The general $\beta > 0$ Hard Edge

Based on:

Diffusion at the random matrix hard edge,”
Ramírez, J., Rider, B. (*Comm. Math. Phys.) (2009).*

Basic hard edge result

With $\lambda_1(\beta,a) < \lambda_2(\beta,a) < \cdots$ the ordered points under the law

$$P_{\beta,a}(\lambda_1,\ldots,\lambda_n) = \frac{1}{Z_{\beta,a}} \prod_{j<k} |\lambda_j - \lambda_k|^\beta \times \prod_{k=0}^{n-1} \lambda_k^{\frac{\beta}{2}(a+1)-1} e^{-\frac{\beta}{2}\lambda_k},$$

for any finite k the family

$$\{n\lambda_1(\beta,a), n\lambda_2(\beta,a), \ldots, n\lambda_k(\beta,a)\}$$

converges in distribution to the bottom k eigenvalues of the (positive) operator

$$\mathfrak{G}_{\beta,a} = -e^x \left(\frac{d^2}{dx^2} - (a + \frac{2}{\sqrt{\beta}} b'(x)) \frac{d}{dx} \right)$$

acting on $\mathbb{R}^+$ with Dirichlet conditions at the origin.

This is however a completely formal (but suggestive) way to write $\mathfrak{G}_{\beta,a}$.

Making sense of $\mathfrak{G}_{\beta,a}$

The operator should really be viewed in conjugate form

$$\begin{aligned} -\mathfrak{G}_{\beta,a} &= \exp\left[(a+1)x + \frac{2}{\sqrt{\beta}}b(x)\right]\frac{d}{dx}\left\{\exp\left[-ax - \frac{2}{\sqrt{\beta}}b(x)\right]\frac{d}{dx}\right\} \\ &= \frac{d}{dm}\frac{d}{ds} \end{aligned}$$

where

$$m(dx) = \exp\left[-(a+1)x - \frac{2}{\sqrt{\beta}}b(x)\right]dx \quad \text{and} \quad s(dx) = \exp\left[ax + \frac{2}{\sqrt{\beta}}b(x)\right]dx.$$

(So What?)

This connects to the classical (pre-Itô) theory of one-dimensional diffusion: the operator $d/dm\, d/ds$ will unambiguously (pathwise) define a process, which you might take as a way to understand the thing spectrally.

A second aside on diffusions

We've discussed diffusions $t \mapsto X_t$ defined by Itô equations

$$dX_t = \sigma(X_t)db_t + f(X_t)dt.$$

Connected to any such there is a second-order elliptic operator, or generator:

$$\mathfrak{G} = \frac{1}{2}\sigma(x)^2\frac{d^2}{dx^2} + f(x)\frac{d}{dx},$$

the important fact being $p(t,x,dy) := P(X_t \in dy | X_0 = x)$ is the fundamental solution of $\partial_t \Box = \mathfrak{G}\Box$.

Now define the following

$$s(x) = \int^x e^{-\int^{x'} 2f(z)/\sigma(z)dz} dx' \quad \text{and} \quad m'(x) = \sigma(x)^{-2} e^{\int^x 2f(z)/\sigma(z)dz},$$

and note

$$\mathfrak{G} = \frac{1}{2}\frac{1}{m'(x)}\frac{d}{dx}\frac{1}{s'(x)}\frac{d}{dx}.$$

These are the scale function and speed measure, respectively.

Speed and scale

The scale satisfies $\mathfrak{G}s(x) = 0$ and provides a "roadmap" for the process. The speed density satisfies $\mathfrak{G}^* m'(x) = 0$, and is the "reversing" measure of the process ($\mathfrak{G}$ is symmetric in $L^2[m']$).

More to the point: for a Brownian motion $t \mapsto B(t)$ set

$$t \mapsto X_t = s^{-1}\Big(B(T_m^{-1}(t))\Big)$$

in which

$$T_m(t) := \int_0^t m'(s^{-1}(B(t))s'(s^{-1}(B(t)))\,dt.$$

This is a faithful path-wise construction of X_t.

