: For $-\psi''(x) + V(x)\psi(x) = \lambda\psi(x)$ on $[0, L]$ with nice V, show that

$$\#\Big\{\text{Dirichlet eigenvalues } \leq \lambda\Big\} = \#\Big\{\text{zeros of the sine-like } \psi(x) \text{ on } (0, L]\Big\}.$$

Homework Problem: At $\beta = \infty$, for the soft edge we have

$$dp(x) = (x - \lambda - p^2(x))dx$$

and

$$dp(x) = (ap(x) - p^2(x) - \lambda e^{-x})dx.$$

at the hard edge.

Check that these ODE's explode or not depending on whether λ is greater or smaller than the first root of the Airy function or J_a Bessel function respectively.