Now you realize that there is no need for (s, m) to have come from nice functions (σ, f); they can even be measures more unpleasant than the ones we have for $-\mathfrak{G}_{\beta,a}$:

$$m(dx) = \exp\left[-(a+1)x - \frac{2}{\sqrt{\beta}}b(x)\right]dx \quad \text{and} \quad s(dx) = \exp\left[ax + \frac{2}{\sqrt{\beta}}b(x)\right]dx.$$

The inverse operator

The same probabilistic theory also provides: with $\tau_0 = \inf\{t > 0 : X(t) = 0\}$,

$$\mathbb{E}_x\left[\int_0^{\tau_0} f(X_t)dt\right] = \int_0^\infty s(x \wedge y) f(y) m(dy).$$

The analytic content of which is

$$(-\mathfrak{G}_{\beta,a}^{-1} f)(y) = \int_0^\infty \left[\int_0^{x\wedge y} e^{az + \frac{2}{\sqrt{\beta}} b(z)} dz\right] f(y)\, e^{-(a+1)y - \frac{2}{\sqrt{\beta}} b(y)} dy$$

for $f \in L^2[\mathbb{R}_+, m]$.

Homework Problem: Check this is really the inverse of $-\mathfrak{G}_{\beta,a}$.

Homework Problem: With the help of the law of the iterated logarithm, $P(|b_x| \le C\sqrt{x \log x}, x \uparrow \infty) = 1$, prove that $(-\mathfrak{G}_{\beta,a})^{-1}$ is a.s. trace class.

This second point makes all the compactness issues we saw in the soft-edge case moot: we can go through the (compact) inverse operators.

Discrete Resolvent

An equivalent tri-diagonal model is $M_{\beta,a}M_{\beta,a}^T$ for

$$M_{\beta,a} = \frac{1}{\sqrt{\beta}} \begin{bmatrix} \chi_{(a+1)\beta} & & & & \\ -\tilde{\chi}_{\beta} & \chi_{(a+2)\beta} & & & \\ & -\tilde{\chi}_{2\beta} & \chi_{(a+3)\beta} & & \\ & & \ddots & \ddots & \\ & & & -\tilde{\chi}_{(n-1)\beta} & \chi_{(a+n)\beta} \end{bmatrix},$$

where the tilde's are to emphasize the independence of the processes along the main and lower diagonals.

For any (lower) bi-diagonal $B = b_{ij}$, the inverse is (lower) triangular with

$$[B^{-1}]_{ij} = \frac{(-1)^{i+j}}{b_{ii}} \prod_{k=j}^{i-1} \frac{b_{k+1,k}}{b_{kk}} \quad \text{for} \quad j \leq i.$$

So,

$$\left[M_{\beta,a}^{-1}v\right]_i = \frac{\sqrt{\beta}}{\chi_{\beta(i+a)}} \sum_{k=1}^{i} \exp\left[\sum_{m=k}^{i-1} \ln(\tilde{\chi}_{\beta m}/\chi_{\beta(m+a)})\right] v_k.$$

Norm convergence

View $i/n \to x \in [0,1]$ and embed into $L^2[0,1]$ in the obvious way. The inverse acts on an $f \in L^2$ via

$$\left[\frac{1}{\sqrt{n}} M_{\beta,a}^{-1} f\right](x) = \frac{\sqrt{\beta n}}{\chi_{\beta([nx]+a)}} \sum_{k=1}^{[nx]} \exp\left[\sum_{m=k}^{[nx]-1} \ln(\tilde{\chi}_{\beta m}/\chi_{\beta(m+a)})\right] \int_{\Delta_k} f(z)dz.$$

where Δ_k indicates the k^{th} subinterval.

Plainly,

$$\frac{\sqrt{\beta n}}{\chi_{\beta([nx]+a)}} \to \frac{1}{\sqrt{x}}$$

and almost as simply

$$\sum_{m=[ny]}^{[nx]} \ln \tilde{\chi}_{\beta m} - \ln \chi_{\beta(m+a)} \Rightarrow \frac{2}{\sqrt{\beta}} \int_y^x \frac{db(z)}{\sqrt{z}} + \frac{a}{2} \log(y/x).$$

(You need to know here that $\int_0^t f(s)db(s) \sim N(0, \int_0^t f^2(s)ds)$.)

Norm convergence, con't

We can in fact build a probability space on which, with again

$$\left[\frac{1}{\sqrt{n}}M_{\beta,a}^{-1}f\right](x) = \frac{\sqrt{\beta n}}{\chi_{\beta([nx]+a)}}\sum_{k=1}^{[nx]}\exp\left[\sum_{m=k}^{[nx]-1}\ln(\tilde{\chi}_{\beta m}/\chi_{\beta(m+a)})\right]\int_{\Delta_k} f(z)dz,$$

and

$$(\mathcal{L}_{\beta,a}^{-1}f)(x) = x^{-1/2}\int_0^x \exp\left[\frac{2}{\sqrt{\beta}}\int_y^x \frac{db(z)}{\sqrt{z}} + \frac{a}{2}\log(y/x)\right] f(y)dy.$$

it holds that

$$||\frac{1}{\sqrt{n}}M_{\beta,a}^{-1} - \mathcal{L}_{\beta,a}^{-1}||_{HS} \to 0 \text{ with probability one.}$$

Thus, $(nM_{\beta,a}M_{\beta,a}^T)^{-1}$ converges a.s. in trace norm to the operator

$$f \mapsto \int_0^1 (xy)^{a/2}\left(\int_{x\vee y}^1 e^{-2\int_z^1 \frac{db_s}{\sqrt{\beta s}}} z^{-(a+1)}dz\right) e^{\int_x^1 \frac{db_s}{\sqrt{\beta s}}} e^{\int_y^1 \frac{db_s}{\sqrt{\beta s}}} f(y)\, dy.$$

Loose ends

We have identified the scaled, limiting inverse of our tridiagonals as the integral operator with kernel

$$K(x,y) = (xy)^{a/2} \left(\int_{x \vee y}^{1} e^{-2\int_z^1 \frac{db_s}{\sqrt{\beta s}}} z^{-(a+1)} dz \right) e^{\int_x^1 \frac{db_s}{\sqrt{\beta s}}} e^{\int_y^1 \frac{db_s}{\sqrt{\beta s}}}$$

acting on $L^2[0,1]$. The advertised limit has kernel

$$K(x,y) = \int_0^{x \wedge y} e^{az + \frac{2}{\sqrt{\beta}} b(z) dz},$$

acting on $L^2[\mathbb{R}_+, m]$. (Recall $m(dx) = e^{-(a+1)x - \frac{2}{\sqrt{\beta}} b(x)} dx$.)

Homework Problem: Check that the above random integral operators have the same eigenvalues. You will need to know the Brownian time change trick, $\int_0^t f_s db_s = \tilde{b}(\int_0^t f_s^2 ds)$ for a new Brownian motion $\tilde{b}$ and deterministic f.

Homework Problem: We have proved a sequence of non-negative, symmetric compact operators converge in trace norm to some other nice, non-negative, symmetric compact operator. Convince yourself the eigenvalues must converge along the way.

Hard to Soft transition

In the classical $\beta = 1, 2, 4$ and integer a cases, you will recall that you see soft edge if $a = O(n)$ as $n \to \infty$. Can we recover the soft edge by now taking $a \to \infty$ after the fact?

Still in those classical cases, one has the explicit Fredholm expressions: at $\beta = 2$ in particular

$$\mathbb{P}\left(\Lambda(-\mathfrak{G}_{\beta,a}) > \lambda\right) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_0^{\lambda} dx_1 \cdots \int_0^{\lambda} dx_k \,\det\left[K_{Bessel}(x_i, x_j)\right]_{i,j=1,\dots,k},$$

while

$$\mathbb{P}\left(TW_2 < \lambda\right) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{\lambda}^{\infty} dx_1 \cdots \int_{\lambda}^{\infty} dx_k \,\det\left[K_{Airy}(x_i, x_j)\right]_{i,j=1,\dots,k}.$$

Using the converges of the J_a appearing in the Bessel kernel to the Airy function, Borodin-Forrester prove the bottom hard-edge law converges to the top soft-edge law. (Also for $\beta = 1$ and $\beta = 2$.)

General transition

We prove: for any $\beta > 0$,

$$\frac{a^2 - \Lambda_0(-\mathfrak{G}_{\beta,2a})}{a^{4/3}} \Rightarrow TW_\beta$$

as $a \uparrow \infty$.

Our proof uses the Riccati diffusions: you will recall...

(Soft Edge) $$dq(x) = \frac{2}{\sqrt{\beta}} db(x) + (\lambda + x - q^2(x))dx,$$

and

(Hard Edge) $$dp(x) = \frac{2}{\sqrt{\beta}} p(x) db(x) + \Big((a + 2/\beta)p(x) - p^2(x) - \lambda e^{-x} \Big) dx.$$

Then, with $\Xi_0 < \Xi_1 < \cdots$ standing in for either limiting soft/hard eigenvalues we have that

$$\mathbb{P}\Big(\Xi_k \leq \lambda\Big) = \mathbb{P}\left(\begin{array}{c} \text{corresponding process hits } -\infty \text{ at least k+1 times} \\ \text{after starting, and having all restarts, at } +\infty \end{array} \right).$$

Comment on hard edge Riccati map

Our integrated eigenvalue problem is

$$\psi(x) = \lambda \int_0^\infty \left(\int_0^{x \wedge y} \mathrm{exp}\left[az + \frac{2}{\sqrt{\beta}} b(z)\right] dz \right) \psi(y) \, \mathrm{exp}\left[-(a+1)y - \frac{2}{\sqrt{\beta}} b(y)\right] dy.$$

Properties of the kernel show that the RHS, and so ψ, lie in C^1. Then compute

$$\psi'(x) = \lambda \, \mathrm{exp}[ax + \frac{2}{\sqrt{\beta}} b(x)] \int_x^\infty \psi(y) \, \mathrm{exp}[-(a+1)y - \frac{2}{\sqrt{\beta}} b(y)] \, dy,$$

to find that ψ is actually in $C^{3/2-}$.

Continue by taking Itô differentials to arrive at the system

$$\begin{aligned} d\psi'(x) &= \frac{2}{\sqrt{\beta}} \psi'(x) db(x) + \Big((a + 2/\beta)\psi'(x) - \lambda e^{-x} \psi(x) \Big) dx, \\ d\psi(x) &= \psi'(x) dx, \end{aligned}$$

which is the appropriate way to interpret $\mathfrak{G}_{\beta,a}\psi = \lambda\psi$ (and defines a Markov process for all time.)

Now can bring in Riccati's map, $p(x) = \psi'(x)/\psi(x)$, valid away from the zeros in the above.

Sketch of the transition proof

The proof of

$$\mathbf{Theorem}: \quad \frac{\boldsymbol{\eta^2 - \Lambda_0(-\mathfrak{G}_{\beta,2\eta-2/\beta})}}{\boldsymbol{\eta^{4/3}}} \Rightarrow \mathbf{TW}_\beta \quad \text{as} \quad \eta \to \infty$$

is made by showing that: with the scalings

$$a = 2\eta^2 - \frac{2}{\beta} > -1 \quad \text{and} \quad \lambda = \eta^2 - \eta^{4/3}\mu,$$

the chance that the process

$$dp(x) = \frac{2}{\sqrt{\beta}} p(x) db(x) + \left((a + 2/\beta) p(x) - p^2(x) - \lambda e^{-x} \right) dx$$

never hits $-\infty$ goes over into the chance that

$$dq(x) = \frac{2}{\sqrt{\beta}} db(x) + (x + \mu - q^2(x)) dx$$

never hits $-\infty$.

Mechanism: soft process as tunneling time

Set $\mu = 0$ for a moment. The scaled p solves

$$dp(x) = \frac{2}{\sqrt{\beta}} p(x)db(x) + (2\eta p(x) - p^2(x) - \eta^2 e^{-x})dx,$$

and obviously $\mathfrak{p}(x) = p(x)/\eta$ explodes or not with p while satisfying

$$d\mathfrak{p}(x) = \frac{2}{\sqrt{\beta}} \mathfrak{p}(x)db(x) + \eta\,(2\mathfrak{p}(x) - \mathfrak{p}^2(x) - e^{-x})dx.$$

For $\eta \uparrow \infty$, $\mathfrak{p}$ comes quickly to the place $\mathfrak{p} = 1$ as

$$(2\mathfrak{p} - \mathfrak{p}^2 - e^{-x}) \approx -(1-\mathfrak{p})^2$$

over short times. And, if it manages to tunnel through this place quickly enough, explosion is hard to avoid.

So, within this (possible) excursion from 1^+ to 1^- in a small x-window, the q-motion must emerge.

Some details

Step 1 is to show

$$\mathbb{P}_{0,+\infty}\Big(\mathfrak{p}\text{ never explodes}\Big) = \mathbb{P}_{\delta,1+\varepsilon}\Big(\mathfrak{p}\text{ never explodes}\Big) + o(1)$$

for

$$\delta = \delta(\eta) = \frac{1}{K}\eta^{-2/3}, \quad \varepsilon = \varepsilon(\eta) = M\eta^{-1/3}.$$

and whatever large M and K.

Certainly,

$$\mathbb{P}_{0,+\infty}\Big(\mathfrak{p}\text{ never explodes}\Big) \geq \mathbb{P}_{0,1+\epsilon}\Big(\mathfrak{p}\text{ never explodes}\Big)$$

for whatever $\varepsilon > 0$. Also,

$$\mathbb{P}_{0,+\infty}\Big(\mathfrak{p}\text{ never explodes}\Big) \leq \mathbb{P}_{\delta,1+\varepsilon}\Big(\mathfrak{p}\text{ never explodes}\Big) + \mathbb{P}_{0,+\infty}\Big(\mathfrak{m}_{1+\epsilon} \geq \delta\Big),$$

and you need an estimate on the second term.

Step 2 introduces

$$q_\eta(x) = \eta^{1/3}\Big(\mathfrak{p}(\eta^{-2/3}x) - 1\Big) \text{ which starts at } (1/K, M)$$

(a fixed time and place) when $\mathfrak{p}$:

$$d\mathfrak{p}(x) = \frac{2}{\sqrt{\beta}}\mathfrak{p}(x)db(x) + \eta\,(2\mathfrak{p}(x) - \mathfrak{p}^2(x) - e^{-x})dx$$

begins at

$$(\delta, 1+\varepsilon) = \left(\frac{1}{K}\eta^{-2/3}, M\eta^{-1/3}\right).$$

Further, $q_\eta(x)$ solves the Itô equation:

$$\begin{aligned} dq_\eta(x) \quad &= \quad \frac{2}{\sqrt{\beta}}\Big[1 + \eta^{-1/3}q_\eta(x)\Big]d\tilde{b}(x) \\ &\quad\quad + \Big[-q_\eta^2(x) + \eta^{2/3}\Big(1 - (1-\eta^{-2/3}\mu)e^{-\eta^{-2/3}x}\Big)\Big]dx \end{aligned}$$

with a new Brownian motion $\tilde{b}$.

Step 3 is to notice that, for $q_\eta(x)$ defined by

$$\begin{aligned} dq_\eta(x) &= \frac{2}{\sqrt{\beta}}\left[1+\eta^{-1/3}q_\eta(x)\right]d\tilde{b}(x) \\ &\quad + \left[-q_\eta^2(x)+\eta^{2/3}\left(1-(1-\eta^{-2/3}\mu)e^{-\eta^{-2/3}x}\right)\right]dx, \end{aligned}$$

one has:

$$\frac{2}{\sqrt{\beta}}\left[1+\eta^{-1/3}q\right] \to \frac{2}{\sqrt{\beta}}$$

and

$$-q^2+\eta^{2/3}\left(1-(1-\eta^{-2/3}\mu)e^{-\eta^{-2/3}x}\right) \to -q^2+\mu+x,$$

the desired limiting diffusion and drift coefficients.

Then one employs general martingale machinery to conclude

$$\mathbb{P}^{q_\eta} \Rightarrow \mathbb{P}^{q}$$

for all finite starting points. A version of the continuous mapping theorem gives you convergence of the passage time, and then you can let $K, M \to \infty$ by continuity.